General Block Design.

Def^n:
General Block Design is an arrangement of v treatments in b blocks of size $k_1, k_2, \dots, k_b$ such that the ith treatment appears r_i times in the design $(i = 1(1)v)$.
Here k_j denotes the number of experimental units within the j th block $j = 1(1)b$.

Notation:

v: Number of treatments.

b: Number of blocks

$\underline{r}'$: Replication vector for the treatments: $(r_1, r_2, \dots, r_v)$

$\underline{k}'$: Block size vector $(k_1, k_2, \dots, k_b)$.

Incidence Matrix:

Let n_{ij} = number of times the i th treatment appear in j th block.
$(i = 1(1)v, \; j = 1(1)b)$

Then $\underset{v \times b}{N} = ((n_{ij}))$ is called the incidence matrix of the design.

Example:

$d =$
1 2 3
2 4
3 5 4 6

Here $v = 6$, $b = 3$, $\underline{r}' = [1\ 2\ 2\ 2\ 1\ 1]$, $\underline{k}' = [3\ 2\ 4]$

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$d =$
1 1 3
2 3 2 3
4 1 1 2 3
3 1 2 2 4
2 1 1 2 3

Here, $v = 4$, $b = 5$

$\underline{r}' = [7\ 7\ 6\ 2]$ $\underline{k}' = [3\ 4\ 5\ 5\ 5]$

$$N = \begin{bmatrix} 2 & 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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$$\sum_{i=1}^{v} n_{ij} = k_j$$

$$\sum_{j=1}^{b} n_{ij} = r_i$$

$$\sum_{i=1}^{v}\sum_{j=1}^{b} n_{ij} = n \text{ (Total no. of experimental units)}$$

$$\underset{v\times b}{N}\ \underset{b\times 1}{\underline{1}} = \underset{v\times 1}{\underline{r}}$$

$$\underset{1\times v}{\underline{1}'}\ \underset{v\times b}{N} = \underset{1\times b}{\underline{K}'}$$

General Block Design is called proper if k_j's are all equal $\forall j$

It is called equi replicate if r_i's are equal $\forall i$

It is called incomplete if $k_j < v$ for atleast one

Analysis:

We consider the fixed effects additive model

$$y_{iju} = \mu + \tau_i + \beta_j + e_{iju}, \quad \begin{array}{l} i=1(1)v \\ j=1(1)b \\ u=1(1)n_{ij} \end{array}$$

Where μ = general effect

τ_i = effect of ith treatment.

β_j = effect of jth block.

e_{iju} = error effect ; y_{iju} = observation corresponding to the u th occurrence of ith treatment in jth block

In matrix notation,

$$\underset{n\times 1}{\underline{Y}} = \mu \underset{n\times 1}{\underline{1}} + \underset{n\times v}{X_\tau}\ \underset{v\times 1}{\tau} + \underset{n\times b}{X_\beta}\ \underset{b\times 1}{\beta} + \underset{n\times 1}{e}$$

$$X_\tau = ((x_{ij}^{\tau})) \quad \text{and} \quad X_\beta = ((x_{ij}^{\beta}))$$

Where, $x_{ij}^{\tau} = \begin{cases} 1 & \text{if ith experimental unit receives jth treatment} \\ 0 & \text{otherwise} \end{cases}$

and $x_{ij}^{\beta} = \begin{cases} 1, & \text{if ith experimental unit comes from jth block} \\ 0 & \text{otherwise} \end{cases}$

Reference to example 1

$$y_{111} = \mu + \tau_1 + \beta_1 + e_{111}$$

$$y_{211} = \mu + \tau_2 + \beta_1 + e_{211}$$

$$y_{221} = \mu + \tau_2 + \beta_2 + e_{221}$$

$$y_{311} = \mu + \tau_3 + \beta_1 + e_{311}$$

$$y_{331} = \mu + \tau_3 + \beta_3 + e_{331}$$

$$y_{421} = \mu + \tau_4 + \beta_2 + e_{421}$$

$$y_{431} = \mu + \tau_4 + \beta_3 + e_{431}$$

$$y_{531} = \mu + \tau_5 + \beta_3 + e_{531}$$

$$y_{631} = \mu + \tau_6 + \beta_3 + e_{631}$$

$$X_\tau = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_2 \\ \tau_3 \\ \tau_3 \\ \tau_4 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_1 \\ \beta_2 \\ \beta_1 \\ \beta_3 \\ \beta_2 \\ \beta_3 \\ \beta_3 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_2 \\ \tau_3 \\ \tau_3 \\ \tau_4 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix}$$

$$X_\tau' X_\tau = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 9} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{9\times 6}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{Diag}(b_1, b_2, \ldots, b_v) = D_b$$

$X_\beta' X_\beta = \text{Diag}(k_1, k_2, \dots, k_b) = D_k$

(i,j) th element of $AB = \sum_{k=1}^{n} a_{ik} b_{kj}$

$A = X_\tau'$ and $B = X_\tau$

(i,j) th element of $X_\tau' X_\tau$

$$= \sum_{k=1}^{n} (x_{ki}^{\tau})\, x_{kj}^{\tau}$$

$$= \begin{cases} \sum_{k=1}^{n} (x_{ki}^{\tau})(x_{kj}^{\tau}) & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$x_{ki}^{\tau}\, x_{ki}^{\tau} = 1$ for those units k which receive treatment i; $(i = 1(1)v)$

$$\sum_{k=1}^{n} x_{ki}^{\tau} x_{ki}^{\tau} = r_i$$

(i,j) th element of $X_\tau' X_\tau$ is r_i if $i=j$

is 0 if $i \neq j$

$$(X_\tau' X_\beta)_{ij} = \sum_{k=1}^{n} (x_{ki}^{\tau})(x_{kj}^{\beta})$$

= Number of experimental units receiving treatment i and block j.

$= n_{ij}$

$$X_\tau' X_\beta = N$$

Define,

$$T_i = \sum_{j=1}^{b} \sum_{u=1}^{n_{ij}} y_{iju}$$

= Total number of all units receiving treatment i $(i=1(1)v)$

$$\underset{v\times n}{X_\tau'} \underset{n\times 1}{\underset{\sim}{y}} = x_{ki}^{\tau}$$

$$= X_\tau' (\mu \underset{\sim}{1} + X_\tau \cdot \underset{\sim}{\tau} + X_\beta \cdot \underset{\sim}{\beta} + \underset{\sim}{\epsilon})$$

$$= \mu . \underset{v\times n}{X_\tau'} \underset{n\times 1}{\underset{\sim}{1}} + X_\tau' X_\tau \underset{\sim}{\tau} + X_\tau' X_\beta \underset{\sim}{\beta} + X_\tau' \underset{\sim}{\epsilon}$$

$$B_j = \sum_{i=1}^{v} \sum_{u=1}^{n_{ij}} y_{iju}$$

$$= \mu . \underset{v\times 1}{\underset{\sim}{r}} + \underset{v\times v}{D_r} \underset{v\times 1}{\underset{\sim}{\tau}} + \underset{v\times b}{N} \underset{b\times 1}{\underset{\sim}{\beta}} + \underset{v\times n}{X_\tau'} \underset{n\times 1}{\underset{\sim}{\epsilon}}$$

= Total number of all units from block j $(j=1(1)b)$

$$\begin{bmatrix} x_{11}^{\tau} & x_{21}^{\tau} & \dots & x_{n1}^{\tau} \\ \vdots & & & \\ x_{1v}^{\tau} & x_{2v}^{\tau} & \dots & x_{nv}^{\tau} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} y_k x_{k1}^{\tau} \\ \sum_{k=1}^{n} y_k x_{k2}^{\tau} \\ \vdots \\ \sum_{k=1}^{n} y_k x_{kv}^{\tau} \end{bmatrix} = \begin{bmatrix} \text{Sum of those } y_i\text{'s having treatment } 1 \\ \text{Sum of those } y_i\text{'s having treatment } 2 \\ \vdots \\ \text{Sum of those } y_i\text{'s having treatment } v \end{bmatrix}$$

$$X_\tau' Y = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_v \end{bmatrix} \qquad X_\beta' Y = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_b \end{bmatrix}$$

$$\underline{Y} = \mu \cdot \underline{1} + X_\tau \underset{\sim}{\tau} + X_\beta \cdot \underset{\sim}{\beta} + \underset{\sim}{e}$$

$$\underset{\sim}{Y}_{n\times 1} = \begin{bmatrix} \underline{1} & X_\tau & X_\beta \end{bmatrix}_{n\times b+v+1} \begin{bmatrix} \mu \\ \underset{\sim}{\tau} \\ \underset{\sim}{\beta} \end{bmatrix}_{b+v+1\times 1} + \underset{\sim}{e}_{n\times 1}$$

$$= X \underline{\theta} + \underline{e}$$

Least square equation

$$X'X\theta = X'\underline{Y}$$

$$X'X = \begin{bmatrix} \underline{1}' \\ X_\tau' \\ X_\beta' \end{bmatrix} \begin{bmatrix} \underline{1} & X_\tau & X_\beta \end{bmatrix} = \begin{bmatrix} \underline{1}'\underline{1} & \underline{1}'X_\tau & \underline{1}'X_\beta \\ X_\tau' & X_\tau'X_\tau & X_\tau'X_\beta \\ X_\beta' & X_\beta'X_\tau & X_\beta'X_\beta \end{bmatrix}$$

$$= \begin{bmatrix} n & \underset{\sim}{r}' & \underset{\sim}{k}' \\ \underset{\sim}{r} & D_r & N \\ \underset{\sim}{k} & N' & D_k \end{bmatrix}$$

$$X'\underline{Y} = \begin{bmatrix} \underline{1}'\underline{Y} \\ X_\tau' Y \\ X_\beta' Y \end{bmatrix} = \begin{bmatrix} G \\ \underset{\sim}{T} \\ \underset{\sim}{B} \end{bmatrix}$$

$$\begin{bmatrix} n & \underset{\sim}{r}' & \underset{\sim}{k}' \\ \underset{\sim}{r} & D_r & N \\ \underset{\sim}{k} & N' & D_k \end{bmatrix} \begin{bmatrix} \mu \\ \underset{\sim}{\tau} \\ \underset{\sim}{\beta} \end{bmatrix} = \begin{bmatrix} G \\ \underset{\sim}{T} \\ \underset{\sim}{B} \end{bmatrix}$$

$$n\hat{\mu} + \underset{\sim}{r}'\hat{\underset{\sim}{\tau}} + \underset{\sim}{k}'\hat{\underset{\sim}{\beta}} = G \quad \ldots (1)$$

$$\underset{\sim}{r}\hat{\mu} + D_r\hat{\underset{\sim}{\tau}} + N\hat{\underset{\sim}{\beta}} = \underset{\sim}{T} \quad \ldots (2)$$

$$\underset{\sim}{k}\hat{\mu} + N'\hat{\underset{\sim}{\tau}} + D_k\hat{\underset{\sim}{\beta}} = \underset{\sim}{B} \quad \ldots (3)$$

Premultiply (3) by ND_k^{-1} & subtract from (2)

$$(\underset{\sim}{r} - ND_k^{-1}\underset{\sim}{k})\hat{\mu} + (D_r - ND_k^{-1}N')\hat{\underset{\sim}{\tau}} + (N - ND_k^{-1}D_k)\hat{\underset{\sim}{\beta}} = \underset{\sim}{T} - ND_k^{-1}\underset{\sim}{B}$$

$$\Rightarrow (\underset{\sim}{r} - ND_k^{-1}\underset{\sim}{k})\hat{\mu} + (D_r - ND_k^{-1}N')\hat{\underset{\sim}{\tau}} = \underset{\sim}{T} - ND_k^{-1}\underset{\sim}{B}$$

$$D_k^{-1}\underset{\sim}{k} = \begin{bmatrix} 1/k_1 & & & \\ & 1/k_2 & & \\ & & \ddots & \\ & & & 1/k_b \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \underset{\sim}{1}_{b\times 1}$$

$$N \cdot \underline{1} = \underset{\sim}{r}$$

$$\underset{\sim}{r} - ND_k^{-1}\underset{\sim}{k} = \underset{\sim}{r} - \underset{\sim}{r} = 0$$

$$(D_r - ND_k^{-1}N')\hat{\underset{\sim}{\tau}} = \underset{\sim}{T} - ND_k^{-1}\underset{\sim}{B}$$

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Or, $C\hat{\tau} = Q'$ (say) $\rightarrow$ Reduced Normal Equations

Where,
$$C = D_r - N D_k^{-1} N'$$ is called the C-matrix of the design for the treatment effects.

$\underset{\sim}{Q} = T - N D_k^{-1} \underset{\sim}{B}$ is called vector of adjusted treatment totals.

$$d = \begin{matrix} 1 & 2 & 3 & \\ 2 & 4 & & \\ 3 & 4 & 5 & 6 \end{matrix}$$

$$D_r = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_k^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$N D_k^{-1} N' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 5/6 & 1/3 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 7/12 & 1/4 & 1/4 & 1/4 \\ 0 & 1/2 & 1/4 & 3/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

$$D_r - N D_k^{-1} N' = \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\ -1/3 & 7/6 & -1/3 & -1/2 & 0 & 0 \\ -1/3 & -1/3 & \frac{17}{12} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & -1/2 & -1/4 & \frac{5}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & -1/4 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

i) $C\underset{\sim}{1} = \underset{\sim}{0}$ (Row sum zero)

$(D_r - N D_k^{-1} N')\underset{\sim}{1}$

$= D_r \underset{\sim}{1} - N D_k^{-1} N' \underset{\sim}{1}$

$= \underset{\sim}{r} - N D_k^{-1} \underset{\sim}{k}$

$= \underset{\sim}{r} - N\underset{\sim}{1} \quad [\because D_k^{-1} \underset{\sim}{k} = \underset{\sim}{1}]$

$= \underset{\sim}{r} - \underset{\sim}{r}$

$= \underset{\sim}{0}$

ii) $\underset{\sim}{Q}'\underset{\sim}{1} = \underset{\sim}{0}$

$$Q = T - N D_K^{-1} \underset{\sim}{B}$$

$$Q' = \underset{\sim}{T}' - \underset{\sim}{B}' D_K^{-1} N'$$

$$\underset{\sim}{Q}'\underset{\sim}{1} = \underset{\sim}{T}'\underset{\sim}{1} - \underset{\sim}{B}' D_K^{-1} N'\underset{\sim}{1}$$

$$= \underset{\sim}{T}'\underset{\sim}{1} - \underset{\sim}{B}' D_K^{-1} \underset{\sim}{K} \quad [\because N'\underset{\sim}{1} = \underset{\sim}{K}]$$

$$= \underset{\sim}{T}'\underset{\sim}{1} - \underset{\sim}{B}'\underset{\sim}{1} \quad [\because D_K^{-1}\underset{\sim}{K} = \underset{\sim}{1}]$$

$$= G - G$$

$$= 0$$

iii) $E(\underset{\sim}{Q}) = C\underset{\sim}{\tau}$

$$Q = X_\tau' \underset{\sim}{Y} - N D_K^{-1} X_\beta' \underset{\sim}{Y}$$

$$= [X_\tau' - N D_K^{-1} X_\beta'] \underset{\sim}{Y}$$

$$E(Q) = [X_\tau' - N D_K^{-1} X_\beta'] [\underset{\sim}{1}\mu + X_\tau \underset{\sim}{\tau} + X_\beta \underset{\sim}{\beta}]$$

$$= X_\tau'\underset{\sim}{1}\mu - N D_K^{-1} X_\beta' \underset{\sim}{1}\mu + X_\tau' X_\tau \underset{\sim}{\tau} - N D_K^{-1} X_\beta' X_\tau \underset{\sim}{\tau} + X_\tau' X_\beta \underset{\sim}{\beta} - N D_K^{-1} X_\beta' X_\beta \underset{\sim}{\beta}$$

$$= \underset{\sim}{b}\mu - N D_K^{-1} \underset{\sim}{K}\mu + D_r \underset{\sim}{\tau} - N D_K^{-1} X_\beta' X_\tau \underset{\sim}{\tau} + X_\tau' X_\beta \underset{\sim}{\beta} - N D_K^{-1} D_K \underset{\sim}{\beta}$$

$$= \underset{\sim}{b}\mu - \underset{\sim}{b}\mu + [D_r - N D_K^{-1} N'] \underset{\sim}{\tau} + N\underset{\sim}{\beta} - N\underset{\sim}{\beta}$$

$$= C\underset{\sim}{\tau}$$

(iv) $Disp(\underset{\sim}{Q}) = C\sigma^2$

$$Q = (X_\tau' - N D_K^{-1} X_\beta') \underset{\sim}{Y}$$

$$Disp(Q) = (X_\tau' - N D_K^{-1} X_\beta') \, Disp(\underset{\sim}{Y}) \, (X_\tau' - N D_K^{-1} X_\beta')'$$

$$= (X_\tau' - N D_K^{-1} X_\beta') \, \sigma^2 \, (X_\tau - X_\beta D_K^{-1} N')$$

$$= \sigma^2 (D_r - N D_K^{-1} N' - N D_K^{-1} N' + N D_K^{-1} D_K D_K^{-1} N')$$

$$= \sigma^2 (D_r - N D_K^{-1} N')$$

$$= C\sigma^2$$

- C is symmetric
- $C\underset{\sim}{1} = \underset{\sim}{0}$

rank $(C) \le v-1$

Reduced normal equations for the block effects

$$D\hat{\underset{\sim}{\beta}} = \underset{\sim}{P}$$

$$D_{b\times b} = D_K - N' D_r^{-1} N \; ; \; P = B - N' D_r^{-1} T$$

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$p'\tau$:- Linear combination of the treatment effects

If $p'\underset{\sim}{1} = 0$, it is called a contrast of the treatment effects.

Necessary condition for $p'\tau$ to be estimable if $p'\underset{\sim}{1}=0$

$p'\tau$ is called an elementary treatment contrast if the vector p has only two non-zero entries, 1 and -1, the other entries being zero.

Def^n (Connectedness of a design)

A block design is said to be connected if all elementary treatment contrasts are estimable.

Theorem: (Rank - definition of connectedness)

A block design is connected iff rank(C) = $v-1$.

Proof:

<u>If part</u>

Suppose rank(C) = $v-1$

Let $\xi_1, \xi_2, \ldots, \xi_{v-1}$ be a set of orthonormal eigenvectors corresponding to the non-zero eigen values $\theta_1, \theta_2, \ldots, \theta_{v-1}$ of C.

$$\therefore C\underset{\sim}{\xi_i} = \theta_i \underset{\sim}{\xi_i}, \quad i = 1(1)v-1$$

$$E(\xi_i' \underset{\sim}{Q}) = \xi_i' C\tau$$

$$= \theta_i \xi_i' \tau$$

$\dfrac{\xi_i' \underset{\sim}{Q}}{\theta_i}$ is an unbiased estimator of $\xi_i'\tau$.

Since each ξ_i is orthogonal to $\underset{\sim}{1}_{v\times 1}$ and also they are mutually orthogonal, any contrast belongs to the vector space spanned by ξ_is ($i=1(1)v-1$)

$$\therefore \underset{\sim}{p} = \sum_{i=1}^{v-1} \lambda_i \underset{\sim}{\xi_i}$$

$$\therefore E\left[\sum_{i=1}^{v-1} \frac{\lambda_i \xi_i' Q}{\theta_i}\right] = \sum_{i=1}^{v-1} \lambda_i \xi_i' \tau$$

$$= p'\tau$$

Thus, $p'\tau$ is estimable.

Only if part

Suppose the design is connected. ~~Cond~~ consider a set of $(v-1)$ linearly independent treatment contrasts $\tau_1 - \tau_j$ $(j = 2(1)v)$

Let us denote the contrasts by $c_2'\tau, c_3'\tau, \ldots, c_v'\tau$

The vectors $\underset{\sim}{c}_2, \underset{\sim}{c}_3, \ldots, \underset{\sim}{c}_v$ forms a basis of dimension $(v-1)$. ⊛

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 1 & 0 & 0 & & & & -1 \end{bmatrix}$$

Also $p'\underset{\sim}{\tau}$ is estimable iff $\underset{\sim}{p} \in$ column space (C)

Thus the dimension of column space of C must be the same as that of the vector space spanned by the vectors $c_2, c_3, \ldots, c_v$ i.e equal to $v-1$.

Hence rank $(C) = v-1$.

Def^n (structural def^n of connectedness)

A treatment i and a block j in a block design are said to be associated if the treatment i appears in block j.

Two treatments are said to be connected if it is possible to pass from one to the other through a chain consisting alternatively of treatments and blocks such that any two members of a chain are associated. Finally a design is said to be connected if every pair of treatments is connected.

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$$C = \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\ -1/3 & 7/6 & -1/3 & -1/2 & 0 & 0 \\ -1/3 & -1/3 & \frac{17}{12} & -1/4 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & -1/2 & -1/4 & 5/4 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & -1/4 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$\frac{7}{6} - \frac{1}{6}$

$-\frac{1}{3} - \frac{1}{6}$

$= \frac{-2-1}{6} = -\frac{1}{2}$

$-\frac{1}{3} - \frac{1}{6}$

$\frac{17}{12} - \frac{1}{6}$

$= \frac{17-2}{12} = \frac{15}{12}$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{5}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & -1/2 & -1/4 & 5/4 & -1/4 & -\frac{1}{4} \\ 0 & 0 & -1/4 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$\frac{5}{4} - \frac{1}{4}$

$=$

$-\frac{1}{4} - \frac{1}{4}$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & -1/2 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & -1/4 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$\frac{5}{8} - \frac{9}{64}$

$= \frac{40-9}{64} =$

$1 - \frac{1}{4}$

$\frac{3}{4}$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & -1/4 & -1/4 \\ 0 & 0 & 0 & \frac{3}{4} & -\frac{3}{8} & -\frac{3}{8} \\ 0 & 0 & 0 & -\frac{3}{8} & \frac{5}{8} & -\frac{3}{8} \\ 0 & 0 & 0 & -\frac{3}{8} & -\frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

$-\frac{1}{4} - \frac{1}{8}$

$= \frac{-2-1}{8} =$

$-\frac{1}{4} - \frac{1}{8}$

$=$

$\frac{3}{4} - \frac{1}{8} = \frac{6-1}{8}$

$-\frac{3}{8} - \frac{9}{64}$

$= \frac{24-9}{64}$

$-\frac{1}{4} - \frac{1}{8} =$

$= \frac{-2-1}{8}$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 31/64 & -\frac{33}{64} \\ 0 & 0 & 0 & 0 & -\frac{33}{64} & \frac{31}{64} \end{bmatrix}$$

$\frac{3}{8} \times \frac{8}{3}$

$\frac{31}{64} \times \frac{64}{}$

$\frac{33}{64} \times$

$\mathcal{C}(A) = \mathcal{C}(A'A)$

Suppose

$x \in \mathcal{C}(A)$

$\underset{\sim}{x} = A\underset{\sim}{y}$

$\Rightarrow \underset{\sim}{x} = A'A\underset{\sim}{z}$

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Lemma :

A linear parametric function $p'\tau$ is estimable iff $\underset{\sim}{p} \in \mathcal{C}(C)$ (under the block design $\underset{\sim}{d}$) / C is the C matrix of the design d.

Proof :

If part

Let $\underset{\sim}{p} \in \mathcal{C}(C)$

$\therefore \underset{\sim}{p} = C\underset{\sim}{\lambda}$ for some vector $\underset{\sim}{\lambda}$.

Let $\hat{\tau}$ be a solution of the normal equation.

$$p'\hat{\tau} = \underset{\sim}{\lambda}' C \hat{\tau} \quad [C = D_n - N D_K^{-1} N' \text{ is symmetric}]$$

$$= \underset{\sim}{\lambda}' Q$$

$$E(\underset{\sim}{p}'\hat{\tau}) = \underset{\sim}{\lambda}' E(Q) = \underset{\sim}{\lambda}' C\tau = p'\tau$$

$\therefore \underset{\sim}{p}'\tau$ is estimable.

Only if part

Let $p'\tau$ is estimable.

Then $\exists$ a linear function $\underset{\sim}{l}'\underset{\sim}{y}$ such that

$$E(\underset{\sim}{l}'\underset{\sim}{y}) = \underset{\sim}{p}'\underset{\sim}{\tau}$$

$$\therefore E(y) = \mu \cdot \underset{\sim}{1} + X_\tau \underset{\sim}{\tau} + X_\beta \cdot \underset{\sim}{\beta}$$

$$\therefore \underset{\sim}{l}' [\mu \cdot \underset{\sim}{1} + X_\tau \underset{\sim}{\tau} + X_\beta \cdot \underset{\sim}{\beta}] = p'\underset{\sim}{\tau} \quad \forall \underset{\sim}{\mu} \in \mathbb{R},\ \forall \underset{\sim}{\tau} \in \mathbb{R}^v,\ \forall \underset{\sim}{\beta} \in \mathbb{R}^b$$

$$\text{or } \mu \underset{\sim}{l}' \underset{\sim}{1} + (\underset{\sim}{l}' X_\tau)\underset{\sim}{\tau} + (l' X_\beta)\underset{\sim}{\beta} = p'\tau \quad \forall \underset{\sim}{\mu} \in \mathbb{R}, \underset{\sim}{\tau} \in \mathbb{R}^v, \underset{\sim}{\beta} \in \mathbb{R}^b$$

$$\therefore \underset{\sim}{l}'\underset{\sim}{1} = 0 \quad \ldots (1)$$

$$\underset{\sim}{l}' X_\tau = \underset{\sim}{p}' \quad \ldots (2)$$

$$\underset{\sim}{l}' X_\beta = 0 \quad \ldots (3)$$

$$\underset{\sim}{p} = X_\tau' \underset{\sim}{l} - X_\tau' X_\beta (X_\beta' X_\beta)^{-1} X_\beta' \underset{\sim}{l} \quad [\because X_\beta' \underset{\sim}{l} = \underset{\sim}{0}]$$

$$= X_\tau' (\underset{\sim}{l} - X_\beta (X_\beta' X_\beta)^{-1} X_\beta')\underset{\sim}{l}$$

$$= X_\tau' (I - X_\beta (X_\beta' X_\beta)^{-1} X_\beta'$$

$$\Rightarrow \underset{\sim}{p} \in \mathcal{C}(X_\tau' Z) \text{ where } Z = I - X_\beta (X_\beta' X_\beta)^{-1} X_\beta'.$$

$$\mathcal{C}(X_\tau' Z) = \mathcal{C}(C)$$

$$C = D_n - N D_K^{-1} N'$$

$$= X_\tau' X_\tau - X_\tau' X_\beta (X_\beta' X_\beta)^{-1} X_\beta' X_\tau$$

$= X_\tau' \left[I - X_\beta (X_\beta' X_\beta)^{-1} X_\beta \right] X_\tau$

$\underset{\sim}{p} \, \mathcal{C} = C \underset{\sim}{1}^*$

$= X_\tau' \left[I - X_\beta (X_\beta' X_\beta)^{-1} X_\beta' \right] X_\tau 1^*$

$= X_\tau' Z \underset{\sim}{1}.$

$\therefore \underset{\sim}{p} \in \mathcal{C}(C)$

<u>Result</u>:

$C = \mathcal{D}_b - N \mathcal{D}_k^{-1} N'$

$\mathcal{D} = \mathcal{D}_k - N' \mathcal{D}_b^{-1} N$

For any block design with v treatments and b blocks,
$\text{Rank}(C) + b = \text{Rank}(\mathcal{D}) + v.$

<u>Result</u>:

Structural Connectedness $\Leftrightarrow$ Rank $(C) = v-1$.

<u>Proof</u>:

First we assume that the design is structurally connected.

Let y_{ij} be the yield in any plot in block j receiving treatment i. $(i = 1(1)v, \; j = 1(1)b)$

Suppose (i, i_1) are contained in the j_1 th block, (i_1, i_2) are contained in the j_2 th block, $\ldots, (i_m, i')$ in the j_{m+1} th block.

Consider the following linear function of the yields

$(y_{ij_1} - y_{i_1 j_1}) + (y_{i_1 j_2} - y_{i_2 j_2}) + \ldots + (y_{i_m j_{m+1}} - y_{i' j_{m+1}})$

Expectation of the above quantity is

$(\tau_i - \tau_{i_1}) + (\tau_{i_1} - \tau_{i_2}) + \ldots + (\tau_{i_m} - \tau_{i'})$

$= \tau_i - \tau_{i'}$

$\therefore \tau_i - \tau_{i'}$ is estimable because i & i' are arbitrary

$\Rightarrow$ All treatment contrasts of the form $\tau_i - \tau_j$ are estimable.

$\Rightarrow$ Rank $(C) = v-1$ (Using the previous result) conversely,

Suppose Rank $(C) = v-1$. but the design is structurally disconnected, i.e. the treatments can be subdivided into $t(\geq 2)$ groups, so that the treatment within each group are connected but between groups are disconnected.

Example:

Block 1   1 2 3

Block 2   2 3 4

Block 3   5 6

$$\underset{v\times b}{N} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Diag

$$D_r = [1, 2, 2, 1, 1, 1]$$

$$D_k = \text{Diag}(3, 3, 2)$$

$$C = D_r - N D_k^{-1} N'$$

$$N D_k^{-1} N' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 1/3 & 2/3 & 2/3 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 2/3 & 2/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

$$C = D_r - N D_k^{-1} N'$$

$$= \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} -$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

This implies that C is a block diagonal matrix of the form

$$C = \begin{bmatrix} C_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & C_2 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & C_t \end{bmatrix}$$

$$\therefore \text{Rank}(C) = \text{Rank}(C_1) + \text{Rank}(C_2) + \ldots + \text{Rank}(C_t)$$
$$= (v_1 - 1) + (v_2 - 1) + \ldots + (v_t - 1)$$
$$= \sum_{i=1}^{t} v_i - t = v - t,$$ which is a contradiction.

$\therefore$ The design is structurally connected.

Rules for checking connectedness:

① If every element of C is non-zero, the design is connected.

② If C contains a row or column of non-zero elements, the design is connected.

③ Fixed treatment 1 and check connectedness with all other treatments.

Orthogonality of design :

Two or more things are orthogonal if changes in one donot effect any of the other. In the context of design, orthogonality means ability of the design to provide estimate of the effects of the independent variable.

Defn :

A block design is said to be orthogonal if

$cov(Q_i, P_j) = 0 \ \forall \ i, j$

where Q_i is the adjusted total of the ith treatmean and P_j is adjusted total of the jth block.

As a result the BLUE of every estimable treatment contrast is uncorrelated with the BLUE of every estimable block contrast.

We have, $\underset{\sim}{Q} = \underset{\sim}{T} - N \mathcal{D}_K^{-1} \underset{\sim}{B}$ → Adjusted treatment total vector

$\underset{\sim}{P} = \underset{\sim}{B} - N' \mathcal{D}_r^{-1} \underset{\sim}{T}$ → Adjusted block total vector.

$$
\begin{aligned}
cov(\underset{\sim}{Q}, \underset{\sim}{P}) &= Cov(\underset{\sim}{T} - N\mathcal{D}_K^{-1}\underset{\sim}{B}, \underset{\sim}{B} - N'\mathcal{D}_r^{-1}\underset{\sim}{T}) \\
&= cov(X_\tau' \underset{\sim}{Y} - N\mathcal{D}_K^{-1} X_\beta' \underset{\sim}{Y}, X_\beta' \underset{\sim}{Y} - N'\mathcal{D}_r^{-1} X_\tau' \underset{\sim}{Y}) \\
&= cov((X_\tau' - N\mathcal{D}_K^{-1}X_\beta')\underset{\sim}{Y}, (X_\beta' - N'\mathcal{D}_r^{-1}X_\tau')\underset{\sim}{Y}) \\
&= (X_\tau' - N\mathcal{D}_K^{-1}X_\beta')(X_\beta - X_\tau\mathcal{D}_r^{-1}N)\sigma^2 \quad [\because Disp(\underset{\sim}{Y}) = \sigma^2 . I] \\
&= (X_\tau' X_\beta - N\mathcal{D}_K^{-1}X_\beta' X_\beta - X_\tau' X_\tau \mathcal{D}_r^{-1} N + N\mathcal{D}_K^{-1}X_\beta' X_\tau \mathcal{D}_r^{-1} N)\sigma^2 \\
&= (N - N\mathcal{D}_K^{-1}\mathcal{D}_K - \mathcal{D}_r\mathcal{D}_r^{-1}N + N\mathcal{D}_K^{-1}N'\mathcal{D}_r^{-1}N)\sigma^2 \\
&= -(N - N\mathcal{D}_K^{-1}N'\mathcal{D}_r^{-1}N)\sigma^2 \\
&= -(\mathcal{D}_r\mathcal{D}_r^{-1}N - N\mathcal{D}_K^{-1}N'\mathcal{D}_r^{-1}N)\sigma^2 \\
&= -(\mathcal{D}_r - N\mathcal{D}_K^{-1}N')\mathcal{D}_r^{-1}N\sigma^2 \\
&= -C\mathcal{D}_r^{-1}N\sigma^2
\end{aligned}
$$

$\therefore$ A block design is orthogonal if

$$cov(Q, P) = -C \mathcal{D}_b^{-1} N \sigma^2 = 0$$

$$or, \; C \mathcal{D}_b^{-1} N = 0$$

Result:

A necessary and sufficient condition for a connected block design is that the cell frequencies are proportional i.e $n_{ij} = \frac{b_i k_j}{n}$ $\forall$ $i = 1(1)v$, $j = 1(1)b$

Proof:

only if part:

Assume the design to be connected and orthogonal

We know that $C \underset{\sim}{1}_v = \underset{\sim}{0}$

Since the design is orthogonal, it follows that $C \mathcal{D}_b^{-1} N = 0$

We may let $\mathcal{D}_b^{-1} N = \underset{\sim}{1}_v \underset{\sim}{S}'$ so that $C \mathcal{D}_b^{-1} N = C \underset{\sim}{1}_v S' = 0$

$$\therefore N = \mathcal{D}_b \underset{\sim}{1}_v \underset{\sim}{S}' = \underset{\sim}{b} \underset{\sim}{S}' \quad [\because \mathcal{D}_b \underset{\sim}{1}_v = \underset{\sim}{b}]$$

$$\therefore \underset{\sim}{1}_v' N = \underset{\sim}{1}_v' \underset{\sim}{b} \underset{\sim}{S}'$$

$$or, \; \underset{\sim}{k}' = n \underset{\sim}{S}'$$

$$or, \; \underset{\sim}{S}' = \frac{\underset{\sim}{k}'}{n}$$

$$\therefore N = \frac{\underset{\sim}{b} \underset{\sim}{k}'}{n} \quad or \; N_{ij} = \frac{b_i k_j}{n} \quad \forall \; i = 1(1)v, \; \forall \; j = 1(1)b$$

If part

Assume that the cell frequencies are proportional, i.e. $n_{ij} = \frac{b_i k_j}{n}$ $\forall$ $i = 1(1)v$, $j = 1(1)b$

$$or, \; N = \frac{\underset{\sim}{b} \underset{\sim}{k}'}{n}$$

$$C = \mathcal{D}_b - N \mathcal{D}_k^{-1} N'$$

$$= \mathcal{D}_b - \frac{\underset{\sim}{b} \underset{\sim}{k}'}{n} \mathcal{D}_k^{-1} \frac{\underset{\sim}{k} \underset{\sim}{b}'}{n}$$

$$C = D_r - \frac{\underset{\sim}{r}\,\underset{\sim}{k}'}{n} D_k^{-1} \frac{\underset{\sim}{k}\,\underset{\sim}{r}'}{n}$$

$$= D_r - \frac{\underset{\sim}{r}}{n}(k_1\ k_2 \ldots k_b)\begin{bmatrix} 1/k_1 & 0 & 0 & \ldots & 0 \\ 0 & 1/k_2 & 0 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \ldots & & 1/k_b \end{bmatrix} \frac{\underset{\sim}{k}\,\underset{\sim}{r}'}{n}$$

$$= D_r - \frac{\underset{\sim}{r}}{n}\,\underset{\sim}{1}_b' \frac{\underset{\sim}{k}\,\underset{\sim}{r}'}{n}$$

$$= D_r - \frac{\underset{\sim}{r}}{n}\, n\, \frac{\underset{\sim}{r}'}{n} \quad \left[\because \underset{\sim}{1}_b'\,\underset{\sim}{k} = \sum_{j=1}^{b} k_j = n\right]$$

$$= D_r - \frac{\underset{\sim}{r}\,\underset{\sim}{r}'}{n}$$

It can be shown(*) that $\operatorname{rank}\left(D_r - \frac{\underset{\sim}{r}\,\underset{\sim}{r}'}{n}\right) = v-1$

So the design is connected.

(*) $$\begin{bmatrix} r_1 & 0 & \ldots & 0 \\ 0 & r_2 & \ldots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \ldots & r_v \end{bmatrix} - \frac{1}{n}\begin{bmatrix} r_1^2 & r_1 r_2 & \ldots & r_1 r_v \\ r_1 r_2 & r_2^2 & \ldots & r_2 r_v \\ \vdots & & & \\ r_1 r_v & r_2 r_v & \ldots & r_v^2 \end{bmatrix}$$

$$= \begin{bmatrix} r_1 - \frac{r_1^2}{n} & -\frac{r_1 r_2}{n} & \ldots & -\frac{r_1 r_v}{n} \\ -\frac{r_1 r_2}{n} & r_2 - \frac{r_2^2}{n} & \ldots & -\frac{r_2 r_v}{n} \\ \vdots & & & \\ -\frac{r_1 r_v}{n} & -\frac{r_2 r_v}{n} & \ldots & r_v - \frac{r_v^2}{n} \end{bmatrix}$$

$$= r_1 - \frac{r_1}{n}(r_1 + r_2 + \ldots + r_v)$$
$$= r_1 - \frac{r_1}{n}\, n$$
$$= 0$$

Now $$C D_r^{-1} N = \left(D_r - \frac{\underset{\sim}{r}\,\underset{\sim}{r}'}{n}\right) D_r^{-1} N$$

$$= N - \frac{\underset{\sim}{r}}{n}\,\underset{\sim}{r}' D_r^{-1} N$$

$$= N - \frac{\underset{\sim}{r}\,\underset{\sim}{1}_v' N}{n}$$

$$= N - \frac{\underset{\sim}{r}\,\underset{\sim}{k}'}{n} \quad \left[\because \underset{\sim}{1}_v' N = \underset{\sim}{k}'\right]$$

$$= 0 \text{ [Proved]}$$

Note:

If at least one of the elements of N is zero, the design is non-orthogonal.

Note:

A design with atleast one zero element is called incomplete.

Testing of the hypothesis $H_0 :- \tau_1 = \tau_2 = \dots = \tau_v$

Against $H_1 :-$ Atleast two of them are unequal.

Define $R_0^2 = \underset{\text{Model}}{\text{Min}} \, (\underset{\sim}{y} - X\underset{\sim}{\theta})'(\underset{\sim}{y} - X\underset{\sim}{\theta})$

Model
$\underset{\sim}{y} = X\underset{\sim}{\theta} + \underset{\sim}{\epsilon}$

$$R_1^2 = \underset{\text{Model}+H_0}{\text{Min}} \, (\underset{\sim}{y} - X\underset{\sim}{\theta})'(\underset{\sim}{y} - X\underset{\sim}{\theta})$$

$$\underset{\sim}{\theta} = \begin{bmatrix} \mu \\ \underset{\sim}{\tau} \\ \underset{\sim}{\beta} \end{bmatrix}$$

$$X = [\underset{\sim}{1} \;\; X_\tau \;\; X_\beta]$$

$$\epsilon \sim N_n(\underset{\sim}{0}, \sigma^2 I)$$

$$R_0^2 = \sum_i \sum_j \sum_k (y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j)^2$$

Where $\hat{\mu}, \hat{\tau}_i, \hat{\beta}_j$ s $(i = 1(1)v, \; j = 1(1)b)$ are the solutions of the normal equations.

$$= \sum_i \sum_j \sum_k y_{ijk}^2 - \sum_{j=1}^{b} \left(\frac{B_j^2}{k_j}\right) - \sum_{i=1}^{v} \hat{\tau}_i Q_i$$

Solution of the reduced normal equation.

$$R_1^2 = \sum_i \sum_j \sum_k y_{ijk}^2 - \sum_j \left(\frac{B_j^2}{k_j}\right)$$

$$R_1^2 - R_0^2 = \sum_{i=1}^{v} \hat{\tau}_i Q_i = \hat{\underset{\sim}{\tau}}' \underset{\sim}{Q} \rightarrow \text{Sum of squares due to } H_0.$$

$$\text{Test statistic} = \frac{(R_1^2 - R_0^2)/(v-1)}{R_0^2/(n-b-v+1)} \sim F_{v-1,\, n-b-v+1} \text{ under } H_0.$$

$\rightarrow$ Adjusted treatment SS

ANOVA TABLE

Source	df	SS
Between Treatment (Adjusted)	$v-1$	$\sum_{i=1}^{v} \hat{\tau}_i Q_i$
Between Blocks (Unadjusted)	$b-1$	$\sum_{j=1}^{b} B_j^2/k_j - \frac{\left(\sum_{j=1}^{b} B_j\right)^2}{n}$
Error	$n-b-v+1$	By subtraction
Total	$n-1$	$\sum_i \sum_j \sum_k y_{ijk}^2 - \frac{G^2}{n}$

$G = \sum_{i=1}^{v} T_i = \sum_{j=1}^{b} B_j$

For $H_0 :- \beta_1 = \beta_2 = \ldots = \beta_b$ against

H_1 : Atleast two of them are unequal.

ANOVA TABLE

Source	df	SS
Between Blocks (Adjusted)	$b-1$	$\sum_{j=1}^{b} \hat{B}_j P_j$ → Solution of $D\underset{\sim}{\hat{\beta}} = \underset{\sim}{P}$
Between treatments (unadjusted)	$v-1$	$\sum_{i=1}^{v} T_i^2/k_i - \frac{G^2}{n}$
Error	$n-b-v+1$	By subtraction
Total	$n-1$	$\sum_i \sum_j \sum_k y_{ijk}^2 - \frac{G^2}{n}$

Unadjusted block SS + Adjusted treatment SS = Adjusted block SS + Unadjusted treatment SS

The intra-block analysis of block designs boils down to the computation of a g-inverse of the C matrix of the design leading to a solution of equ^n $C\underset{\sim}{\tau} = \underset{\sim}{Q}$. Some of the methods of finding the G-inverse are as follows. In each of the cases, the design is assumed to be connected.

① Let Ω^{-1} be a matrix defined as

$$\Omega^{-1} = C + \frac{\underset{\sim}{k}\underset{\sim}{k}'}{n}$$

Then Ω^{-1} is non singular & a solu^n of the equ^n $C\underset{\sim}{\tau} = \underset{\sim}{Q}$ is given by $\hat{\underset{\sim}{\tau}} = \Omega \underset{\sim}{Q}$

The adjusted treatment SS is $Q'\Omega Q$.

② If a doubly-centered matrix A of order m has rank $m-1$, then the unique doubly-centered g inverse of A is the Moore-Penrose inverse A^+.

Rao & Mitra (1971) have shown that

$$A^+ = \left(A + \frac{1}{m}\underset{\sim}{1}\underset{\sim}{1}'\right)^{-1}$$

Since the C matrix of a block design is doubly-centered of order v and for connected designs rank $(C) = v-1$, the doubly-centered g-inverse of C is given by

$$C^+ = \left(C + \frac{1}{v}\underset{\sim}{1}\underset{\sim}{1}'\right)^{-1}$$

③ Let $\theta_1, \theta_2, \ldots, \theta_n$ $(n \le v-1)$ be the distinct non-zero eigen values of C and let $\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2, \ldots, \underset{\sim}{\xi}_n$ be the corresponding orthonormal eigen vectors. Then a g-inverse is given by

$$C^- = \sum_{i=1}^{n} \theta_i^{-1}\, \underset{\sim}{\xi}_i \underset{\sim}{\xi}_i'$$

Balancing in Block Design

Two types of balance
① Variance balanced
② Efficiency balanced

Def^n of Variance balanced :

1. A connected block design is said to be variance balanced if $var(\hat{\tau}_i - \hat{\tau}_j)$ are the same for all $i \neq j$
2. A connected block design is said to be variance balanced if all the normalized treatment contrast are estimated by their blues with the same variance.

Theorem ≫
A connected block design is variance balanced iff the non-zero eigen values of the C matrix are same.

Proof :

only if part

Let $\theta_1, \theta_2, \ldots, \theta_{v-1}$ be the $(v-1)$ non-zero eigen values of the C-matrix of the design and $\underset{\sim}{\xi_1}, \underset{\sim}{\xi_2}, \ldots, \underset{\sim}{\xi_{v-1}}$ be the corresponding orthonormal eigen vectors.
Then the spectral decomposition of C is $C = \sum_{i=1}^{v-1} \theta_i \underset{\sim}{\xi_i} \underset{\sim}{\xi_i}'$
One particular choice of G-inverse is $C^- = \sum_{i=1}^{v-1} \frac{1}{\theta_i} \underset{\sim}{\xi_i} \underset{\sim}{\xi_i}'$

& $\hat{\underset{\sim}{\tau}} = C^- \underset{\sim}{Q}$

Now, $Disp(\underset{\sim}{Q}) = C\sigma^2$

$\therefore Disp(\hat{\underset{\sim}{\tau}}) = Disp(C^- \underset{\sim}{Q}) = \sigma^2 C^- C C^{-'} = \sigma^2 C^-$

Let $C^- = ((\bar{c}_{ij}))$

$\therefore Var(\hat{\tau}_i - \hat{\tau}_j) = \sigma^2 \left[\bar{c}_{ii} + \bar{c}_{jj} - 2\bar{c}_{ij} \right]$

Average variance of all elementary treatment contrasts is given by

$$\frac{1}{v(v-1)} \sum_{i \neq j}\sum Var(\hat{\tau}_i - \hat{\tau}_j) = \frac{\sigma^2}{v(v-1)} \sum_{\substack{i=1}}^{v} \sum_{\substack{j=1 \\ i \neq j}}^{v} (\bar{c}_{ii} + \bar{c}_{jj} - 2\bar{c}_{ij})$$

$$= \frac{\sigma^2}{v(v-1)} \left[(v-1)\sum_{i=1}^{v} \bar{c}_{ii} + (v-1)\sum_{j=1}^{v} \bar{c}_{jj} + 2\sum_{i=1}^{v} \bar{c}_{ii} \right]$$

$$2\sum_i \sum_{j \neq i} \bar{c}_{ij} = 2\sum_{i=1}^{v}\left[\sum_{j=1}^{v} \bar{c}_{ij} - \bar{c}_{ii}\right]$$

$[\because C\underline{1} = \underline{0} \Rightarrow \sum_{j=1}^{v} \bar{c}_{ij} = 0 \quad \forall\ i = 1(1)v]$

[Doubly centered G-inverse]

$$\therefore 2\sum_i \sum_{j \neq i} \bar{c}_{ij} = -2\sum_{i=1}^{v} \bar{c}_{ii}$$

$$\therefore \frac{\sum\sum_{i \neq j} Var(\hat{\tau}_i - \hat{\tau}_j)}{v(v-1)}$$

$$= \frac{\sigma^2}{v(v-1)} \left[2(v-1)\sum_{i=1}^{v} \bar{c}_{ii} + 2\sum_{i=1}^{v} \bar{c}_{ii} \right]$$

$$= \frac{2\sigma^2}{(v-1)} \sum_{i=1}^{v} \bar{c}_{ii}$$

$$= \frac{2\sigma^2}{v-1} \operatorname{trace}(\bar{C}^-)$$

$$= \frac{2\sigma^2}{v-1} \sum_{i=1}^{v} \frac{1}{\theta_i}$$

Since the design is balanced,

$$Var(\hat{\tau}_i - \hat{\tau}_j) = \frac{2\sigma^2}{v-1} \sum_{i=1}^{v-1} \frac{1}{\theta_i} = \sigma_a^2 \text{ (say)}$$

$$var(\hat{\tau}_i - \hat{\tau}_j) = Var\left[(\hat{\tau}_i - \hat{\tau}_k) - (\hat{\tau}_j - \hat{\tau}_k)\right] \ \forall\ i \neq j$$

$$= Var(\hat{\tau}_i - \hat{\tau}_k) + Var(\hat{\tau}_j - \hat{\tau}_k) - 2\,cov(\hat{\tau}_i - \hat{\tau}_k, \hat{\tau}_j - \hat{\tau}_k)$$

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$\therefore\ \sigma_a^2 = \sigma_a^2 + \sigma_a^2 - 2\,\text{cov}\,(\hat{\tau}_i - \hat{\tau}_k, \hat{\tau}_j - \hat{\tau}_k)$

or, $\text{cov}\,(\hat{\tau}_i - \hat{\tau}_k, \hat{\tau}_j - \hat{\tau}_k) = \dfrac{\sigma_a^2}{2}$

We consider the normalized treatment contrast $\underset{\sim}{\xi}_i' \underset{\sim}{\tau}$.

We write $\underset{\sim}{\xi}_i'\underset{\sim}{\tau}$ as $a_1(\hat{\tau}_1 - \hat{\tau}_v) + a_2(\hat{\tau}_2 - \hat{\tau}_v) + \dots + a_{v-1}(\hat{\tau}_{v-1} - \hat{\tau}_v)$

Where $(a_1^2 + a_2^2 + \dots + a_{v-1}^2) + (a_1 + a_2 + \dots + a_{v-1})^2 = 1$

$\sum_{i=1}^{v-1} a_i^2 + \sum_{i=1}^{v-1} a_i^2 + 2\sum\sum_{i<j} a_i a_j = 1$

or, $2\sum_{i=1}^{v-1} a_i^2 + 2\sum\sum_{i<j} a_i a_j = 1$

or, $\sum_{i=1}^{v-1} a_i^2 + \sum\sum_{i<j} a_i a_j = \frac{1}{2}$ $\quad\dots\circledast$

$\text{var}\,(\underset{\sim}{\xi}_i'\hat{\tau}) = \sum_{i=1}^{v-1} a_i^2\, \text{Var}\,(\hat{\tau}_i - \hat{\tau}_v) + 2\sum\sum_{i<j} a_i a_j\, \text{cov}(\hat{\tau}_i - \hat{\tau}_v, \hat{\tau}_j - \hat{\tau}_v)$

$= \sigma_a^2 \sum_{i=1}^{v-1} a_i^2 + 2\sum\sum_{i<j} a_i a_j \dfrac{\sigma_a^2}{2}$

$= \dfrac{\sigma_a^2}{2}$ (From $*$)

Alternatively,

$\text{Var}\,(\underset{\sim}{\xi}_i'\hat{\tau}) = \underset{\sim}{\xi}_i'\, \text{Disp}\,(\hat{\tau})\, \underset{\sim}{\xi}_i$

$= \sigma^2\, \underset{\sim}{\xi}_i'\, C^-\, \underset{\sim}{\xi}_i$

$= \sigma^2\, \underset{\sim}{\xi}_i' \left[\sum_{i=1}^{v-1} \frac{1}{\theta_i} \underset{\sim}{\xi}_i \underset{\sim}{\xi}_i'\right] \underset{\sim}{\xi}_i$

$= \dfrac{\sigma^2}{\theta_i} (\underset{\sim}{\xi}_i' \underset{\sim}{\xi}_i)^2$ [$\because \underset{\sim}{\xi}_i' \underset{\sim}{\xi}_j = 0\ \forall j \neq i$ as $\underset{\sim}{\xi}_i$ s are orthonormal eigenvectors]

$= \dfrac{\sigma^2}{\theta_i}$ ($\because \underset{\sim}{\xi}_i' \underset{\sim}{\xi}_i = 1$, because of orthonormality)

Thus, $\dfrac{\sigma_a^2}{2} = \dfrac{\sigma^2}{\theta_i}\ \forall\ i = 1(1)v-1$

or, $\dfrac{2\sigma^2}{2(v-1)} \sum_{i=1}^{v-1} \dfrac{1}{\theta_i} = \dfrac{\sigma^2}{\theta_i}$

or, $\dfrac{1}{\theta_i} = \dfrac{1}{v-1} \sum_{i=1}^{v-1} \dfrac{1}{\theta_i}$ $\quad \forall\ i = 1(1)v-1$

$\Rightarrow \theta_i$ s are all equal.

If part:

Suppose $\theta_i = \theta \quad \forall\ i = 1(1)v-1$

So, $C = \theta \sum_{i=1}^{v-1} \underset{\sim}{\xi_i} \underset{\sim}{\xi_i}'$

$$C^- = \frac{1}{\theta} \sum_{i=1}^{v-1} \underset{\sim}{\xi_i} \underset{\sim}{\xi_i}'$$

$$Var(\hat{\underset{\sim}{\tau}}) = \sigma^2 C^-$$

So, $var(\hat{\tau}_i - \hat{\tau}_j) = \sigma^2 \left[\bar{c}_{ii} + \bar{c}_{jj} - 2\bar{c}_{ij}\right]$

$$\underset{\sim}{\xi_i} = \begin{bmatrix} \xi_{i1} \\ \xi_{i2} \\ \vdots \\ \xi_{iv} \end{bmatrix} \qquad \underset{\sim}{\xi_i} \underset{\sim}{\xi_i}' = \begin{bmatrix} \xi_{i1}^2 & \xi_{i1}\xi_{i2} & \dots & \xi_{i1}\xi_{iv} \\ \vdots & & & \\ \xi_{i1}\xi_{iv} & \xi_{i2}\xi_{iv} & \dots & \xi_{iv}^2 \end{bmatrix}$$

So, $\bar{C}_{ii} = \frac{1}{\theta} \sum_{i=1}^{v-1} \xi_{ii}^2$

$$\begin{aligned} var(\xi_i' \hat{\underset{\sim}{\tau}}) &= \sigma^2 \left[\xi_i' C^- \xi_i\right] \\ &= \sigma^2 \left[\xi_i' \left(\sum_{i=1}^{v-1} \frac{1}{\theta} \xi_i \xi_i'\right) \xi_i\right] \\ &= \frac{\sigma^2}{\theta} \left[\xi_i' \xi_i\right]^2 \quad \left[\because \xi_i' \xi_j = 0 \ \forall\ i \neq j\right] \\ &= \frac{\sigma^2}{\theta} \quad \forall\ i = 1(1) v-1 \end{aligned}$$

All elementary treatment contrasts are estimated with the same variance.

Corollary:
A connected block design is variance balanced iff its C-matrix has all its diagonal elements equal and all its off diagonal elements are equal i.e. C is of the form

$$C = (a-b) I + bJ \quad \left[J = \underset{\sim}{1}\underset{\sim}{1}' = \text{Matrix with all entries } 1\right]$$

A necessary and sufficient condition for a symmetric matrix A of order n to have all its diagonal entries equal and all its off diagonal entries equal is that A has only two eigen values, 1 with multiplicity $(n-1)$ and the vector $\underline{1}$ is an eigen vector corresponding to the other eigen value.

Since the design is connected it follows that rank$(C) = v-1$. Thus the no. of non-zero eigen values of C is $v-1$. Again a necessary and sufficient condition for the variance balance is non-zero eigen values of C are all equal. Therefore a design is connected iff it has two eigen values, the 1st one is zero and the 2nd one is equal to θ $(\neq 0)$ which has multiplicity $v-1$. Therefore the design is variance balanced iff its diagonal entries are all equal and off diagonal entries are all equal.

Proof of the statement used to prove the corollary

Since A is symmetric, there exists an orthogonal matrix $B = \begin{bmatrix} \alpha_{1\times n} \\ \hdashline P_{n-1\times n} \end{bmatrix}$, where $\alpha' = n^{-\frac{1}{2}} \underline{1}'$

such that $BAB' = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 I_{n-1} \end{bmatrix}$

Where θ_1 & θ_2 are the eigen values of A & the multiplicity of θ_2 is $n-1$.

We have,

$$B'B\,AB'B = A = [\underset{\sim}{\alpha}' \,\vdots\, P'] \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 I_{n-1} \end{bmatrix} \begin{bmatrix} \alpha \\ \hdashline P \end{bmatrix}$$

$$= [\underset{\sim}{\alpha}'\theta_1 \,\vdots\, \theta_2 P'] \begin{bmatrix} \alpha \\ \hdashline P \end{bmatrix}$$

$$= \theta_1 \underset{\sim}{\alpha}'\underset{\sim}{\alpha} + \theta_2 P'P$$

$$= \theta_1 n^{-1} J_n + \theta_2 I_n$$

$$\underset{\sim}{\alpha}'\underset{\sim}{\alpha} = n^{-1}\underline{1}\,\underline{1}' = n^{-1} J$$

$BB' = B'B = I$

$$= \begin{bmatrix} \alpha \\ \hdashline P \end{bmatrix} [\alpha' \,\vdots\, P'] = I_n$$

$$\left[\begin{array}{c|c} \underset{1\times 1}{\alpha\alpha'} & \alpha P' \\ \hline P\alpha' & \underset{n-1\times n-1}{PP'} \end{array}\right] = I_n$$

The matrix is of the form $a I_n + b J_n$ which has all of its diagonal entries same and all off diagonal entries same.

Corollary:

$$C = (a-b) I + bJ$$

Since C is a doubly centered matrix, so, for a variance-balanced design

$$C = \theta \left(I - \frac{J}{v}\right), \text{ where } \theta \text{ is the non-zero eigenvalue of } C.$$

* Proper $\rightarrow$ $k_j = k \quad \forall\, j = 1(1)b \Rightarrow D_k = k I_b$
* Equireplicate $\Rightarrow r_i = r \quad \forall\, i = 1(1)v \Rightarrow D_r = r I_v$
* Binary if $n_{ij} = 1$ or 0.

$$\sum_{i=1}^{v} n_{ij} = k_j \;;\; \sum_{j=1}^{b} n_{ij} = r_i$$

$$\underset{v\times b}{N}\ \underset{b\times 1}{1_b} = \underset{v\times 1}{r}$$

$$\underset{1\times v}{1_v'}\ \underset{v\times b}{N} = \underset{1\times b}{k'}$$

Binary Design

$\Downarrow$

$$\sum_{i=1}^{v} n_{ij}^2 = \sum_{i=1}^{v} n_{ij} = k_j \quad \forall\ j = 1(1)b$$

$$\sum_{j=1}^{b} n_{ij}^2 = \sum_{i=1}^{b} n_{ij} = r_i \quad \forall\ i = 1(1)v$$

$$(NN')_{i,j} = \sum_{k=1}^{b} n_{ik} (n'_{kj})$$

$$= \sum_{k=1}^{b} n_{ik} n_{jk}$$

If $j = i$

$$\sum_{k=1}^{b} n_{ik}^2 = \sum_{k=1}^{b} n_{ik} = r_i$$

<u>Result</u>:

Theorem $\Rightarrow$

For an equireplicate proper binary variance balanced design the incidence matrix N satisfies

$$NN' = (r-\lambda) I + \lambda J$$

Where λ is a scalar such that

$$r(k-1) = \lambda (v-1)$$

Proof »

The C-matrix is given by $C = D_r - N D_k^{-1} N'$

Since the design is proper and equireplicate

$$D_r = r I_v$$
$$D_k = k I_b$$
$$\therefore \quad C = r I_v - \frac{1}{k} N N'$$

Design is binary

$\therefore$ The diagonal entries of NN' are all equal to r.

$$tr(NN') = rv$$
$$\therefore tr(C) = rv - \frac{rv}{k}$$

Total no. of experimental units

$$n = rv = bk$$
$$\Rightarrow b = \frac{rv}{k}$$
$$\therefore tr(C) = rv - b$$

Since the design is variance balanced

$$C = \theta\left(I - \frac{J}{v}\right)$$
$$\therefore tr(C) = \theta(v-1)$$

So, $\quad rv - b = \theta(v-1)$

$$\theta = \frac{rv-b}{v-1}$$

Again, $\left(\frac{rv-b}{v-1}\right)\left(I_v - \frac{J_v}{v}\right) = r I_v - \frac{1}{k} NN'$

$$\Rightarrow \frac{1}{k} NN' = I_v\left(r - \frac{rv-b}{v-1}\right) + \left(\frac{rv-b}{v(v-1)}\right) J_v$$

$$NN' = \frac{k(-r+b)}{(v-1)} + \frac{k(rv-b)}{v(v-1)} J_v$$

$$NN' = \left[r - \frac{r(k-1)}{v-1}\right] I_v + \frac{r(k-1)}{v-1} J_v$$

$$= \frac{(rv - rk)}{v-1}$$

$$NN' = \frac{vb - bk}{v-1} I_v + \frac{kbv - vb}{v(v-1)} J_v$$

$$= \frac{b(v-k)}{(v-1)} I_v + \frac{vb(k-1)}{v(v-1)} J_v$$

$$= b\left(\frac{(v-1)-(k-1)}{(v-1)}\right) I_v + \frac{b(k-1)}{v-1} J_v$$

$$= \left(b - \frac{b(k-1)}{v-1}\right) I_v + \lambda J_v$$

$$= (b-\lambda) I_v + \lambda J_v$$

Theorem:

Barring trivial exceptions, the equality $b \geq v$ holds for all connected, equireplicate, variance balanced designs. (Fisher's inequality)

Proof:

For a connected variance-balanced design, we have

$$C = \theta\left(I - \frac{1}{v} J\right)$$

The eigenvalues of C are 0 (with multiplicity 1) & θ (with multiplicity $v-1$).

The eigenvalues of D_b are b (with multiplicity v) [$\because$ Equireplicate]

So, the eigenvalues of $P = D_b - C$ $= N D_k^{-1} N'$ are b (with multiplicity 1) and $b-\theta$ (with multiplicity $v-1$)

$\therefore$ rank $(P) = v$

P is singular iff $b = \theta$

In this case P is of rank 1. and hence the rank of N is also 1.

The columns of P, and hence those of N are spanned by the vector $\underline{1}$, which is the eigen vector corresponding to the zero eigen value of C. Thus, it follows that in case $b = \theta$, the rows of N are identical.

If we exclude designs with ~~inder~~ incidence matrices having identical rows, then we find that for every other equireplicate, variance balanced design, P is non-singular.

Thus, $v = \text{rank}(P) = \text{rank}(N) \leq b$ (Proved)

Recovery of inter block information

While discussing the intra block analysis of block design it was stated that the block as well as the treatment effects are fixed. If the block effect is regarded as the random variable, the analysis is ~~as~~ termed ~~stunned~~ as inter block analysis or recovery of inter block information.

In the context of incomplete block design. Yates noticed that since the allocation of treatment to incomplete blocks is made at random, it is reasonable to assume that the block effects themselves are ~~at~~ random variables instead of fixed. If the experimental material is farely heterogeneous, treating the block effect as fixed results in the loss of information contained in the block totals.

(Binary, Proper Designs)

Model:

$$y_{ij} = \mu + \tau_i + \beta_j + e_{ij} \quad , \quad \begin{matrix} i = 1(1)v \\ j = 1(1)b \end{matrix}$$

Assumptions:

(i) e_{ij} s iid with $E(e_{ij}) = 0$, $V(e_{ij}) = \sigma^2 \ \forall i, j$

(ii) β_j s are random variables with $E(\beta_j) = 0$, $Var(\beta_j) = \sigma_b^2$, $cov(\beta_j, \beta_{j'}) = 0$, $j \neq j'$, $j = 1(1)b$.

(iii) β_j s are uncorrelated with the error terms e_{ij} s.

We may regard the block totals as observations

$$B_j = \sum_i y_{ij} = K\mu + \sum_i n_{ij}\tau_i + \left(K\beta_j + \sum_i e_{ij}\right)$$

New error terms

$$d_j = K\beta_j + \sum_i e_{ij}$$

$$E(d_j) = 0 \quad , \quad Var(d_j) = K^2\sigma_b^2 + K\sigma_e^2$$

Inter block estimates are obtained by minimizing the SS due to new errors

$$S = \sum_{j=1}^{b} d_j^2 = \sum_{j=1}^{b}\left(B_j - K\mu - \sum_i n_{ij}\tau_i\right)^2$$

$$= \left(\underset{\sim}{B} - K\mu \underset{\sim}{1}_{b\times 1} - \underset{b\times v}{N'}\underset{\sim}{\tau}_{v\times 1}\right)'\left(\underset{\sim}{B} - K\mu\underset{\sim}{1} - N'\underset{\sim}{\tau}\right)$$

Normal Equ^n

$$\frac{\partial S}{\partial \mu} = 0 \Rightarrow K\sum_{j=1}^{b}\left(B_j - K\mu - \sum_i n_{ij}\tau_i\right) = 0$$

$$\Rightarrow bK\mu + \sum_j\sum_i n_{ij}\tau_i = \sum_{j=1}^{b} B_j \quad \ldots (1)$$

$$\frac{\partial S}{\partial \tau_i} = 0 \Rightarrow \sum_{j=1}^{b} n_{ij}\left(B_j - K\mu - \sum_i n_{ij}\tau_i\right) = 0$$

$$\Rightarrow \mu K\sum_{j=1}^{b} n_{ij} + \sum_{j=1}^{b} n_{ij}\left(\sum_i n_{ij}\tau_i\right) = \sum_{j=1}^{b} n_{ij}B_j \quad \ldots (2)$$

1st equ^n

$$bK\mu + \underset{\sim}{1}_b' N'\underset{\sim}{\tau} = G \quad \text{or} \quad bK\mu + \underset{\sim}{1}_v' D_R\underset{\sim}{\tau} = G$$

2nd equ^n

$$\mu K\sum_{j=1}^{b} n_{1j} + \sum_{j=1}^{b} n_{1j}\left(\sum_i n_{ij}\tau_i\right) = \sum_{j=1}^{b} n_{1j}B_j$$

$$\mu K\sum_{j=1}^{b} n_{2j} + \sum_{j=1}^{b} n_{2j}\left(\sum_i n_{ij}\tau_i\right) = \sum_{j=1}^{b} n_{2j}B_j$$

$$\vdots$$

$$\mu K\sum_{j=1}^{b} n_{vj} + \sum_{j=1}^{b} n_{vj}\left(\sum_i n_{ij}\tau_i\right) = \sum_{j=1}^{b} n_{vj}B_j$$

$$\mu K\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_v \end{bmatrix} + \ldots = \begin{bmatrix} n_1'B \\ n_2'B \\ \vdots \\ n_v'B \end{bmatrix} = \begin{bmatrix} n_1' \\ n_2' \\ \vdots \\ n_v' \end{bmatrix}$$

$$K \underset{v\times v}{D_n} \underset{v\times 1}{\underset{\sim}{1}_v} \mu + NN'\tau = N\underline{B}$$

In matrix notation

$$\begin{bmatrix} bK & \underset{\sim}{1}_v' D_R \\ K D_n \underset{\sim}{1}_v & NN' \end{bmatrix} \begin{bmatrix} \mu \\ \underset{\sim}{\tau} \end{bmatrix} = \begin{bmatrix} G \\ N\underline{B} \end{bmatrix}$$

Premultiplying both sides by the non-singular matrix $\begin{bmatrix} 1 & 0 \\ -\frac{1}{b} D_n \underset{\sim}{1}_v & I \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{b} D_n \underset{\sim}{1}_v & I \end{bmatrix} \begin{bmatrix} bK & \underset{\sim}{1}_v' D_n \\ K D_n \underset{\sim}{1}_v & NN' \end{bmatrix} \begin{bmatrix} \mu \\ \underset{\sim}{\tau} \end{bmatrix} = \begin{bmatrix} G \\ N\underline{B} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{b} D_n \underset{\sim}{1}_v & I \end{bmatrix} \begin{bmatrix} G \\ N\underline{B} \end{bmatrix}$$

$$\begin{bmatrix} bK & \underset{\sim}{1}_v' D_n \\ -K D_n \underset{\sim}{1}_v + K D_n \underset{\sim}{1}_v & -\frac{1}{b} D_n \underset{\sim}{1}_v \underset{\sim}{1}_v' D_n + NN' \end{bmatrix} \begin{bmatrix} \mu \\ \underset{\sim}{\tau} \end{bmatrix} = \begin{bmatrix} G \\ N\underline{B} - \frac{G}{b} D_n \underset{\sim}{1}_v \end{bmatrix}$$

A solution of the above system of equ^{n}s is obtained by taking the side condition $\underset{\sim}{1}_v' D_n \underset{\sim}{\tau} = 0$ & assuming that NN' is non-singular

$$bK\mu + 0 = G$$

$$NN'\tau = N\underline{B} - \frac{G}{b} D_n \underset{\sim}{1}_v$$

$$\therefore \hat{\mu} = \frac{G}{bK}$$

$$\hat{\tau} = (NN')^{-1} \left(N\underline{B} - \frac{G}{b} D_n \underset{\sim}{1}_v \right)$$

$$= (NN')^{-1} \left(N\underline{B} - \frac{G}{bK} NK \underset{\sim}{1}_b \right)$$

$$= (NN')^{-1} \left(N\underline{B} - \frac{G}{bK} NK \underset{\sim}{1}_b \right)$$

We now have two estimators of τ,

i) The intra block estimator $\hat{\tau} = C^{-}\underset{\sim}{Q}$

ii) The inter block estimator $\hat{\hat{\tau}} = (NN')^{-1}\left[N\underset{\sim}{B} - \frac{G}{bK} NK\underset{\sim}{1}_b\right]$

$$= (NN')^{-1}\left[N\underset{\sim}{B} - \frac{G\,\underset{\sim}{1}_v}{bK}\right]$$

Let $\psi = p'\tau$ be a contrast of treatment effects.

The intra block estimator of ψ, say ψ_1 is $\psi_1 = p'\hat{\tau} = p'C^{-}\underset{\sim}{Q}$

with variance $Var(\psi_1) = \sigma^2 \underset{\sim}{p}' C^{-} \underset{\sim}{p}$

The inter block estimator of ψ, say ψ_2 is

$$\psi_2 = \underset{\sim}{p}'(NN')^{-1} N \left(\underset{\sim}{B} - \frac{G\,\underset{\sim}{1}_b}{b}\right)$$

$$= \underset{\sim}{p}'(NN')^{-1} N\underset{\sim}{B} - \underset{\sim}{p}'\underset{\sim}{1}_v\left(\frac{G}{bK}\right)$$

$$= \underset{\sim}{p}'(NN')^{-1} N\underset{\sim}{B} \quad [\because \underset{\sim}{p}'\underset{\sim}{1}_v = 0]$$

With variance $Var(\psi_2) = \sigma_d^2\, \underset{\sim}{p}'(NN')^{-1}\underset{\sim}{p}$

Where $\sigma_d^2 = k(k\sigma_b^2 + \sigma^2)$

The two estimators ψ_1 and ψ_2 are uncorrelated because $cov(Q_i, B_j) = 0 \quad \forall\ i, j$. If we want to combine these two estimators to obtain an estimator with smallest variance, then the combined estimator is obtained by taking a weighted average of ψ_1 and ψ_2, weights being the inverse of the variances of the two estimators. Thus the combine estimator is given by

$$\psi^* = \frac{\theta_1\psi_1 + \theta_2\psi_2}{\theta_1 + \theta_2}$$

Where $\theta_1 = \left(\sigma^2\, \underset{\sim}{p}' C^{-} \underset{\sim}{p}\right)^{-1}$

$\theta_2 = \left(\sigma_d^2\, \underset{\sim}{p}'(NN')^{-1}\underset{\sim}{p}\right)^{-1}$

☐ Linear Mixed Effect Model:

Suppose $y_1, y_2, \ldots, y_n$ are n random variables such that

$$y_i = a_{i1}\beta_1 + a_{i2}\beta_2 + \ldots + a_{ip}\beta_p + b_{i1}x_1 + b_{i2}x_2 + \ldots + b_{iq}x_q + \epsilon_i'$$

$(i = 1(1)n)$

Where $\beta_1, \beta_2, \ldots, \beta_p$ are unknown parameters (fixed).

$x_1, x_2, \ldots, x_q$ are random variables.

$a_{i1}, \ldots, a_{ip}, b_{i1}, \ldots, b_{iq}$ are known constants.

Further let $E(\epsilon_i) = 0$, $Var(\epsilon_i) = \sigma^2$ $\forall$ $i = 1(1)n$,

$cov(\epsilon_i, \epsilon_{i'}) = 0$ $\forall$ $i \neq i'$

$E(x_j) = 0$, $Var(x_j) = \sigma_b^2$ $\forall$ $j = 1(1)q$

$cov(x_j, x_{j'}) = 0$ $\forall$ $j \neq j'$

$cov(\epsilon_i, x_j) = 0$ $\forall$ $i = 1(1)n$ & $j = 1(1)q$.

A model of this type is called linear mixed effects model.

Matrix notation:

$$\underset{n\times 1}{\underset{\sim}{y}} = \underset{n\times p}{A}\,\underset{p\times 1}{\underset{\sim}{\beta}} + \underset{n\times q}{B}\,\underset{q\times 1}{\underset{\sim}{x}} + \underset{n\times 1}{\underset{\sim}{\epsilon}}$$

$$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} ; \quad \underset{\sim}{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} ; \quad \underset{\sim}{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix} ; \quad A = \begin{bmatrix} a_{11} & \ldots & a_{1p} \\ & & \\ a_{n1} & \ldots & a_{np} \end{bmatrix} ; \quad B = \begin{bmatrix} b_{11} & \ldots & b_{1q} \\ \vdots & & \\ b_{n1} & \ldots & b_{nq} \end{bmatrix}$$

$\therefore E(\underset{\sim}{\epsilon}) = \underset{\sim}{0}$, $Disp(\underset{\sim}{\epsilon}) = \sigma^2 I$

$E(\underset{\sim}{x}) = \underset{\sim}{0}$, $Disp(\underset{\sim}{x}) = \sigma_b^2 I$; $cov(\underset{\sim}{\epsilon}, \underset{\sim}{x}) = 0$

$\therefore E(\underset{\sim}{y}) = A\underset{\sim}{\beta}$

$Disp(\underset{\sim}{y}) = \sigma_b^2 BB' + \sigma^2 I$

Normal Equation:

$$A'\Sigma^{-1}A\underset{\sim}{\beta} = A'\Sigma^{-1}\underset{\sim}{y}$$

Combined intra and inter block analysis:

Assumption: Design is proper $\begin{bmatrix} \underset{\sim}{k} = K \underset{\sim}{1} \\ D_K = K.I \end{bmatrix}$

Model

$$\underset{\sim}{Y} = \mu \underset{\sim}{1} + X_\tau . \underset{\sim}{\tau} + X_\beta . \underset{\sim}{\beta} + \underset{\sim}{\epsilon}$$

- Treatment effects $\underset{\sim}{\tau}$ are fixed.
- Block effects $\underset{\sim}{\beta}$ are random.

Assumption:

$E(\underset{\sim}{\epsilon}) = \underset{\sim}{0}$, $Disp(\underset{\sim}{\epsilon}) = \sigma^2 I_n$

$E(\underset{\sim}{\beta}) = \underset{\sim}{0}$, $Disp(\underset{\sim}{\beta}) = \sigma_b^2 I_b$

$cov(\underset{\sim}{\epsilon}, \underset{\sim}{\beta}) = 0$

In standard Notation

$$\underset{\sim}{Y} = (\underset{\sim}{1} : X_\tau) \begin{pmatrix} \mu \\ \underset{\sim}{\tau} \end{pmatrix} + X_\beta . \underset{\sim}{\beta} + \underset{\sim}{\epsilon}$$

$$= A \underset{\sim}{\theta} + B \underset{\sim}{\beta} + \underset{\sim}{\epsilon} \quad (\text{Say})$$

Design is proper.

$n = bK$

Assume that the $n = bK$ observations are such that The 1st set of K individuals are from block 1.

2nd " " " " " " " 2.

⋮

bth " K " " " " b.

Now, $\underset{n\times b}{X_\beta} = ((X_{ij}^\beta))$

$$X_{ij}^\beta = \begin{cases} 1 & \text{if the } i\text{ th observation comes from block } j \\ 0 & \text{otherwise} \end{cases}$$

Under this situation

$$X_\beta = \begin{pmatrix} \left.\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ & & & & \\ 1 & 0 & 0 & \cdots & 0 \end{matrix}\right] k \\ \left.\begin{matrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 1 & 0 & \cdots & 0 \end{matrix}\right] k \\ \\ \left.\begin{matrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{matrix}\right] k \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1}_k & \underset{\sim}{0}_k & \cdots & \underset{\sim}{0}_k \\ \underset{\sim}{0}_k & \underset{\sim}{1}_k & \cdots & \underset{\sim}{0}_k \\ \vdots & & & \\ \underset{\sim}{0}_k & \underset{\sim}{0}_k & \cdots & \underset{\sim}{1}_k \end{pmatrix}$$

Thus

$$X_\beta X_\beta' = \begin{bmatrix} \underset{\sim}{1}_k \underset{\sim}{1}_k' & 0 & \cdots & 0 \\ 0 & \underset{\sim}{1}_k \underset{\sim}{1}_k' & & 0 \\ \vdots & & & \\ 0 & 0 & & \underset{\sim}{1}_k \underset{\sim}{1}_k' \end{bmatrix}$$

$$= \begin{bmatrix} J_k & 0 & 0 & \cdots & 0 \\ 0 & J_k & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & J_k \end{bmatrix}$$

$$= Diag\,(J_k, J_k, \ldots, J_k)$$

$$Disp\,(\underset{\sim}{Y}) = \Sigma = \sigma^2 I_n + \sigma_b^2\, X_\beta X_\beta'$$

$$= \sigma^2 I_n + \sigma_b^2\; Diag\,(\underbrace{J_k, J_k, \ldots, J_k}_{b \text{ times}})$$

$$= \sigma^2\, Diag\,(\underbrace{I_k, I_k, \ldots, I_k}_{b \text{ times}}) +$$

$$\sigma_b^2\; Diag(J_k, J_k, \ldots, J_k)$$

$$= Diag\,(\sigma^2 I_k + \sigma_b^2 J_k,\ \sigma^2 I_k + \sigma_b^2 J_k, \ldots, \sigma^2 I_k + \sigma_b^2 J_k)$$

$$= Diag\,(L, L, \ldots, L)$$

Where $L = \sigma^2 I_k + \sigma_b^2 J_k$

$\Sigma^{-1} = \text{Diag}\{$

$aI + bJ$

$a[I + \frac{b}{a} \underset{\sim}{1}\underset{\sim}{1}']$

$= a[A + \underset{\sim}{u}\underset{\sim}{v}']$.

$$(aI+bJ)^{-1} = \frac{1}{a}[A + \underset{\sim}{u}\underset{\sim}{v}']^{-1}$$

$$= \frac{1}{a}\left[A^{-1} - \frac{(A^{-1}u)(v'A^{-1})}{1 + \underset{\sim}{v}'A^{-1}\underset{\sim}{u}}\right]$$

$$= \frac{1}{a}\left[I_K - \frac{\left(\frac{b}{a}\underset{\sim}{1}_K\right)\left(\underset{\sim}{1}_K'\right)}{1 + \frac{b}{a}\underset{\sim}{1}_K'\underset{\sim}{1}_K}\right]$$

$$\Sigma^{-1} = \text{Diag}(L^{-1}, L^{-1}, \dots, L^{-1}) \qquad = \frac{1}{a}\left[I_K - \frac{\frac{b}{a}J_K}{1+\frac{Kb}{a}}\right]$$

$$= \frac{1}{a}I_K - \frac{b}{a(a+Kb)}J_K$$

Here $a = \sigma^2$; $b = \sigma_b^2$

$\therefore L^{-1} = a^* I_K + b^* J_K$; Where $a^* = \frac{1}{a} = \frac{1}{\sigma^2}$

$b^* = -\frac{b}{a(a+Kb)}$

Define,

$W_1 = \frac{1}{\sigma^2}$ (Reciprocal of intra block variance)

$W_2 = \frac{1}{\sigma^2 + K\sigma_b^2}$ (Reciprocal of inter block variance) $= -\frac{\sigma_b^2}{\sigma^2(\sigma^2 + K\sigma_b^2)}$

$a^* = W_1$ and $b^* = -\frac{W_1 - W_2}{K}$

$$\Sigma^{-1} = \text{Diag}(a^* I_K + b^* J_K, a^* I_K + b^* J_K, \dots, a^* I_K + b^* J_K)$$

$$= a^* I_n + b^* \text{Diag}(J_K, J_K, \dots, J_K)$$

$$= W_1 I_n - \left(\frac{W_1 - W_2}{K}\right) X_\beta X_\beta'$$

Normal equations

$$A'\Sigma^{-1}A\underset{\sim}{\theta} = A'\Sigma^{-1}\underset{\sim}{y}$$

$$A = [\underset{\sim}{1}_{n\times 1} : X_{\tau\, n\times v}] ; \theta = \begin{bmatrix} \mu_{1\times 1} \\ \tau_{v\times 1} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \underset{\sim}{1}_n' \\ X_\tau' \end{bmatrix} \Sigma^{-1} (\underset{\sim}{1}_n \; X_\tau)\underset{\sim}{\theta} = \begin{bmatrix} \underset{\sim}{1}_n' \\ X_\tau' \end{bmatrix} \Sigma^{-1}\underset{\sim}{y} \Rightarrow \begin{bmatrix} \underset{\sim}{1}_n'\Sigma^{-1}\underset{\sim}{1}_n & \underset{\sim}{1}'\Sigma^{-1}X_\tau \\ X_\tau'\Sigma^{-1}\underset{\sim}{1} & X_\tau'\Sigma^{-1}X_\tau \end{bmatrix}\underset{\sim}{\theta} = \begin{bmatrix} \underset{\sim}{1}'\Sigma^{-1}y \\ X'\Sigma^{-1}y \end{bmatrix}$$

$$\underset{\sim}{1}_n' \Sigma^{-1} \underset{\sim}{1}_n = \underset{\sim}{1}_n' \left[W_1 I_n - \frac{(W_1 - W_2)}{K} x_\beta x_\beta' \right] \underset{\sim}{1}_n$$

$$= W_1 \underset{\sim}{1}_n' \underset{\sim}{1}_n - \frac{(W_1-W_2)}{K} (\underset{\sim}{1}_n' x_\beta)(x_\beta' \underset{\sim}{1}_n)$$

$$= W_1 n - \frac{(W_1-W_2)}{K} (K \underset{\sim}{1}_b')(\underset{\sim}{1}_b K) \quad [\because \text{Proper}]$$

$$= W_1 n - K(W_1 - W_2)(\underset{\sim}{1}_b' \underset{\sim}{1}_b)$$

$$= W_1 n - bK(W_1 - W_2) = W_2 n$$

$$\underset{\sim}{1}_n' \Sigma^{-1} X_\tau = \underset{\sim}{1}_n' \left[W_1 I_n - \frac{(W_1-W_2)}{K} x_\beta x_\beta' \right] X_\tau$$

$$= W_1 \underset{\sim}{1}_n' X_\tau - \left(\frac{W_1 - W_2}{K}\right) (\underset{\sim}{1}_n' x_\beta)(x_\beta' x_\tau)$$

$$= W_1 \underset{\sim}{r}' - \frac{(W_1-W_2)}{K} (K \underset{\sim}{1}_b') N' \quad [\because \text{Proper}]$$

$$= W_1 \underset{\sim}{r}' - \frac{(W_1-W_2)}{\cancel{K}} (N \underset{\sim}{1}_b)'$$

$$= W_1 \underset{\sim}{r}' - \frac{(W_1-W_2)}{\cancel{K}} \underset{\sim}{r}' = W_2 \underset{\sim}{r}'$$

$$\therefore X_\tau' \Sigma^{-1} \underset{\sim}{1}_n = W_2 \underset{\sim}{r}$$

$$X_\tau' \Sigma^{-1} X_\tau = X_\tau' \left[W_1 I_n - \frac{(W_1-W_2)}{K} x_\beta x_\beta' \right] X_\tau$$

$$= W_1 X_\tau' X_\tau - \frac{(W_1-W_2)}{K} (X_\tau' x_\beta)(x_\beta' x_\tau)$$

$$= W_1 D_r - \frac{(W_1-W_2)}{K} N N'$$

$$\underset{\sim}{1}_n' \Sigma^{-1} \underset{\sim}{y} = \underset{\sim}{1}_n' \left[W_1 I_n - \frac{(W_1-W_2)}{K} x_\beta x_\beta' \right] \underset{\sim}{y}$$

$$= W_1 \underset{\sim}{1}_n' \underset{\sim}{y} - \frac{(W_1-W_2)}{K} (\underset{\sim}{1}_n' x_\beta)(x_\beta' \underset{\sim}{y})$$

$$= W_1 G - \frac{(W_1-W_2)}{K} (K \underset{\sim}{1}_b') \underset{\sim}{B} \quad [\text{Proper}]$$

$$= W_1 G - (W_1 - W_2)(\underset{\sim}{1}_b' B)$$

$$= W_1 G - (W_1 - W_2) G = W_2 G$$

$$x_\tau' \Sigma^{-1} \underset{\sim}{y} = X_\tau' \left[W_1 I_n - \frac{(W_1-W_2)}{K} x_\beta x_\beta' \right] \underset{\sim}{y}$$

$$= W_1 x_\tau' \underset{\sim}{y} - \frac{(W_1-W_2)}{K} (x_\tau' x_\beta)(x_\beta' \underset{\sim}{y})$$

$$= W_1 \underset{\sim}{T} - \frac{(W_1-W_2)}{K} N \underset{\sim}{B}$$

$$= W_1 \underset{\sim}{T} - (W_1 - W_2)(\underset{\sim}{T} - \underset{\sim}{Q}) = W_1 \underset{\sim}{Q} + W_2(\underset{\sim}{T} - \underset{\sim}{Q})$$

Adjusted Treatment Total

$$\underset{\sim}{Q} = \underset{\sim}{T} - N D_K^{-1} \underset{\sim}{B}$$

For a proper design

$$\underset{\sim}{Q} = \underset{\sim}{T} - N D_K^{-1} \underset{\sim}{B} = \underset{\sim}{T} - \frac{1}{K} N \underset{\sim}{B}$$

$$\therefore \frac{1}{K} N \underset{\sim}{B} = \underset{\sim}{T} - \underset{\sim}{Q}$$

The normal equation reduces to

$$\begin{pmatrix} W_2 n & W_2 \underset{\sim}{r}' \\ W_2 \underset{\sim}{r} & W_2 D_r - \frac{(W_1-W_2)}{K} N N' \end{pmatrix} \begin{pmatrix} \mu \\ \underset{\sim}{\tau} \end{pmatrix} = \begin{pmatrix} W_2 G \\ W_1 \underset{\sim}{Q} + W_2(\underset{\sim}{T} - \underset{\sim}{Q}) \end{pmatrix}$$

$$n W_2 \mu + W_2 \underset{\sim}{r}' \underset{\sim}{T} = W_2 G \quad \ldots (1)$$

$$W_2 \underset{\sim}{r} \mu + \left(W_1 D_r - \frac{(W_1 - W_2)}{k} N N' \right) \underset{\sim}{\tau} = W_1 \underset{\sim}{Q} + W_2 (\underset{\sim}{T} - \underset{\sim}{Q}) \ldots (2)$$

Premultiply both sides of equ^n (1) by $\underset{\sim}{r}/n$

$$\underset{\sim}{r} W_2 \mu + W_2 \left(\frac{\underset{\sim}{r}\underset{\sim}{r}'}{n} \right) \underset{\sim}{\tau} = W_2 \frac{\underset{\sim}{r} G}{n} \quad \ldots (3)$$

(2) − (3)

$$\left(W_1 D_r - \frac{(W_1 - W_2)}{k} N N' - W_2 \frac{\underset{\sim}{r}\underset{\sim}{r}'}{n} \right) \underset{\sim}{\tau} = W_1 \underset{\sim}{Q} + W_2 \left(\underset{\sim}{T} - \underset{\sim}{Q} - \frac{\underset{\sim}{r} G}{n} \right)$$

$$\Rightarrow \left[W_1 \left(D_r - \frac{N N'}{k} \right) + W_2 \left(\frac{N N'}{k} - \frac{\underset{\sim}{r}\underset{\sim}{r}'}{n} \right) \right] \underset{\sim}{\tau} = W_1 \underset{\sim}{Q} + W_2 \left(\underset{\sim}{T} - \underset{\sim}{Q} - \frac{\underset{\sim}{r} G}{n} \right)$$

$$\Rightarrow (W_1 C + W_2 C^*) \tau = W_1 \underset{\sim}{Q} + W_2 \underset{\sim}{Q}^* \quad \ldots (*)$$

Where $C^* = \frac{N N'}{k} - \frac{\underset{\sim}{r}\underset{\sim}{r}'}{n}$ & $Q^* = \underset{\sim}{T} - \underset{\sim}{Q} - \frac{\underset{\sim}{r} G}{n}$

Equation (*) is called adjusted inter-intra block normal equation.

Result 1: $E(\underset{\sim}{Q}) = C \underset{\sim}{\tau}$

Proof >> $\underset{\sim}{Q} = \underset{\sim}{T} - \frac{N \underset{\sim}{B}}{k} = X_\tau' \underset{\sim}{y} - \frac{N X_\beta' \underset{\sim}{y}}{k}$

$$= \left(X_\tau - \frac{X_\beta N'}{k} \right)' \underset{\sim}{y}$$

$$\therefore E(\underset{\sim}{Q}) = \left(X_\tau - X_\beta \frac{N'}{k} \right)' E(\underset{\sim}{y})$$

$$= \left(X_\tau - \frac{X_\beta N'}{k} \right)' \left(\mu \underset{\sim}{1}_n + X_\tau \cdot \underset{\sim}{\tau} \right)$$

$$= \mu X_\tau' \underset{\sim}{1}_n - \frac{\mu N (X_\beta' \underset{\sim}{1}_n)}{k} + X_\tau' X_\tau \underset{\sim}{\tau} - \frac{N X_\beta' X_\tau \underset{\sim}{\tau}}{k}$$

$$= \mu \underset{\sim}{r} - \mu \frac{N (k \underset{\sim}{1}_b)}{k} + D_r \underset{\sim}{\tau} - \frac{N N'}{k} \underset{\sim}{\tau}$$

$$= \mu \underset{\sim}{r} - \mu \underset{\sim}{r} + \left(D_r - \frac{N N'}{k} \right) \underset{\sim}{\tau} \quad \left[\because N \underset{\sim}{1}_b = \underset{\sim}{r} \right]$$

$$= C \underset{\sim}{\tau}$$

Result 2: $E(Q^*) = C^* \underset{\sim}{\tau}$

Proof >> $Q^* = \underset{\sim}{T} - \underset{\sim}{Q} - \frac{\underset{\sim}{r} G}{n}$

$$= \frac{1}{k} N \underset{\sim}{B} - \frac{\underset{\sim}{r} G}{n}$$

$$= \frac{1}{k} N X_\beta' \underset{\sim}{y} - \frac{\underset{\sim}{r} \underset{\sim}{1}_n' \underset{\sim}{y}}{n}$$

$$= \left(\frac{N X_\beta'}{k} - \frac{\underset{\sim}{r} \underset{\sim}{1}_n'}{n} \right) \underset{\sim}{y}$$

$$\therefore E(Q^*) = \left(\frac{N X_\beta'}{k} - \frac{\underset{\sim}{r} \underset{\sim}{1}_n'}{n} \right) E(\underset{\sim}{y})$$

$$= \left(\frac{N X_\beta'}{k} - \frac{\underset{\sim}{r} \underset{\sim}{1}_n'}{n} \right) \left(\mu \underset{\sim}{1}_n + X_\tau \underset{\sim}{\tau} \right)$$

$$= \frac{\mu N X_\beta' \underset{\sim}{1}_n}{k} - \frac{\mu \underset{\sim}{r} \underset{\sim}{1}_n' \underset{\sim}{1}_n}{n} + \frac{N X_\beta' X_\tau \underset{\sim}{\tau}}{k} - \frac{\underset{\sim}{r} \underset{\sim}{1}_n' X_\tau \underset{\sim}{\tau}}{n}$$

$$= \mu \frac{N (k \underset{\sim}{1}_b)}{k} - \mu \underset{\sim}{r} \frac{n}{n} + \frac{N N'}{k} \tau - \frac{\underset{\sim}{r} \underset{\sim}{r}'}{n} \underset{\sim}{\tau}$$

$$= \left(\frac{N N'}{k} - \frac{\underset{\sim}{r} \underset{\sim}{r}'}{n} \right) \underset{\sim}{\tau} = C^* \underset{\sim}{\tau} \quad \left[\because N \underset{\sim}{1}_b = \underset{\sim}{r} \right]$$

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Result 3 : (***)

$$\mathcal{D}isp\begin{pmatrix}\underset{\sim}{Q}\\ \underset{\sim}{Q}^*\\ G\end{pmatrix}_{2v+1} = \begin{bmatrix} \underset{v\times v}{C\sigma^2} & \underset{v\times v}{\underset{\sim}{0}} & \underset{v\times v}{\underset{\sim}{0}} \\ 0 & \underset{v\times v}{C^*/W_2} & \underset{v\times 1}{\underset{\sim}{0}} \\ \underset{1\times v}{\underset{\sim}{0}'} & \underset{1\times v}{\underset{\sim}{0}'} & \underset{1\times 1}{n/W_2} \end{bmatrix}_{(2v+1)\times(2v+1)}$$

Proof:

$$\begin{aligned} Var(G) &= Var(\underset{\sim}{1}_n' \underset{\sim}{y}) \\ &= \underset{\sim}{1}_n' \Sigma \underset{\sim}{1}_n \\ &= \underset{\sim}{1}_n'(\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta')\underset{\sim}{1}_n \\ &= \sigma^2 \underset{\sim}{1}_n'\underset{\sim}{1}_n + \sigma_b^2 \underset{\sim}{1}_n' X_\beta X_\beta' \underset{\sim}{1}_n \\ &= \sigma^2 n + \sigma_b^2 (K\underset{\sim}{1}_b')(K\underset{\sim}{1}_b) \\ &= \sigma^2 n + \sigma_b^2 K^2 (\underset{\sim}{1}_b'\underset{\sim}{1}_b) \\ &= \sigma^2 n + bK^2\sigma_b^2 \\ &= \sigma^2 n + nK\sigma_b^2 \\ &= n[\sigma^2 + K\sigma_b^2] = \frac{n}{W_2} \end{aligned}$$

$$\begin{aligned} \mathcal{D}isp(\underset{\sim}{Q}) &= \mathcal{D}isp\left(T - \frac{NB}{K}\right) \\ &= \mathcal{D}isp\left(X_\tau'\underset{\sim}{y} - \frac{NX_\beta'\underset{\sim}{y}}{K}\right) \\ &= \left(X_\tau' - \frac{NX_\beta'}{K}\right)\Sigma\left(X_\tau - \frac{1}{K}X_\beta N'\right) \end{aligned}$$

$= X_\tau'\Sigma X_\tau - \frac{1}{K}NX_\beta'\Sigma X_\tau + \frac{1}{K^2}NX_\beta'\Sigma X_\beta N' - \frac{1}{K}X_\tau'\Sigma X_\beta N'$

$= X_\tau'(\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta')X_\tau - \frac{1}{K}NX_\beta'(\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta')X_\tau$
$- \frac{1}{K}X_\tau'(\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta')X_\beta N' + \frac{1}{K^2}NX_\beta'$
$(\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta')X_\beta N'$

$= \sigma^2[X_\tau'X_\tau - \frac{1}{K}NX_\beta'X_\tau - \frac{1}{K}X_\tau'X_\beta N' + \frac{1}{K^2}NX_\beta'X_\beta N']$
$+ \sigma_b^2[(X_\tau'X_\beta)(X_\beta'X_\tau) - \frac{1}{K}N(X_\beta'X_\beta)(X_\beta'X_\tau) -$
$\frac{1}{K}(X_\tau'X_\beta)(X_\beta'X_\beta)N' + \frac{1}{K^2}N(X_\beta'X_\beta)$
$(X_\beta'X_\beta)N']$

$= \sigma^2[\mathcal{D}_r - \frac{1}{K}NN' - \frac{1}{K}NN' + \frac{1}{K^2}N\mathcal{D}_K N'] +$
$\sigma_b^2[NN' - \frac{1}{K}N\mathcal{D}_K N' - \frac{1}{K}N\mathcal{D}_K N' + \frac{1}{K^2}N\mathcal{D}_K\mathcal{D}_K N']$

$= \sigma^2[\mathcal{D}_r - \frac{NN'}{K} - \frac{NN'}{K} + \frac{N}{K^2}K\underset{\sim}{I}_b N']$
$+ \sigma_b^2[NN' - \frac{1}{K}NK\underset{\sim}{I}_b N' - \frac{1}{K}NKI_b N' + \frac{1}{K^2}N(K\underset{\sim}{I}_b)$
$(K\underset{\sim}{I}_b)N']$

$= C\sigma^2 + \sigma_b^2[NN' - NN']$

$= C\sigma^2$

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$$\mathrm{Disp}(\underset{\sim}{Q}^*) = \mathrm{Disp}\left(\frac{N\underset{\sim}{B}}{K} - \frac{\underset{\sim}{b}\, G}{n}\right)$$

$$= \mathrm{Disp}\left(\frac{N X_\beta' \underset{\sim}{Y}}{K} - \frac{\underset{\sim}{b}\, \underset{\sim}{1}' \underset{\sim}{Y}}{n}\right)$$

$$= \left(\frac{N X_\beta'}{K} - \frac{\underset{\sim}{b}\, \underset{\sim}{1}'}{n}\right) \Sigma \left(\frac{X_\beta N'}{K} - \frac{\underset{\sim}{1}\, \underset{\sim}{b}'}{n}\right)$$

$$= \frac{1}{K^2} N X_\beta' \Sigma X_\beta N' - \frac{1}{nK} \underset{\sim}{b}\, \underset{\sim}{1}' \Sigma X_\beta N'$$

$$- \frac{1}{nK} N X_\beta' \Sigma \underset{\sim}{1}\, \underset{\sim}{b}' + \frac{1}{n^2} \underset{\sim}{b}\, \underset{\sim}{1}' \Sigma \underset{\sim}{1}\, \underset{\sim}{b}'$$

$$\underset{\sim}{1}' \Sigma \underset{\sim}{1} = \underset{\sim}{1}' \left(\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta'\right) \underset{\sim}{1}$$

$$= \sigma^2 \underset{\sim}{1}' \underset{\sim}{1} + \sigma_b^2 \underset{\sim}{1}' X_\beta X_\beta' \underset{\sim}{1}$$

$$= n\sigma^2 + \sigma_b^2 (K \underset{\sim}{1}_b') (K I_b)$$

$$= n\sigma^2 + K^2 b \sigma_b^2$$

$$= n\sigma^2 + nK\sigma_b^2$$

$$= n(\sigma^2 + K\sigma_b^2) = \frac{n}{w_2}$$

$$\underset{\sim}{1}' \Sigma X_\beta = \underset{\sim}{1}' \left(\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta'\right) X_\beta$$

$$= \sigma^2 \underset{\sim}{1}' X_\beta + \sigma_b^2 \underset{\sim}{1}' X_\beta X_\beta' X_\beta$$

$$= K\sigma^2 \underset{\sim}{1}_b' + \sigma_b^2 K \underset{\sim}{1}_b' K I_b$$

$$= K(\sigma^2 + K\sigma_b^2) \underset{\sim}{1}_b' = \frac{K}{w_2} \underset{\sim}{1}_b'$$

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$X_\beta' \Sigma X_\beta$

$= X_\beta' \left(\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta' \right) X_\beta$

$= \sigma^2 X_\beta' X_\beta + \sigma_b^2 (X_\beta' X_\beta)(X_\beta' X_\beta)$

$= \sigma^2 (K I_b) + \sigma_b^2 (K I_b)(K I_b)$

$= K(\sigma^2 + K \sigma_b^2) I_b$

$= \frac{K}{W_2} I_b$

$$= \frac{1}{K^2} N N' \left(\frac{K}{W_2}\right) - \frac{1}{nK} \mathbf{b} \left(\frac{K}{W_2} \mathbf{1}_b'\right) N' - \frac{1}{nK} N \left(\frac{K}{W_2} \mathbf{1}_b\right) \mathbf{b}' + \frac{1}{n^2} \mathbf{b} \left(\frac{n}{W_2}\right) \mathbf{b}'$$

$$= \frac{1}{W_2} \left[\frac{NN'}{K} - \frac{\mathbf{b} \mathbf{1}_b' N'}{n} - \frac{N \mathbf{1}_b \mathbf{b}'}{n} + \frac{\mathbf{b}\mathbf{b}'}{n} \right]$$

$$= \frac{1}{W_2} \left[\frac{NN'}{K} - \frac{\mathbf{b}\mathbf{b}'}{n} - \frac{\mathbf{b}\mathbf{b}'}{n} + \frac{\mathbf{b}\mathbf{b}'}{n} \right]$$

$$= \frac{C^*}{W_2}$$

$$\mathrm{Cov}(\mathbf{Q}, \mathbf{Q}^*) = \mathrm{Cov}\left(\mathbf{T} - \frac{N\mathbf{B}}{K}, \frac{N\mathbf{B}}{K} - \frac{\mathbf{b} G}{n} \right)$$

$$= \mathrm{Cov}\left(X_\tau' \mathbf{Y} - \frac{N X_\beta' \mathbf{Y}}{K}, \frac{N X_\beta' \mathbf{Y}}{K} - \frac{\mathbf{b} \mathbf{1}_n' \mathbf{Y}}{n} \right)$$

$$= \left(X_\tau' - \frac{N X_\beta'}{K} \right) \Sigma \left(\frac{X_\beta N'}{K} - \frac{\mathbf{1}_n \mathbf{b}'}{n} \right)$$

$$= \frac{X_\tau' \Sigma X_\beta N'}{K} - \frac{N X_\beta' \Sigma X_\beta N'}{K^2} - \frac{X_\tau' \Sigma \mathbf{1}_n}{n} \mathbf{b}' + \frac{N X_\beta' \Sigma \mathbf{1}_n \mathbf{b}'}{nK}$$

$$= \frac{NN'}{KW_2} - \frac{NN'}{KW_2} - \frac{1}{nW_2} \mathbf{b}\mathbf{b}' + \frac{1}{nW_2} N \mathbf{1}_b \mathbf{b}'$$

$$= 0 \quad [\because N \mathbf{1}_b = \mathbf{b}]$$

$$X_\tau' \Sigma \underset{\sim}{1}_n = X_\tau' (\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta') \underset{\sim}{1}_n$$
$$= \sigma^2 X_\tau' \underset{\sim}{1} + \sigma_b^2 X_\tau' X_\beta X_\beta' \underset{\sim}{1}_n$$
$$= \sigma^2 \underset{\sim}{b} + K\sigma_b^2 \, N \underset{\sim}{1}_b$$
$$= \sigma^2 \underset{\sim}{b} + K\sigma_b^2 \underset{\sim}{b}$$
$$= (\sigma^2 + K\sigma_b^2)\underset{\sim}{b}$$
$$= \frac{1}{W_2}\underset{\sim}{b}$$

$$X_\tau' \Sigma X_\beta = X_\tau' (\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta') X_\beta$$
$$= \sigma^2 X_\tau' X_\beta + \sigma_b^2 (X_\tau' X_\beta)(X_\beta' X_\beta)$$
$$= \sigma^2 N + \sigma_b^2 \cdot N (K I_b)$$
$$= (\sigma^2 + K\sigma_b^2) N = \frac{1}{W_2} N$$

$$\operatorname{cov}(\underset{\sim}{Q}, G) = \operatorname{cov}\left(T - \frac{NB}{K}, G\right)$$
$$= \operatorname{cov}\left(X_\tau' \underset{\sim}{Y} - \frac{N X_\beta' \underset{\sim}{Y}}{K}, \underset{\sim}{1}_n' \underset{\sim}{Y}\right)$$
$$= \left(X_\tau' - \frac{N X_\beta'}{K}\right) \Sigma \underset{\sim}{1}_n$$
$$= X_\tau' \Sigma \underset{\sim}{1}_n - \frac{N X_\beta' \Sigma \underset{\sim}{1}_n}{K}$$
$$= \frac{\underset{\sim}{b}}{W_2} - \frac{N \underset{\sim}{1}_b}{W_2}$$
$$= \frac{\underset{\sim}{b}}{W_2} - \frac{\underset{\sim}{b}}{W_2} = \underset{\sim}{0}$$

$$\operatorname{cov}(\underset{\sim}{Q}^*, G) = \operatorname{cov}\left(\frac{NB}{K} - \frac{\underset{\sim}{b}\, G}{n}, \underset{\sim}{1}_n' Y\right)$$
$$= \operatorname{cov}\left(\frac{N X_\beta' \underset{\sim}{Y}}{K} - \frac{\underset{\sim}{b}\, \underset{\sim}{1}_n' Y}{n}, \underset{\sim}{1}_n' \underset{\sim}{Y}\right)$$
$$= \left(\frac{N X_\beta'}{K} - \frac{\underset{\sim}{b}\, \underset{\sim}{1}_n'}{n}\right) \Sigma \underset{\sim}{1}_n$$
$$= \frac{N}{K} X_\beta' \Sigma \underset{\sim}{1}_n - \frac{\underset{\sim}{b}\, \underset{\sim}{1}_n' \Sigma \underset{\sim}{1}_n}{n}$$
$$= \frac{N}{\cancel{K}} \frac{\cancel{K}}{W_2} \underset{\sim}{1}_b - \frac{\underset{\sim}{b}}{\cancel{n}} \frac{\cancel{n}}{W_2}$$
$$= \frac{\underset{\sim}{b}}{W_2} - \frac{\underset{\sim}{b}}{W_2} = \underset{\sim}{0}$$

Different Sum of Squares and their expectations under mixed effects model:

Assuming proper & binary design

In the intra block analysis

① Total $SS = S^2 = \sum_i \sum_j y_{ij}^2 - \frac{G^2}{n}$

② Unadjusted block $SS = S_b'^2 = \sum_j \frac{B_j^2}{k} - \frac{G^2}{n}$

③ Unadjusted treatment $SS = S_t'^2 = \sum_{i=1}^{v} \frac{T_i^2}{r_i} - \frac{G^2}{n}$

④ Adjusted treatment $SS = S_t^2 = \underset{\sim}{Q}' \hat{\underset{\sim}{\tau}}$

⑤ Residual $SS = R_0^2$

$$S^2 = S_b'^2 + S_t^2 + R_0^2 = S_b^2 + S_t'^2 + R_0^2$$

$$E(G) = E(\underset{\sim}{1}_n' \underset{\sim}{Y}) = \underset{\sim}{1}_n' (\mu \underset{\sim}{1}_n + X_\tau \underset{\sim}{\tau})$$
$$= \mu \underset{\sim}{1}_n' \underset{\sim}{1}_n + \underset{\sim}{1}_n' X_\tau \underset{\sim}{\tau}$$
$$= n\mu + \underset{\sim}{r}' \underset{\sim}{\tau}$$
$$= n\mu + \sum_{i=1}^{v} r_i \tau_i$$

$$Var(G) = \underset{\sim}{1}_n' \Sigma \underset{\sim}{1}_n = n(\sigma^2 + k\sigma_b^2) \quad \text{(Already proved)}$$

$$E(G^2/n) = \frac{1}{n}\left[Var(G) + (E(G))^2\right]$$
$$= \frac{1}{n}\left[n(\sigma^2 + k\sigma_b^2) + n^2\left(\mu + \frac{\sum_{i=1}^{v} r_i \tau_i}{n}\right)^2\right]$$
$$= (\sigma^2 + k\sigma_b^2) + n(\mu + \bar{\tau})^2 \quad \left[\bar{\tau} = \sum_{i=1}^{v} \frac{r_i \tau_i}{n}\right]$$

$$E(\underset{\sim}{T}) = E(X_\tau' \underset{\sim}{Y})$$
$$= X_\tau' (\mu \underset{\sim}{1}_n + X_\tau \underset{\sim}{\tau})$$
$$= \mu X_\tau' \underset{\sim}{1}_n + X_\tau' X_\tau \underset{\sim}{\tau}$$
$$= \mu \underset{\sim}{r} + D_r \underset{\sim}{\tau}$$

$$D_r \underset{\sim}{\tau} = \begin{bmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_v \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_v \end{bmatrix} = \begin{bmatrix} r_1\tau_1 \\ r_2\tau_2 \\ \vdots \\ r_v\tau_v \end{bmatrix}$$

$$E(T_i) = \mu r_i + r_i \tau_i$$
$$= r_i(\mu + \tau_i)$$

$$Disp(\underset{\sim}{T}) = X_\tau' \Sigma X_\tau$$
$$= X_\tau' (\sigma^2 I_n + \sigma_b^2 X_\beta X_\beta') X_\tau$$
$$= \sigma^2 X_\tau' X_\tau + \sigma_b^2 X_\tau' X_\beta X_\beta' X_\tau$$
$$= \sigma^2 D_r + \sigma_b^2 NN'$$

$$NN' = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ \vdots & & & \\ n_{v1} & n_{v2} & \cdots & n_{vb} \end{bmatrix} \begin{bmatrix} n_{11} & \cdots & n_{v1} \\ \vdots & & \\ n_{1b} & \cdots & n_{vb} \end{bmatrix}$$

ith diagonal element $\sum_{j=1}^{b} n_{ij}^2 = \sum_{j=1}^{b} n_{ij} = r_i$ [$\because$ binary design]

$$\therefore Var(T_i) = \sigma^2 r_i + \sigma_b^2 r_i = r_i(\sigma^2 + \sigma_b^2)$$

$$E\left(\frac{T_i^2}{r_i}\right) = \frac{1}{r_i}\left(Var(T_i) + \{E(T_i)\}^2\right)$$
$$= \sigma^2 + \sigma_b^2 + r_i(\mu + \tau_i)^2, \quad i = 1(1)v$$

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$$E\left[\sum_{i=1}^{v}\frac{T_i^2}{b_i}\right] - E\left[\frac{G^2}{n}\right] = v(\sigma^2+\sigma_b^2) + \sum_{i=1}^{v} b_i(\mu+\tau_i)^2 - (\sigma^2+k\sigma_b^2) - n(\mu+\bar{\tau})^2$$

$$= (v-1)\sigma^2 + (v-k)\sigma_b^2 + \sum_{i=1}^{v} b_i(\mu^2+\tau_i^2+2\mu\tau_i) - n(\mu^2+\bar{\tau}^2+2\mu\bar{\tau})$$

$$= (v-1)\sigma^2 + (v-k)\sigma_b^2 + \sum_{i=1}^{v} b_i\mu^2 - n\mu^2 + \sum_{i=1}^{v} b_i\tau_i^2 - n\bar{\tau}^2 + 2\mu\sum_{i=1}^{v} b_i\tau_i - 2\mu n\bar{\tau}$$

$$= (v-1)\sigma^2 + (v-k)\sigma_b^2 + \sum_{i=1}^{v} b_i(\tau_i-\bar{\tau})^2$$

$$\left[\because \sum_{i=1}^{v} b_i = n,\ \bar{\tau} = \sum_{i=1}^{v} b_i\tau_i/n\right]$$

$E(R_0^2) = (n-v-b+1)\sigma^2$ (connected design)

$$S^2 = S_b^2 + S_t'^2 + R_0^2$$

$$E(S_b^2) = E(S^2) - E(S_t'^2) - E(R_0^2)$$

$$S^2 = \sum_{i=1}^{v}\sum_{j=1}^{b} y_{ij}^2 - \frac{G^2}{n}$$

$$E(y_{ij}^2) = Var(y_{ij}) + \{E(y_{ij})\}^2$$

$$y_{ij} = \mu + \tau_i + \beta_j + e_{ij} \quad ; \ i=1(1)v,\ j=1(1)b$$

$$\therefore E(y_{ij}^2) = (\sigma_b^2+\sigma^2) + (\mu+\tau_i)^2 \quad ; \ i=1(1)v,\ j=1(1)b$$

$$E\left[\sum_i\sum_j y_{ij}^2\right] = n(\sigma^2+\sigma_b^2) + \sum_{i=1}^{v}\sum_{j=1}^{b} n_{ij}(\mu+\tau_i)^2$$

$$E(S^2) = n(\sigma^2+\sigma_b^2) + \sum_{i=1}^{v}\sum_{j=1}^{b} n_{ij}(\mu+\tau_i)^2 - (\sigma^2+k\sigma_b^2) - n(\mu+\bar{\tau})^2$$

$$= (n-1)\sigma^2 + (n-k)\sigma_b^2 + \sum_{i=1}^{v} b_i(\mu+\tau_i)^2 - n(\mu+\bar{\tau})^2 \quad \left[\because \sum_{j=1}^{b} n_{ij} = b_i\right]$$

$$= (n-1)\sigma^2 + (n-k)\sigma_b^2 + \sum_{i=1}^{v} \cancel{b_i\mu^2} + 2\sum_{i=1}^{v} b_i\mu\tau_i + \sum_{i=1}^{v} b_i\tau_i^2 - \cancel{n\mu^2} - 2n\mu\bar{\tau} - n\bar{\tau}^2$$

$$= (n-1)\sigma^2 + (n-k)\sigma_b^2 + \sum_{i=1}^{v} b_i(\tau_i-\bar{\tau})^2$$

$$E(S_b^2) = E(S^2) - E(S_t'^2) - E(R_0^2)$$

$$= \cancel{(n-1)\sigma^2 + (n-k)\sigma_b^2 + \sum_{i=1}^{v} b_i(\tau_i-\bar{\tau})^2 - (v-1)\sigma^2 - (v-k)\sigma_b^2}$$

$$\cancel{-(n-v-b+1)\sigma^2 - \sum_{i=1}^{v} b_i(\tau_i-\bar{\tau})^2 + (\sigma^2+k\sigma_b^2) + n(\mu+\bar{\tau})^2}$$

$$= \cancel{(n-1-v+1+1-n+v+b-1)\sigma^2 + (n-k-v+k+k)\sigma_b^2 + n(\mu+\bar{\tau})^2}$$

$$= \cancel{(b-1)\sigma^2 + (n-v)\sigma_b^2}$$

$$= (n-1)\sigma^2 + (n-k)\sigma_b^2 + \cancel{\sum_{i=1}^{v} b_i(\tau_i-\bar{\tau})^2} - (v-1)\sigma^2 - (v-k)\sigma_b^2$$

$$- \cancel{\sum_{i=1}^{v} b_i(\tau_i-\bar{\tau})^2} - (n-v-b+1)\sigma^2$$

$$= (n-1-v+1-n+v+b-1)\sigma^2 + (n-k-v+k)\sigma_b^2$$

$$= (b-1)\sigma^2 + (n-v)\sigma_b^2$$

$$E\left[\frac{S_b^2}{b-1} - \frac{R_0^2}{n-b-v+1}\right] = \frac{n-v}{b-1}\,\sigma_b^2$$

$$\therefore\ \hat{\sigma}^2 = \frac{R_0^2}{n-b-v+1} \quad ; \quad \hat{\sigma}_b^2 = \frac{(b-1)}{(n-v)}\left[\frac{S_b^2}{b-1} - \frac{R_0^2}{n-b-v+1}\right]$$

Groups

A non-empty set G is said to form a group with respect to a binary composition $\circ$, if

i) G is closed under the composition $\circ$, viz., if $a \in G$, $b \in G$, then $a \circ b \in G$

ii) $\circ$ is associative.

iii) $\exists$ an element e in G such that $e \circ a = a \circ e = a$ $\forall a \in G$ (e is the identity element under operation $\circ$)

iv) For each $a \in G$, $\exists$ an element $a' \in G$ such that $a \circ a' = a' \circ a = e$ (a' is the inverse of a)

▫ Abelian Group : A group $(G, \circ)$ is said to be a commutative group or an Abelian group if the operation $\circ$ is commutative.

For ~~an~~ Abelian group, we use operation '+'. Then

i) $\forall a, b \in$ modul (or group), $\exists$ a unique element $s \in$ modul $\ni a + b = s \in$ modul

ii) $a + (b + c) = (a + b) + c$, $\forall a, b, c \in$ modul.

iii) $\exists$ a unique element 0 (zero) $\ni$ $\forall c \in$ modul, $c + 0 = 0 + c = c$. 0 is called the identity element.

v) For any two $a, b \in$ modul, $\exists x \in a + x = b$, x is unique.

) $a + b = b + a$, $a, b \in$ modul

Ring

A non-empty set R is said to form a ring with respect to two binary operations, addition (+) and multiplication (·) defined on it, if the following conditions are satisfied.

i) $(R, +)$ is a commutative group or Abelian group, viz., all the properties of Abelian group hold for $(R, +)$.

ii) $(R, \cdot)$ is a semigroup (In which each of the equations $ax = b$ & $y.a = b$ has a solution in R for all $a, b \in R$).

iii) $\forall a, b \in R$, $\exists$ a unique element $p \in R \ni a.p = b$

iv) For any three elements $a, b, c \in R$

③ $$a.(b.c) = (a.b).c$$

④ $$a.(b+c) = (a.b) + (a.c)$$

⑤ In addition, if $a.b = b.a$, $\forall a, b \in R$, then it is called a commutative ring.

Field:

A non-trivial ring unity is a field if it is and $c.1 = c$

i) For each non-zero element $c \in$ Field, (1 is called the unit element)

ii) For $a, b \in$ Field, $\exists y \in$ Field such that $y.a = b$

Congruence:

a and b are said to be congruent to each other (mod p), p is the positive non-zero integer if

$$a = b \pmod p$$

For example, $15 \equiv 3 \pmod 4$

Let all integers congruent to one another (mod p), where p is the positive non zero integer, be considered to belong to the same class and let the class to which a belongs be denoted by (a), then $(a) = (b)$ when and only when ~~a is equivalent to~~ $a \equiv b \pmod p$. The addition of these classes may be denoted by $(a) + (b) = (a+b)$. These p-classes then form a modul, called modul of residue class (mod p). The 0-element being (0). There is only one non-negative integer $a_1 (< p)$

such that $a \equiv a_1 \pmod p$. Then a_1 may be called the standard representative of (a). The p- classes of $(\bmod\ p)$ are $(0), (1), (2), \ldots, (p-1)$.

It can be proved that the residue class $(\bmod\ p)$ satisfies the properties of field. The field is called $GF(p)$. It is the simplest example of Galois Field (A field with finite number of element).

Fermat's Theorem »

If $x \in GF(p)$, then $x^{p-1} - 1 \equiv 0 \bmod (p)$
p is prime or prime power.

Note:
Suppose h is the least positive integer such that $x^h \equiv 1 \pmod p$, then h is called the order of the element x.

Example:
mod 5

$x \backslash h$	1	2	3	4
1	1	1	1	1
2	2	4	3	1
3	3	4	2	1
4	4	1	4	1

2 and 3 are primitive

When h has the value $p-1$. Hence x is said to be a primitive element of $GF(p)$.

To find the primitive element of $GF(3)$

From Fermat's Theorem, if $x \in GF(p)$

$$x^{p-1} \equiv 1 \pmod p$$

$$x^2 - 1 = 0 \pmod 3$$

$$(x+1)(x-1) = 0$$

$x = 1$
$x = -1 = 2 \pmod 3$

$[-1+3=2]$

Since x is the primitive element it follows that the value of b (order of x) is 2. Thus $x^{b_1} \neq 1$ for $b_1 < 2$. Hence $x-1=0$ is not possible. So the primitive element is $x=2$.

To find the primitive element of GF(5).

From Fermat's Theorem, if $x \in GF(p)$

$$x^{p-1} \equiv 1 \pmod 5$$

$$\Rightarrow x^4 - 1 = 0 \pmod 5$$

$$(x^2+1)(x^2-1) = 0$$

Either $x^2+1=0$

Or $x^2-1=0$

Since x is the primitive element of GF(5), it follows that $x^{b_1} \neq 1$ for $b_1 < 4$. Thus $x^2-1 \neq 0$

Hence $x^2+1=0$.

$$x^2+1=0$$

$$x^2 = -1 \pmod 5$$

$$= 4 \pmod 5$$

$$x = 2$$

$$\& \; x = -2 \equiv 3 \pmod 5$$

Thus, the primitive elements are $x=2$ and $x=3$

To find the primitive element of GF(7)

From Fermat's Theorem, if $x \in GF(7)$

$$x^{7-1} \equiv 1 \pmod 7$$

$4^3 = 64 \equiv 1 \bmod 7$

$5^3 = 125 = 6 \bmod 7$

$6^3 = 216 = 6 \bmod 7$

$$\Rightarrow x^6 - 1 = 0 \pmod 7$$

$$\Rightarrow (x^3-1)(x^3+1) = 0$$

Since x is the primitive element of GF(7), it follows that $x^{b_1} \neq 1$ for $b_1 < 6$. Thus $x^3-1 \neq 0$.

Hence $x^3 \cancel{x^6} +1 = 0$.

Hence $x^3+1=0$

$$x^3 = -1 \pmod 7 \equiv 13 \pmod 7$$

$$= 6 \pmod 7 \equiv 20 \pmod 7$$

$$\equiv 27 \pmod 7$$

Thus, the primitive element is $x=3$

To find the primitive element of GF(11).

From Fermat's Theorem, if $x \in GF(11)$

$$x^{11-1} \equiv 1 \ (\text{mod } 11)$$

$$\Rightarrow x^{10} - 1 = 0 \ (\text{mod } 11)$$

$$\Rightarrow (x^5+1)(x^5-1) = 0$$

Since x is the primitive element of GF(11), it follows that $x^{h_1} \neq 1$ for $h_1 < 10$. Thus $x^5 - 1 \neq 0$. Hence $x^5 + 1 = 0$

$$x^5 + 1 = 0$$

$$x^5 = -1 \ (\text{mod } 11)$$

$$\equiv 10 \ (\text{mod } 11)$$

$$\equiv 21 \ (\text{mod } 11)$$

$$\equiv 32 \ (\text{mod } 11) = 2^5 \ (\text{mod } 11)$$

$\therefore x^5 = 2^5$

$\therefore x = 2$ Smallest primitive

$3^5 = 243 \equiv 1 \ (\text{mod } 11)$

$4^5 = 1024 \equiv 1 \ (\text{mod } 11)$

$5^5 = 3125 \equiv 1 \ (\text{mod } 11)$

$6^5 \equiv 7776 \equiv 10 \ (\text{mod } 11)$

$7^5 \equiv 16807 \equiv 10 \ (\text{mod } 11)$

$8^5 \equiv 10 \ (\text{mod } 11)$

$9^5 = 1 \ (\text{mod } 11)$

$10^5 = 10 \ (\text{mod } 11)$

$9^2 = 4 \ (\text{mod } 11)$

$9^4 = 16 \ (\text{mod } 11) \equiv 5 \ (\text{mod } 11)$

$9^5 = 45 \ (\text{mod } 11) \equiv 1 \ (\text{mod } 11)$

$2^0 = 1$ $\quad 2^1 = 2$ $\quad 2^2 = 4$ $\quad 2^3 = 8$ $\quad 2^4 = 16 = 5 \ \text{mod}(11)$

$2^5 = 32 \ 10 \ (\text{mod}(11))$ $\quad 2^6 = 9 \ (\text{mod } 11)$ $\quad 2^7 = 7 \ \text{mod}(11)$

$2^8 = 3 \ \text{mod}(11)$ $\quad 2^9 = 6 \ \text{mod}(11)$ $\quad 2^{10} = 12 \ \text{mod}(11) = 1 \ \text{mod}(11)$

$7^0 = 1$ $\quad 7^1 = 7$ $\quad 7^2 = 49 \equiv 5 \ \text{mod}(11)$ $\quad 7^3 = 2 \ \text{mod}(11)$

$7^4 = 3 \ \text{mod}(11)$ $\quad 7^5 = 10 \ \text{mod}(11)$ $\quad 7^6 = 4 \ \text{mod}(11)$

$7^7 = 6 \ \text{mod}(11)$ $\quad 7^8 = 9 \ \text{mod}(11)$ $\quad 7^9 = 8 \ \text{mod}(11)$

$7^{10} = 56 \equiv 1 \ \text{mod}(11)$

Find the primitive number of GF(13)

$x^{13-1} \equiv 1 \pmod{13}$

$\Rightarrow x^{12} - 1 \equiv 0 \pmod{13}$

$\Rightarrow (x^6-1)(x^6+1) = 0$

$x^6+1=0$

$\Rightarrow x^6 = -1 \pmod{13}$

$= 12 \pmod{13}$

$= 25 \pmod{13}$

$= 38 \pmod{13}$

$= 51 \pmod{13}$

$= 64 \pmod{13} = 2^6 \pmod{13}$

$\therefore x = 2$ Smallest primitive

$3^6 = 729 \equiv 1 \bmod (13)$

$4^6 = 4096 \equiv 1 \bmod (13)$

$5^6 = 15625 = \cancel{15613}\ 12 \bmod (13)$

$6^6 = 46656 \equiv 12 \bmod (13)$

$7^6 = \cancel{117649}\ (10)^3 \pmod{13} = (-3)^3 \pmod{13} = -27 \pmod{13}$

$\equiv -1 \pmod{13}$

$= 12 \pmod{13}$

$8^6 = (64)^3 \pmod{13} = (-1)^3 \pmod{13}$

$= -1 \pmod{13} = 12 \pmod{13}$

$9^6 = 3^3 \pmod{13} \equiv 1 \pmod{13}$

$10^6 = 9^3 \pmod{13} \equiv (-4)^3 \pmod{13} = -64 \pmod{13}$

$= 1 \pmod{13}$

$11^6 = 4^3 \pmod{13} = 64 \pmod{13} = -1 \pmod{13}$

$12^6 = 1^3 \pmod{13}$

Primitive Elements

p	Primitive Elements
3	2
5	2, 3
7	3, 5, 6
11	2, 6, 7, 8, 10
13	2, 5, 6, 7, 8, 11
17	3, 5, 6, 7, 10, 11, 12, 14
19	2, 3, 8, 10, 12, 13, 14, 15, 18

$x^{16} - 1 = 0$

$\Rightarrow (x^8 - 1)(x^8 + 1) = 0$

$x^8 + 1 = 0$

$\Rightarrow x^8 = -1 \pmod{17}$

$= 16 \pmod{17}$

$= 33 \pmod{17}$

$= 50 \pmod{17}$

$=$

$3^8 = 6561 \equiv 16 \bmod (17)$

$4^8 = 65536 \equiv 1 \bmod (17)$

$5^8 = 390625 \equiv 16 \bmod (17)$

$6^8 = 1679616 \equiv 16 \bmod (17)$

$7^8 = 5764801 \equiv 16 \bmod (17)$

$8^8 = 16777216 \equiv 1 \bmod (17)$

$9^8 = 43046721 \equiv 1 \bmod (17)$

$10^8 = 100000000 = 16 \bmod (17)$

$11^8 = 214358881 \equiv 16 \bmod (17)$

$12^8 = 429981696 \equiv 16 \bmod (17)$

$13^8 = 1 \bmod (17)$

$14^8 = 1475789056 \equiv 16 \bmod (17)$

$15^8 = 1 \bmod (17)$

$16^8 = 1 \bmod (17)$

$x^{18} - 1 = 0$

$\Rightarrow (x^9 - 1)(x^9 + 1) = 0$

$x^9 + 1 = 0$

$\Rightarrow x^9 = -1 \bmod (19)$

$= 18 = \cancel{20} \bmod (19)$

$2^9 = 18 \bmod (19)$

$3^9 = 18 \bmod (19)$

$4^9 = 1 \bmod (19)$

$5^9 = 1 \bmod (19)$

$6^9 = 1 \bmod (19)$

$7^9 = 1 \bmod (19)$

$8^9 =$

If x is a primitive element of $GF(p)$, then all the elements of $GF(p)$ can be written as

$$\alpha_0 = 0$$
$$\alpha_1 = 1$$
$$\alpha_2 = x$$
$$\alpha_3 = x^2$$
$$\vdots$$
$$\alpha_{p-1} = x^{p-2}$$

Example $GF(17)$

$x=3$ (Primitive Element)

$\alpha_0 = 0$
$\alpha_1 = 1$
$\alpha_2 = 3$
$\alpha_3 = 9$
$\alpha_4 = 27 \equiv 10$
$\alpha_5 = 13$
$\alpha_6 = 5$
$\alpha_7 = 15$
$\alpha_8 = 11$
$\alpha_9 = 16$

$\alpha_{10} = 14$
$\alpha_{11} = 8$
$\alpha_{12} = 7$
$\alpha_{13} = 4$
$\alpha_{14} = 12$
$\alpha_{15} = 2$
$\alpha_{16} = 6$

$x=2$ (Non - Primitive Element)

$\alpha_0 = 0$
$\alpha_1 = 1$
$\alpha_2 = 2$
$\alpha_3 = 4$
$\alpha_4 = 8$
$\alpha_5 = 16$
$\alpha_6 = 15$
$\alpha_7 = 13$
$\alpha_8 = 9$

$\alpha_9 = 1$
$\alpha_{10} = 2$
$\alpha_{11} = 9$
$\alpha_{12} = 8$
$\alpha_{13} = 16$
$\alpha_{14} = 15$

$GF(19)$

$x=3$ (Primitive Element)

$\alpha_0 = 0$
$\alpha_1 = 1$
$\alpha_2 = 3$
$\alpha_3 = 9$
$\alpha_4 = 27 = 8$
$\alpha_5 = 5$
$\alpha_6 = 15$
$\alpha_7 = 7$
$\alpha_8 = 2$
$\alpha_9 = 6$
$\alpha_{10} = 18$

$\alpha_{11} = 16$
$\alpha_{12} = 10$
$\alpha_{13} = 11$
$\alpha_{14} = 14$
$\alpha_{15} = 4$
$\alpha_{16} = 12$
$\alpha_{17} = 17$
$\alpha_{18} = 13$

$x=5$ (Non- Primitive Element)

$\alpha_0 = 0$
$\alpha_1 = 1$
$\alpha_2 = 5$
$\alpha_3 = 25 \equiv 6$
$\alpha_4 = 11$
$\alpha_5 = 17$
$\alpha_6 = 9$
$\alpha_7 = 7$
$\alpha_8 = 16$
$\alpha_9 = 4$
$\alpha_{10} = 1$

Form of GF (p^n)
(p is a prime number)

If $x \in GF(p^n)$, $x^{p^n-1} - 1 = 0 \pmod p$

① GF (2^2)

$$x^{2^2-1} - 1 = 0$$

$$x^3 - 1 = 0$$

$$(x-1)(x^2+x+1) = 0$$

$x - 1 \neq 0$ since x is the primitive elements.

$\therefore x^2+x+1 = 0$

It is called the minimal polynomial in x.

② GF (2^3)

$$x^{2^3-1} - 1 = 0$$

$$\Rightarrow x^7 - 1 = 0$$

$$\Rightarrow (x-1)(x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$\begin{array}{l} x-1 \,)\, x^7 - 1 \,(\, x^6+x^5+x^4+x^3+x^2+x+1 \\ \quad x^7 - x^6 \\ \quad (-)\;(+) \\ \hline \quad x^6 - 1 \\ \quad x^6 - x^5 \\ \hline \quad x^2 - 1 \\ \quad x^2 - x \end{array}$$

$x - 1 \neq 0$ since x is the primitive element.

$$x^6+x^5+x^4+x^3+x^2+x+1 = 0$$

$(x^2)^3 + 3(x^2)^2$.

$$x^6+x^5+x^4+x^3+x^2+x+1 = 0.$$

$$\Rightarrow x^6+x^5+x^4+3x^3+x^2+x+1 = 0$$

x^3+x+1

$x^3, x^2 + \cancel{x}\, 1$

$$\Rightarrow x^6+x^5+x^4+x^3+x^3+x^3+x^2+x+1$$

$$\Rightarrow x^6 + x^5 + x^4 + x^3 + x^3 + x^3 + x^2 + x + 1 = 0.$$

$$\Rightarrow x^4(x^3 + x + 1) + x^6 + x^3 + x$$

$$\Rightarrow x^3+x+1 + x^6 + x^4 + x^5 + x^3 + x^3 + x^2 = 0.$$

$$\Rightarrow x^3+x+1 + x^3(x^3+x+1) + x^2(x^2+\ldots$$

$$\Rightarrow x^3+x+1 + x^6+x^4+x^3 + x^5+x^3+x^2 = 0$$

$$\Rightarrow x^3+x+1 + x^3(x^3+x+1) + x^2(x^3+x+1) = 0$$

$$\Rightarrow (x^3+x+1) + x^3(x^3+x+1)$$

$$\Rightarrow (1+x^2+x^3)(x^3+x+1) = 0$$

3. GF(3^2)

$$x^{p^n-1}-1=0 \pmod 3$$

$\Rightarrow x^8-1=0$

$\Rightarrow (x^4-1)(x^4+1)=0$

$x^4+1=0$

$\Rightarrow x^4+4x^2+4-4x^2=0$

$\Rightarrow (x^2+2)^2-4x^2=0$

$\Rightarrow (x^2+2x+2)(x^2-2x+2)=0$

a^2+b^2
$=(a-b)^2+$

x^4+2x^2+1
$-2x^2$

x^2+2x+2
x^2-2x+2

4. GF(3^3)

$x^{p^n-1}-1=0 \pmod{3}$

$\Rightarrow x^{3^3-1}-1=0$

$\Rightarrow x^{26}-1=0$

$\Rightarrow (x^{13})^2-1=0$

$\Rightarrow (x^{13}-1)(x^{13}+1)=0$

x^3+2x+1

$\cancel{x-1})\ \cancel{x^{13}}+1(\cancel{x^{12}}$

$(\cancel{x^{13}}+\cancel{1})=(\cancel{x-1})(\cancel{x^{12}}+x^{11}+\cancel{x^{10}}+\cancel{x^9}$

$(\cancel{x^{13}}+\cancel{1})$

$=(\cancel{x^{13}}\cancel{+(13)^2})(x^3+2x+1)$

$=\cancel{x}$

$x^{13}+1=0$

$\Rightarrow (x^{13}+2x^{11}+x^{10})+(x^{11}+2x^9+x^8)\ \cancel{+}$
$-(x^{10}+2x^8+x^7)+(x^9+2x^7+x^6)$
$+(x^8+2x^6+x^5)-(x^7+2x^5+x^4)$
$+(x^5+2x^3+x^2)+(x^4+2x^2+x)+$
$(x^3+2x+1)=0$

7
11
1
-2
-5
-8
-11

$x^3+2x+1)\ x^{13}+1(\ x^{10}-2x^8+4x^{\dots}$
$x^{13}\quad +2x^{11}+x^{10}$
$(-)\quad (-)\quad (-)$

$-2x^{11}-x^{10}+1$
$-2x^{11}\quad -4x^9-\dots$
$(+)\quad (+)\ (+)$

$4x^9+2x^8-x^{10}+1 \quad +8x^{\dots}$
$4x^9-$

Elements of GF (p^n)

$\alpha_0 = 0$
$\alpha_1 = 1$
$\alpha_2 = x$
$\alpha_3 = x^2$
$\vdots$
$\alpha_{p^n-1} = x^{p^n-2}$

GF (2^2)

Minimal polynomial

x^2+x+1

Primitive element x satisfies

$x^2+x+1=0 \pmod 2$

$x^2 = -x-1$
$= -1(x+1)$
$\equiv (x+1) \pmod 2$
$-1 \equiv -1+2 \equiv 1 \pmod 2$

$\therefore \alpha_0 = 0$
$\alpha_1 = 1$
$\alpha_2 = x$
$\alpha_3 = x^2 \equiv (x+1)$

GF(2^3)

Minimal polynomial

$x^3+x+1=0$

$\Rightarrow x^3 = -x-1$
$\equiv (x+1) \pmod 2$

$x^3+x = -1 \equiv 1 \pmod 2$

$\alpha_0 = 0$
$\alpha_1 = 1$
$\alpha_2 = x$
$\alpha_3 = x^2$ ~~$\equiv (x+1)$~~
$\alpha_4 = x^3 = (x+1)$
$\alpha_5 = x^4 \equiv x^2+x$
$\alpha_6 = x^5 = x^3+x^2 \equiv x^2+x+1$
$\alpha_7 = x^6 = x^3+x^2+x \equiv x^2+1$

$1+x^2+x^3 = 0$

$\Rightarrow x^3 = -1-x^2$
$= +1(x^2+1)$

$\alpha_0 = 0$
$\alpha_1 = 1$
$\alpha_2 = x$
$\alpha_3 = x^2$
$\alpha_4 = x^3 \equiv x^2+1$
$\alpha_5 = x^4 = x^3+x$
$= x^2+x+1$
$\alpha_6 = x^5 = x^3+x^2+x$
$= x+1$
$\alpha_7 = x^6 = x^2+x$

$-x-2$
$= -1(x+2)$
$= 2(x+2)$

GF(3^2)

Minimal polynomial

$x^2+2x+2=0$

$\Rightarrow x^2 = -2-2x$
$= -2(1+x)$

$2(x^2+$

$x^2+x \equiv 2(2+x)$

$=$

$\alpha_0 = 0$
$\alpha_1 = 1$
$\alpha_2 = x$
$\alpha_3 = x^2 = x+1$
$\alpha_4 = x^3 = x^2+x$
$= 1+2x$

$\alpha_5 = 4x+2x^2 =$
$= 2(x^2+2x)$
$\equiv 2$
$\alpha_6 = 2x$
$\alpha_7 = 2x^2 = 2x+2$
$\alpha_8 = 2x^2+2x$
$= 2x+2+2x = x+2$

Minimal polynomial

$x^2-2x+2=0$

$GF(5^2)$

Minimal polynomial

$x^2+2x+3=0$

$\Rightarrow x^2=-2x-3$

$=-1(2x+3)$

$=4(2x+3)$

$8x^2+2$

$x^2+x=-3-x$

$=-1(x+3)$

$=$

$\alpha_{22}=x^2+4x=(3x+2)+4x$
$=2x+2$

$\alpha_{23}=2x^2+2x=2(3x+2)+2x$
$=3x+4$

$\alpha_{24}=3x^2+4x=3(3x+2)+4x$
$=3x+1$

$\alpha_0=0$

$\alpha_1=1$

$\alpha_2=x$

$\alpha_3=x^2=4(2x+3)=8x+12=3x+2$

$\alpha_4=\cancel{8x^2}+12x=3x^2+2x$
$=3[8x+12]+2x$
$=x+1$

$\alpha_5=x^2+x=4x+2$

$\alpha_6=4x^2+3x=4[8x+12]+3x$
$=4x+3$

$\alpha_7=3x^2+3x=3[8x+12]+3x$
$=2x+3$

$\alpha_8=\cancel{2x^2+2x}\ 3x$

$\alpha_9=3x^2=9x+6=4x+1$

$\alpha_{10}=4x^2+x=4[3x+2]+x=3x+3$

$\alpha_{11}=3x^2+3x=3[3x+2]+3x=2x+1$

$\alpha_{12}=2x^2+x=2[3x+2]+x=2x+4$

$\alpha_{13}=2x^2+4x=2[3x+2]+4x=4$

$\alpha_{14}=4x$

$\alpha_{15}=4x^2=4[3x+2]=2x+3$

$\alpha_{16}=2x^2+3x=2[3x+2]+3x=4x+4$

$\alpha_{17}=4x^2+4x=4[3x+2]+4x=x+3$

$\alpha_{18}=x^2+3x=(3x+2)+3x=x+2$

$\alpha_{19}=x^2+2x=(3x+2)+2x=2$

$\alpha_{20}=2x$

$\alpha_{21}=2x^2=2(3x+2)=x+4$

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Mutually Orthogonal Latin Squares :

Two latin squares of the same order are said to be orthogonal if when one is super imposed on the other ~~case~~ each symbol of one false on every symbol of the other exactly once. A set of latin squares is said to be mutually orthogonal if every pair of these squares is orthogonal.

1

A	B	C
B	C	A
C	A	B

2

A	B	C
C	A	B
B	C	A

3

B	C	A
C	A	B
A	B	C

4

B	C	A
A	B	C
C	A	B

5

C	A	B
A	B	C
B	C	A

6

C	A	B
B	C	A
A	B	C

1 ⊃ 2

AA BB CC
BC CA AB
CB AC BA

1 & 2 are mutually orthogonal

1 ⊃ 3

AB BC CA
BC CA AB
CB AC BA

1 & 3 are not mutually orthogonal.

1 & 4 are orthogonal each other

1 & 6 " " "

But 2 & 4 are not orthogonal each other

2 & 6 " " " "

So, 1, 2, 4, 6 are not mutually orthogonal.

1 & 2 only mutually orthogonal.

Substandard form:

A latin square is said to be in a substandard form if the symbols occur in ~~each~~ its 1st row in the natural order. If the symbols used in the 1st row they are the alphabets, then in the alphabetic order to make the latin occur in the square substandard. It is always possible to write orthogonal latin squares in the substandard form by having a suitable permutation over the symbols.

For example consider the following latin square

D A C B
A C B D
B D A C
C B D A

It can be written in the substandard form by permuting D to A, A to B, C to C and B to D

A B C D
B C D A
D A B C
C D A B

Result:
The number of orthogonal latin squares of order b can be at most $b-1$.

Construction of mutually orthogonal Latin square:
Let $b = s_1 \cdot s_2 \dots s_p$ where each factor s_i is either a prime or prime power. We shall use the s_i element of $GF(s_i)$; $i = 1(1)p$ to form combination of elements of p differents fields as follows.

Let us combine the elements from the p different fields taking one from each field in all possible ways. There are evidently b such combination. If b is a prime or prime power, then $p=1$ and each such combination is just an element of its field. We shall use such combination of the p field elements as the symbols for writing the latin square.

Let the b combinations be written in a row and again in a column so as to obtain the summation table of all possible sums, two by two of the row column combination. These column will be the principal column and the row will be principle row. By addition and multiplication of two combinations means addition or multiplication of each pair of corresponding elements (occuring in the same position) in the two combinations in the respective fields.

It can be easily seen that the summation table gives a latin square. Next, each combination in the principle column is multiplied by a combination say $(a_1, a_2, \ldots, a_p)$ where $a_i \neq 0 \; \forall \; i=1(1)p$ The resultant column is the 2nd principal column. Again another summation table is formed by using the principal column and the 1st principal row. This table gives a 2nd latin square which is orthogonal to the previous one.

Again a 3rd principal column is obtained by multiplying the different elements in the 1st principal column by another multiplier, say $(b_1, b_2, \ldots, b_p)$ where $b_i \neq a_i$, $b_i \neq 0$ and $b_i \neq 1$ $\forall \; i = 1(1)p$ i.e. the multipliers are so chosen that no element in any field is repeated in the different multiplier. A 3rd latin square is obtained by ~~adding~~ using the 3rd principal column and 1st principal row. This square is orthogonal to the previous two. This process is contineed till suitable multipliers are available.

At each time we have to introduce a new element in each field in the multiplier combinations. We cannot get more than $(s-1)$ multipliers where s is the minimum factor of n. Each of these multipliers contains a different non zero element of the field of s and so not more than $(s-1)$ can be taken without repeating an element in this field. If n is a prime or prime power, each multiplier combination consists of only one element. We can therefore get $(n-2)$ multipliers which n the different non zero elements in the field other than unity.

MOLS(4)

Elements of $GF(2^2)$

$\alpha^3 - 1 = 0$

$0, 1, \alpha, \alpha^2 = \alpha + 1$

1st principal column	Principal Row			
	0	1	α	α^2
0	0	1	α	α^2
1	1	0	α^2	α
α	α	α^2	0	1
α^2	α^2	α	1	0

2nd principal column obtained by multiplying by α	Principal Row			
	0	1	α	α^2
0	0	1	α	α^2
α	α	α^2	0	1
α^2	α^2	α	1	0
1	1	0	α^2	α

3rd principal column obtained by multiplying by α^2	principal Row			
	0	1	α	α^2
0	0	1	α	α^2
α^2	α^2	α	1	0
1	1	0	α^2	α
α	α	α^2	0	1

MOLS(12)

Factors of 12 are 3 & 4

Elements of GF(3) are 0, 1, 2

Elements of GF(4) are 0, 1, α, α^2

12 combinations will be used as symbols

Principal row

1st principal column	00	01	0α	$0\alpha^2$	10	11	1α	$1\alpha^2$	20	21	2α	$2\alpha^2$
00	00	01	0α	$0\alpha^2$	10	11	1α	$1\alpha^2$	20	21	2α	$2\alpha^2$
01	01	00	$0\alpha^2$	0α	11	10	$1\alpha^2$	1α	21	20	$2\alpha^2$	2α
0α	0α	$0\alpha^2$	00	01	1α	$1\alpha^2$	10	11	2α	$2\alpha^2$	20	21
$0\alpha^2$	$0\alpha^2$	0α	01	00	$1\alpha^2$	1α	11	10	$2\alpha^2$	2α	21	20
10	10	11	1α	$1\alpha^2$	20	21	2α	$2\alpha^2$	00	01	0α	$0\alpha^2$
11	11	10	$1\alpha^2$	1α	21	20	$2\alpha^2$	2α	01	00	$0\alpha^2$	0α
1α	1α	$1\alpha^2$	10	11	2α	$2\alpha^2$	20	21	0α	$0\alpha^2$	00	01
$1\alpha^2$	$1\alpha^2$	1α	11	10	$2\alpha^2$	2α	21	20	$0\alpha^2$	0α	01	00
20	20	21	2α	$2\alpha^2$	00	01	0α	$0\alpha^2$	10	11	1α	$1\alpha^2$
21	21	20	$2\alpha^2$	2α	01	00	$0\alpha^2$	0α	11	10	$1\alpha^2$	1α
2α	2α	$2\alpha^2$	20	21	0α	$0\alpha^2$	00	01	1α	$1\alpha^2$	10	11
$2\alpha^2$	$2\alpha^2$	2α	21	20	$0\alpha^2$	0α	01	00	$1\alpha^2$	1α	11	10

Principal row

2nd principal column Obtained by multiplying 1α	00	01	0α	$0\alpha^2$	10	11	1α	$1\alpha^2$	20	21	2α	$2\alpha^2$
00	00	01	0α	$0\alpha^2$	10	11	1α	$1\alpha^2$	20	21	2α	$2\alpha^2$
0α	0α	$0\alpha^2$	00	01	1α	$1\alpha^2$	10	11	2α	$2\alpha^2$	20	21
$0\alpha^2$	$0\alpha^2$	0α	01	00	$1\alpha^2$	1α	11	10	$2\alpha^2$	2α	21	20
01	01	00	$0\alpha^2$	0α	11	10	$1\alpha^2$	1α	21	20	$2\alpha^2$	2α
10	10	11	1α	$1\alpha^2$	20	21	2α	$2\alpha^2$	00	01	0α	$0\alpha^2$
1α	1α	$1\alpha^2$	10	11	2α	$2\alpha^2$	20	21	0α	$0\alpha^2$	00	01
$1\alpha^2$	$1\alpha^2$	1α	11	10	$2\alpha^2$	2α	21	20	$0\alpha^2$	0α	01	00
11	11	10	$1\alpha^2$	1α	21	20	$2\alpha^2$	2α	01	00	$0\alpha^2$	0α
20	20	21	2α	$2\alpha^2$	00	01	0α	$0\alpha^2$	10	11	1α	$1\alpha^2$
2α	2α	$2\alpha^2$	20	21	0α	$0\alpha^2$	00	01	1α	$1\alpha^2$	10	11
$2\alpha^2$	$2\alpha^2$	2α	21	20	$0\alpha^2$	0α	01	00	$1\alpha^2$	1α	11	10
21	21	20	$2\alpha^2$	2α	01	00	$0\alpha^2$	0α	11	10	$1\alpha^2$	1α

3^n Factorial Experiment

Start with 3^2 experiment
Suppose an experiment involves two factors A and B, each at 3 level, viz.. 0 (low), 1 (intermediate), and 2 (high). There are total $3^2 = 9$ treatment combinations.

$$\begin{matrix} a_0b_0 & a_0b_1 & a_0b_2 \\ a_1b_0 & a_1b_1 & a_1b_2 \\ a_2b_0 & a_2b_1 & a_2b_2 \end{matrix} \quad \text{or} \quad \begin{matrix} 00 & 01 & 02 \\ 10 & 11 & 12 \\ 20 & 21 & 22 \end{matrix}$$

$$\text{or} \quad \begin{matrix} (1) & (b) & (b^2) \\ (a) & (ab) & (ab^2) \\ (a^2) & (a^2b) & (a^2b^2) \end{matrix}$$

There will be 8 degrees of freedom for the treatments. They are partitioned as follows
(i) 2 & 2 independent contrasts each having 2 df representing the main effect of A & B.

ii) 4 independent contrasts having 4 degrees of freedom, representing the interaction $A \times B$.

Effect	d.f
A	2
B	2
$A \times B$	4

$$A \longrightarrow \underset{2df}{A} \begin{cases} A^L \text{ (Linear)} & 1\,df \\ A^Q \text{ (quadratic)} & 1\,df \end{cases}$$

2 degrees of freedom belonging to A is partioned into 2 mutually exclusive contrasts each having 1 degrees of freedom and the contrasts representing A^Q and A^L are orthogonal.

$$SSA = SSA^L + SSA^Q$$
$$2\,df = 1\,df + 1\,df$$

4 degrees of freedom belonging to $A \times B$ is partitioned into 4 mutually orthogonal contrasts, each having 1 d.f as follows

$$A \times B \begin{cases} A^L B^L & 1\ df \\ A^L B^Q & 1\ df \\ A^Q B^L & 1\ df \\ A^Q B^Q & 1\ df \end{cases}$$

$$SS\ (\text{Treatment}) = SSA + SSB + SS(A \times B)$$
$$= (SSA^L + SSA^Q) + (SSB^L + SSB^Q) + (SSA^L B^L + SSA^L B^Q + SSA^Q B^L + SSA^Q B^Q)$$

$$\text{d.f} \quad 8 = (1+1) + (1+1) + (1+1+1+1)$$

Define
$[X]_i$ = Total yield from all replicates receiving the factor X at level i $(i = 0, 1, 2)$

If $X = A$

$$[A]_0 = [00] + [01] + [02] \quad i = 0$$
$$[A]_1 = [10] + [11] + [12] \quad i = 1$$
$$[A]_2 = [20] + [21] + [22] \quad i = 2$$

mod 3

$[A]$ = Factorial effect total due to $A = \sum_{i=0}^{2} [A]_i$

Effect		Equation	
AB	$A^L B^L$	$x_1 + x_2 = i$	$(0, 1, 2 \bmod 3)$
A^2B	$A^Q B^L$	$2x_1 + x_2 = i$	$(0, 1, 2 \bmod 3)$
AB^2	$A^L B^Q$	$x_1 + 2x_2 = i$	$(0, 1, 2 \bmod 3)$
A^2B^2	$A^Q B^Q$	$2x_1 + 2x_2 = i$	$(0, 1, 2 \bmod 3)$

Yate's Table

Treatment Combinations	Treatment Totals	
(1)	[1]	[1] + [a] + [a²]
(a)	[a]	[b] + [ab] + [a²b]
(a²)	[a²]	[b²] + [ab²] + [a²b²]
(b)	[b]	[a²] − [1]
(ab)	[ab]	[a²b] − [b]
(a²b)	[a²b]	[a²b²] − [b²]
(b²)	[b²]	[a²] − 2[a] + [1]
(ab²)	[ab²]	[a²b] − 2[ab] + [b]
(a²b²)	[a²b²]	[a²b²] − 2[ab²] + [b²]

[1] + [a] + [a²] + [b] + [ab] + [a²b] + [b²] + [ab²] + [a²b²]

[a²] − [1] + [a²b] − [b] + [a²b²] − [b²]

[a²] − 2[a] + 1 + [a²b] − 2[ab] + [b] + [a²b²] − 2[ab²] + [b²]

[b²] + [ab²] + [a²b²] − [1] − [a] − [a²]

[a²b²] − [b²] − [a²] + [1]

[a²b²] − 2[ab²] + [b²] − [a²] + 2[a] − [1]

[b²] + [ab²] + [a²b²] − 2[b] − 2[ab] − 2[a²b] + [1] + [a] + [a²]

[a²b²] − [b²] − 2[a²b] + 2[b] + [a²] − [1]

[a²b²] − 2[ab²] + [b²] − 2[a²b] + 2[ab] − 2[b] + [a²] − 2[a] + [1]

$A^L = \frac{1}{6}(a^2-1)(b^2+b+1)$ $\quad p=1\ ;\ q=2-1=1$

$A^Q = \frac{1}{18}(a^2-2a+1)(b^2+b+1)$ $\quad p=1\ ;\ q=2-0=2$

$B^L = \frac{1}{6}(a^2+a+1)(b^2-1)$ $\quad p=1\ ;\ q=2-1=1$

$A^L B^L = \frac{1}{4}(a^2-1)(b^2-1)$ $\quad p=2$

$A^Q B^L = \frac{1}{12}(a^2-2a+1)(b^2-1)$

$B^Q = \frac{1}{6}(1+a+a^2)(b^2-2b+1)$

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$A^L B^Q = \frac{1}{12} (a^2-1)(b^2-2b+1)$

$A^Q B^Q = \frac{1}{36} (a^2-2a+1)(b^2-2b+1)$

Sum of squares due to any factorial effect x is given by

$$SS[x] = \frac{[x]^2}{2^p \, 3^q \cdot r}$$

Where r = number of replicates used in the design
p = number of factors present in the effect under consideration.
q = (number of factors in the experiment) − (number of linear factors present in the effect under consideration)

ANOVA TABLE

Source of variation	d.f	SS
Replicate	$r-1$	$\sum_{i=1}^{r} \frac{R_i^2}{9} - CF$
A^L	1	$SS(A^L) = \frac{[A^L]^2}{6r}$
A^Q	1	$SS(A^Q) = \frac{[A^Q]^2}{18r}$
B^L	1	$SS(B^L) = \frac{[B^L]^2}{6r}$
B^Q	1	$SS(B^Q) = \frac{[B^Q]^2}{18r}$
$A^L B^L$	1	$SS(A^L B^L) = \frac{[A^L B^L]^2}{4r}$
$A^L B^Q$	1	$SS(A^L B^Q) = \frac{[A^L B^Q]^2}{12r}$
$A^Q B^L$	1	$SS(A^Q B^L) = \frac{[A^Q B^L]^2}{12r}$
$A^Q B^Q$	1	$SS(A^Q B^Q) = \frac{[A^Q B^Q]^2}{36r}$

$CF = \frac{G^2}{9r}$; R_i = i th replicate total

In a 3^2 experiment, the 9 treatment combinations can be represented by 9 points in the X_1-X_2 plane. Consider the solution of the system of equation

$$\left.\begin{aligned} x_1 &= 0 \\ x_1 &= 1 \\ x_1 &= 2 \end{aligned}\right\} \bmod 3 \quad \cdots (1)$$

The main effect of A is a contrast consisting of the solution of these system of equations. Thus we can represent the main effect of A as a contrast among the solution of the system 1. Similarly the main effect of B is defined by the equation $x_2 = i \pmod 3$ $(i = 0, 1, 2)$. Now consider

$$2x_1 + x_2 = \left.\begin{matrix} 0 \\ 1 \\ 2 \end{matrix}\right\} \bmod 3 \quad \cdots (2)$$

$$\Leftrightarrow \quad 4x_1 + 2x_2 = \left.\begin{matrix} 0 \\ 2 \\ 4 \end{matrix}\right\} \bmod 3$$

$$\Leftrightarrow \quad x_1 + 2x_2 = \left.\begin{matrix} 0 \\ 2 \\ 1 \end{matrix}\right\} \bmod 3$$

The system of equations (2) represent the effect A^2B which is again equivalent to the system representing AB^2. Thus the effects A^2B & AB^2 are identical. By convention we always write it as AB^2. In a similar way AB and A^2B^2 are identical effects. In general any factorial effect in a 3^2 experiment is denoted by

$A^{l_1} B^{l_2}$ where $l_1, l_2 = 0, 1, 2 \pmod 3$.

conventionally we take $l_1 = 0, 1$. The effect is a comparison among the following sets of treatment combinations.

$$l_1 x_1 + l_2 x_2 = 0,\ = 1,\ = 2 \quad \text{mod } 3$$

ANOVA Table

Source of variation	d.f	SS	F_{obs}
Replicate	$r-1$	$\sum_{i=1}^{r} \frac{R_i^2}{9} - CF$	
A	2	$\frac{\sum_{i=0}^{2} [A]^2_{x_1=i}}{3r} - CF$	$F = \frac{MS(A)}{MSE} \sim F_{2,8(r-1)}$ Under H_{01}: Main effect due to A is insignificant
B	2	$\frac{\sum_{i=0}^{2} [B]^2_{x_2=i}}{3r} - CF$	$F = \frac{MS(B)}{MSE}$
AB	2	$\frac{\sum_{i=0}^{2} [AB]^2_{x_1+x_2=i}}{3r} - CF$	
AB^2	2	$\frac{\sum_{i=0}^{2} [AB^2]^2_{x_1+2x_2=i}}{3r} - CF$	
Error	$8(r-1)$	By substraction	

3^3 Experiment :

A, B, C $\rightarrow$ Each at 3 levels.

Treatment combinations

000	001	002	or,	(1)	(c)	(c^2)
010	011	012		(b)	(bc)	(bc^2)
020	021	022		(b^2)	(b^2c)	(b^2c^2)
100	101	102		(a)	(ac)	(ac^2)
110	111	112		(ab)	(abc)	(abc^2)
120	121	122		(ab^2)	(ab^2c)	(ab^2c^2)
200	201	202		(a^2)	(a^2c)	(a^2c^2)
210	211	212		(a^2b)	(a^2bc)	(a^2bc^2)
220	221	222		(a^2b^2)	(a^2b^2c)	$(a^2b^2c^2)$

Main Effect (6)

$A \begin{cases} A^L \\ A^Q \end{cases}$ 2 df $\quad B \begin{cases} B^L \\ B^Q \end{cases}$ 2 df $\quad C \begin{cases} C^L \\ C^Q \end{cases}$ 2 df

Interaction Effect (20)

2 factor (12 df) $\quad$ 3 factor (8 df)

$\left.\begin{matrix} AB & A^2B^2 \\ AB^2 & A^2B \end{matrix}\right\} A\times B$

$\left.\begin{matrix} BC & B^2C^2 \\ BC^2 & B^2C \end{matrix}\right\} B\times C$

$\left.\begin{matrix} AC & A^2C^2 \\ AC^2 & A^2C \end{matrix}\right\} A\times C$

3 factor:

$ABC \quad A^2B^2C^2$
$AB^2C \quad A^2BC^2$
$ABC^2 \quad A^2B^2C$
$AB^2C^2 \quad A^2BC$

Effects	Equations (mod 3 system)	df
A	$x_1 = i\ (0,1,2)$	2
B	$x_2 = i\ (0,1,2)$	2
C	$x_3 = i\ (0,1,2)$	2
$A\times B$ — AB	$x_1 + x_2 = i\ (0,1,2)$	2
$A\times B$ — AB^2	$x_1 + 2x_2 = i\ (0,1,2)$	2
$A\times C$ — AC	$x_1 + x_3 = i\ (0,1,2)$	2
$A\times C$ — AC^2	$x_1 + 2x_3 = i\ (0,1,2)$	2
$B\times C$ — BC	$x_2 + x_3 = i\ (0,1,2)$	
$B\times C$ — BC^2	$x_2 + 2x_3 = i\ (0,1,2)$	
$A\times B\times C$ — ABC	$x_1 + x_2 + x_3 = i\ (0,1,2)$	
$A\times B\times C$ — AB^2C	$x_1 + 2x_2 + x_3 = i\ (0,1,2)$	
$A\times B\times C$ — ABC^2	$x_1 + x_2 + 2x_3 = i\ (0,1,2)$	
$A\times B\times C$ — AB^2C^2	$x_1 + 2x_2 + 2x_3 = i\ (0,1,2)$	

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In general, any factorial effect in 3^3 experiment is defined as $A^{l_1} B^{l_2} C^{l_3}$ where $l_1, l_2, l_3 = i$ (0, 1, 2 (mod 3) system). Conveniently we take $l_1 = 0, 1$. The effect is a comparison among the following sets of treatment combinations $l_1 x_1 + l_2 x_2 + l_3 x_3 = i$ (0, 1, 2 mod3)

The 26 degrees of freedom belonging to the treatments is partitioned according to the following way.

i) 2, 2 and 2 sets of df belonging to the independent contrasts representing the main effects A, B and C.

ii) 4, 4 and 4 sets of df belonging to the independent contrasts representing the interaction $A \times B$, $A \times C$ and $B \times C$.

iii) Again 4 df belonging to $A \times B$ is partitioned into 2 pairs belonging to AB^2 and AB. Similarly for $B \times C$ and $A \times C$.

iv) Also 8 df belonging to $A \times B \times C$ can be partitioned into 4 pairs of 2 df belonging to ABC, AB^2C, ABC^2 and AB^2C^2.

SS due to any factorial effect is given by

$$\sum_{i=0}^{2} \frac{[x]_i^2}{9r} - \frac{G^2}{27r}$$

$[AB^2C]_0$	$[AB^2C]_1$	$[AB^2C]_2$
$x_1 + 2x_2 + x_3 = 0$ (mod3)	$x_1 + 2x_2 + x_3 = 1$	$x_1 + 2x_2 + x_3 = 2$
000	111	010
011	100	002
022	001	200
110	020	021
102	210	120
121	202	112
201	012	211
220	122	222
212	221	101

ANOVA Table

Source of variation	d.f	SS	F
Replicate	$r-1$	$\sum_{i=1}^{r} \frac{R_i^2}{27} - \frac{G^2}{27r}$	
Treatments	26	$SS(x) = \sum_{i=0}^{2} \frac{[x]_i^2}{9r} - \frac{G^2}{27r}$	$MS(x)/MSE \sim F_{2,26(r-1)}$ Under H_0: Factor X is insignificant
Error	$26(r-1)$	By substraction	
Total	$27r-1$	$\sum\sum y_{ij}^2 - \frac{G^2}{27r}$	

$A^L =$

$B^L =$

$C^L =$

$p=1$

$q=3-1$
$=2$

$A^L B^L =$

$A^L C^L =$

$B^L C^L =$

$p=2$
$q=3-2=1$

$A^Q =$

$B^Q =$

$C^Q =$

$p=1$

$q=3-0$
$=3$

$A^L B^Q =$

$A^L C^Q =$

$A^Q B^L =$

$A^Q C^L =$

$B^Q C^L =$

$B^L C^Q =$

$p=2$

$q=3-1=2$

$A^Q B^Q =$

$A^Q C^Q =$

$B^Q C^Q =$

$p=2$

$q=3-0=3$

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$(3^n, 3^k)$ Experiment

No. of treatment combination $= 3^n$

No. " blocks $= 3^k$

Block size $= 3^{n-k}$

In each replicate, we confound $3^k - 1$ df with block differences.

k independent $\rightarrow$ Rest are generalized interaction.

x	y independent
x^2	y^2
xy	x^2y^2
x^2y	xy^2

Construct the layout of a $(3^3, 3^2)$ expt. confounding ABC, BC^2.

$x_1 + x_2 + x_3 = i \quad (0, 1, 2 \text{ mod } 3)$

$x_2 + 2x_3 = i \quad (0, 1, 2 \text{ mod } 3)$

Key block is the combination of 0's in these two equations.

$x_1 + x_2 + x_3 = 0$

$x_2+2x_3=0$	$x_2+2x_3=1$	$x_2+2x_3=2$
000	210	120
111	021	201
222	102	012

$x_1 + x_2 + x_3 = 1$

$x_2+2x_3=0$	$x_2+2x_3=1$	$x_2+2x_3=2$
100	010	220
211	121	001
022	202	112

$x_1 + x_2 + x_3 = 2$

$x_2+2x_3=0$	$x_2+2x_3=1$	$x_2+2x_3=2$
200	110	020
011	221	101
122	002	212

$x_2 = -2x_3 = x_3$

$\Rightarrow x_2 + x_3 = -x_3$

$x_1 - x_3 = 0$

$x_1 = x_3$

$x_2 = 1 - 2x_3$

$= 1 + x_3$

$x_2 + x_3 = 1 + 2x_3$

$x_1 + 1 + 2x_3 = 0$

$\Rightarrow x_1 + 2x_3 = -1 = 2$

$x_1 = 2 - 2x_3$

$= 2 + x_3$

$x_2 + x_3 = 2 - x_3$

$= 2 + 2x_3$

$x_1 + 2 + 2x_3 = 0$

$\Rightarrow x_1 = -2 - 2x_3$

$= 1 + x_3$

$x_2 + 2x_3 = 2$

$x_2 = 2 - 2x_3$

$= 2 + x_3$

$x_2 + 2x_3 = 0$

$x_2 = -2x_3$

$= x_3$

$x_2 + 2x_3 = 1$

$x_2 = 1 - 2x_3$

$= 1 + x_3$

In a 3^3 experiment conducted in 2 replications in blocks of 3^2 plots. The following informations are given below.

Replicate 1 : 100, 201, 112, 210, 002, 121, 020, 011, 222

Replicate 2 : 001, 102, 012, 110, 121, 200, 222, 211, 020.

Identify the confounding effect.

$3^k - 1$
$= 3^1 - 1$
$= 2$
1dep & 1 indep.

Replicate 1:

Multiplying the combination of the given block by C (Adding 001), we obtain the key block.

101, 202, 110, 211, 000, 122, 021, 012, 220.

Take two independent treatment combination from the key block

101 , $x_1 = 1$, $x_2 = 0$, $x_3 = 1$

110 , $x_1 = 1$, $x_2 = 1$, $x_3 = 0$

Suppose the confounded effect is $A^{\alpha_1} B^{\alpha_2} C^{\alpha_3}$

$\alpha_1 + \alpha_3 = 0$ $\quad$ $\alpha_1 + \alpha_2 = 0$

$\alpha_1 = 2\alpha_3$ $\quad$ $\alpha_1 = 2\alpha_2$

$\therefore$ confounded effect is AB^2C^2
A^2BC

Replicate 2:

Multiplying the combination of the given block by C (Adding 100), we obtain the key block.

101, 202, 112, 210, 221, 000, 022, 011, 120.

Take two independent treatment combination from the key block

101 , $x_1 = 1$, $x_2 = 0$, $x_3 = 1$

112 , $x_1 = 1$, $x_2 = 1$, $x_3 = 2$

Suppose the confounded effect is $A^{\alpha_1} B^{\alpha_2} C^{\alpha_3}$

$\alpha_1 + \alpha_3 = 0$

$\Rightarrow \alpha_1 = -\alpha_3 = 2\alpha_3$

$\alpha_1 + \alpha_2 + 2\alpha_3 = 0$

$\Rightarrow \alpha_1 + \alpha_2 = -2\alpha_3 = \alpha_3$

$\Rightarrow 2\alpha_3 + \alpha_2 + 2\alpha_3 = 0$

$\Rightarrow \alpha_3 + \alpha_2 = 0$

$\Rightarrow \alpha_3 = -\alpha_2 = 2\alpha_2$

$\Rightarrow \alpha_2 = -\alpha_3 = 2\alpha_3$

Confounded effect is ABC^2 and A^2B^2C

BIBD (Balanced Incomplete Block Design)

Def^n: A balanced incomplete block design is an arrangement of v treatments in b blocks such that

① each block contains K distinct treatments $(K < v)$

② each treatment appears in r blocks.

③ every pair of treatments appears together in λ blocks.

The integers v, b, r, K, λ are called the parameters of the BIBD. They are related by the following identities ① $bK = vr$

② $r(K-1) = \lambda(v-1)$

BIBD is a binary design as in this case the incidence matrix has only two elements 1 and 0.

» $r(K-1) = \lambda(v-1)$

Proof:
$$N\,\underset{\sim}{1}_b = \underset{\sim}{r} = r\,\underset{\sim}{1}_v$$
$$\underset{\sim}{1}'_v N = \underset{\sim}{K}' = K\,\underset{\sim}{1}'_b$$

$$\left(\underset{v\times b}{N}\,\underset{b\times v}{N'}\right) = \sum_{u=1}^{b} n_{iu} n'_{uj}$$
$$= \sum_{u=1}^{b} n_{iu} n_{ju}$$
$$= \begin{cases} \lambda & \text{for } i \neq j \\ r & \text{for } i = j \end{cases}$$

$$NN' = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix}$$

$$= (r-\lambda) I_v + \lambda J_v$$

$\therefore NN' \underset{\sim}{1}_v = [r + (v-1)\lambda] \underset{\sim}{1}_v \ldots (1)$

$$NN' \underset{\sim}{1}_v = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} r + (v-1)\lambda \\ r + (v-1)\lambda \\ \vdots \\ r + (v-1)\lambda \end{bmatrix}$$

$$\begin{aligned} NN' \underset{\sim}{1}_v &= N(N' \underset{\sim}{1}_v) \\ &= N \underset{\sim}{K} \\ &= N K \underset{\sim}{1}_b \\ &= K (N \underset{\sim}{1}_b) \\ &= K \underset{\sim}{r} \\ &= K r \underset{\sim}{1}_v \quad \ldots (2) \end{aligned}$$

From (1) & (2)

$$rK = r + (v-1)\lambda$$

$$\therefore r(K-1) = \lambda(v-1) \quad \text{(Proved)}.$$

Result:

For a BIBD (v, b, r, K, λ), $b \geq v$

Proof:

Here $NN' = (r-\lambda) I_v + \lambda J_v$

$$\begin{aligned} |NN'| &= (r-\lambda)^{v-1} [r + (v-1)\lambda] \\ &= (r-\lambda)^{v-1} [r + r(K-1)] \quad [\because r(K-1) = \lambda(v-1)] \\ &= rK(r-\lambda)^{v-1} \end{aligned}$$

Since $r > \lambda$

$|NN'| > 0$

$\text{Rank}(NN') = v$

$v = \text{Rank}(NN') = \text{Rank}(N) \leq b$

A BIBD is called symmetric if $b = v$ (consequently $r = K$).

Theorem: (Result)

For a symmetric BIBD with parameters $b = v$ and $r = K$, λ, Any two blocks have exactly λ treatments in common.

Proof :

Since the design is symmetric, N is a square matrix of order v.

$$N \underset{\sim}{1}_b = r \underset{\sim}{1}_v = k \underset{\sim}{1}_v = N' \underset{\sim}{1}_v$$

$$\underset{\sim}{1}_v' N = k \underset{\sim}{1}_v' = r \underset{\sim}{1}_v' = \underset{\sim}{1}_v' N'$$

$$N'NN' = N' \left[(r-\lambda) I_v + \lambda J_v \right]$$

$$= (r-\lambda) N' + \lambda N' \underset{\sim}{1}_v \underset{\sim}{1}_v'$$

$$= (r-\lambda) N' + \lambda (r \underset{\sim}{1}_v) \underset{\sim}{1}_v'$$

$$= (r-\lambda) N' + \lambda \underset{\sim}{1}_v (r \underset{\sim}{1}_v')$$

$$= (r-\lambda) N' + \lambda \underset{\sim}{1}_v \underset{\sim}{1}_v' N'$$

Already proved,

$$\text{rank}(NN') = \text{rank}(N') = v.$$

$N' \rightarrow$ Square matrix of order $v \times v$.
$\rightarrow$ Non singular.

Post multiply both sides by $(N')^{-1}$

$$N'N = (r-\lambda) I_v + \lambda J_v$$

Every off diagonal element of $N'N$ is λ. The (i, j) th element of $N'N$ $(i \neq j)$ is equal to the inner product of the ith and jth column of N. Since the entries of N are 0 or 1, it follows that the inner product of any two different columns of N is nothing but the number of treatment common between these two blocks which the two columns represented. Since we see that every off diagonal element of $N'N$ is λ, it follows that any two blocks of symmetric BIBD have λ treatments in column.

Result :

For a BIBD (v, b, r, k, λ) if b is divisible by r then $b \geq v + r - 1$.

Proof >> Let $b = m.r$ ($m > 1$ is a +ve integer)

$$r(k-1) = \lambda(v-1)$$

$bk = vr$

$mkr = vr$

$\therefore v = mk\ [\because r \neq 0]$

$$\therefore r = \frac{\lambda(v-1)}{(k-1)} = \frac{\lambda(mk-1)}{(k-1)}$$

$$= \frac{\lambda(mk - m + m - 1)}{k-1}$$

$$= \frac{\lambda[m(k-1) + (m-1)]}{k-1}$$

$$= \lambda m + \frac{\lambda(m-1)}{(k-1)}$$

$\therefore \dfrac{\lambda(m-1)}{(k-1)}$ is a +ve integer.

If possible let , $b < v + r - 1$

$$\Rightarrow mr < v + r - 1$$

$$\Rightarrow r(m-1) < v - 1$$

$$\Rightarrow r(m-1) < \frac{r(k-1)}{\lambda}$$

$$\Rightarrow \lambda r < \frac{r(k-1)}{(m-1)}$$

$$\Rightarrow \frac{\lambda(m-1)}{(k-1)} < 1$$

Which is a contradiction. Thus, $b \geq v + r - 1$.

Example :

$b = v = 7$

$r = k = 3$, $\lambda = 1$

1	2	4
2	3	5
3	4	6
4	5	7
5	6	1
6	7	2
7	1	3

Complementary BIBD :

Def^n »

Given a BIBD, D with parameters (v, b, r, k, λ) another BIBD, called the complementary design of D exists. We denote the complementary design of D by $\bar{D}$ which is obtained from D by taking in the j th block of $\bar{D}$ all those treatments which don't occur in the j th block of D ($j = 1(1)b$) [Obviously if $v_1, b_1, r_1, k_1, \lambda_1$ are the parameters of $\bar{D}$, we have $v_1 = v$, $b_1 = b$. The number of treatments in each block is $k_1 = v - k$ and the treatments within a block are distinct. In D, a given treatment θ appears in r blocks and doesnot appear in $(b-r)$ blocks. In $\bar{D}$, these $(b-r)$ blocks each contained θ and thus $r_1 = b - r$.

Consider now a pair of treatments (α, β). This pair of treatments occurs together in λ blocks in D. Further there are $(r-\lambda)$ blocks which contain β but not α and another $(r-\lambda)$ blocks which contain α but not β. Thus the number of blocks which contain neither α nor β is $b - 2(r-\lambda) - \lambda = b - 2r + \lambda$. These number of blocks in $\bar{D}$ contain both α and β. Hence $\lambda_1 = b - 2r + \lambda$.

Residual BIBD :

Suppose D is a symmetric BIBD with parameters (v, k, λ). Choose any block of D and delete from D, the chosen block and ~~or~~ all treatments

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contained in the chosen block. The design D_3 containing the remaining blocks is called the residual design of D.

Example:

$\boxed{1\ 2\ 4}$		
2 3 5		3 5
3 4 6	$\rightarrow$	3 6
4 5 7		5 7
5 6 1		5 6
6 7 2		6 7
7 1 3		7 3

Since one block of D is deleted, D_2 has $b_2 = v-1$ blocks. Also since K treatments are deleted $v_2 = v-K$ treatments are there in D_2. Again the number of treatments common between the deleted block and the remaining blocks of D is λ. Thus λ treatments are deleted from each of the $v-1$ blocks of D to obtain D_2. The block sizes of the design D_2 is therefore $K_2 = K-\lambda$. Finally the replication of the treatments and the pairs of treatments not appearing in the deleted block remains unchanged. Hence in D_2 each treatment occur in $r_2 = K$ blocks and every pair of treatment occurs together in $\lambda_2 = \lambda$ blocks in D_2.

Derived BIBD:

Starting from a symmetric BIBD D, we can delete a block and written only those treatment in other blocks of D which appear in the deleted block. The resultant design D_3 is a BIBD which is called derived design of D.

Example:

$\boxed{1 \; 2 \; 4}$		$b_3 = v-1$
2 3 5	2	$v_3 = k$
3 4 6	4	$r_3 = k-1$
4 5 7	4	$k_3 = \lambda$
5 6 1	1	$\lambda_3 = \lambda - 1$
6 7 2	2	
7 1 3	1	

Clearly the ~~m~~ number of treatments in D_3 is ~~$v_3 = k$~~ $v_3 = k$ and the number of blocks is $b_3 = v-1$. Also since any two blocks of D have λ treatments in common the block size in D_3 is $k_3 = \lambda$. Since each of the retained treatment appears once in the deleted block the replication of each treatment is $r_3 = k-1$. Similarly each pair of treatments appear together in $\lambda_3 = \lambda - 1$ blocks of D_3.

Resolvable Block Design:

The block design is said to be resolvable if its blocks can be grouped into ~~are~~ r set of blocks, each set containing b/r blocks such that every treatment appears in each set precisely once.

Affine Resolvable Design:

A resolvable block design is said to be affine resolvable if any two blocks of two different set have a constant number of treatments in common.

First Fundamental Theorem of the method of differences:

Let M be modul containing the n elements $x^{(0)}$, $x^{(1)}, \ldots, x^{(n-1)}$. To any element $x^{(i)}$, let there correspond m varieties $x_1^{(i)}, \ldots, x_m^{(i)}$. Let it be possible to find a set of t blocks $B_1, B_2, \ldots, B_t$ satisfying the following conditions.

i) Every block contains exactly K-varieties (treatment)

ii) Among the Kt varieties occuring in the t-blocks, exactly r-varieties should belong to each of the m classes ($Kt = mr$)

iii) The differences arising from the t-blocks are symmetrically repeated, each occuring λ-times.

If v be any element of M, then from each block B_l ($l = 1(1)t$) we can form another block B_{l}, θ by taking the variety $x_i^{(v)}$ of class i in B_l, θ corresponding to every variety $x_i^{(n)}$ of the i th class in B_l. Where

$$x_i^{(v)} = x_i^{(n)} + \theta \quad ; \theta = x^{(0)}, x^{(1)}, \ldots, x^{(n-1)}$$

Then, the nt blocks B_l, θ provides us a BIBD with parameters $v = mn$, $b = nt$, r, K, λ.

Second Fundamental Theorem of the Method of Differences:

Let M be a modul containing n elements $x^{(0)}, x^{(1)}, \ldots, x^{(n-1)}$. To any element $x^{(i)}$, let there corresponds m-varieties (treatments) $x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)}$ ($i = 0(1)n-1$) & there mn varieties, let there be adjoined a new variety α. Let it be possible find a set $(t+s)$ blocks $B_1, B_2, \ldots, B_t, B_1', B_2', \ldots, B_s'$ satisfying the following conditions.

i) Each of the blocks $B_1, B_2, \ldots, B_t$ contains exactly K of the varieties $x_i^{(n)}$, while each of the blocks $B_1', B_2', \ldots, B_s'$ contains the adjoined variety α and exactly $(K-1)$ of the varieties $x_i^{(n)}$ (the varieties contained in the same block being different from one another).

ii) Among Kt varieties $x_i^{(n)}$ occurring in the blocks $B_1, B_2, \ldots, B_t$ exactly $ns - \lambda$ varieties should belong to each of the m-classes; while among $s(K-1)$ varieties $x_i^{(n)}$ occurring in the blocks $B_1', B_2', \ldots, B_s'$, exactly λ varieties should belong to each of the m-classes. So, it is clearly necessary that

$$Kt = m(ns - \lambda)$$

$$\Rightarrow (K-1)S = m\lambda$$

The differences arising from $(S+t)$ blocks $B_1, B_2, \ldots, B_t, B_1'', B_2'', \ldots, B_S''$ where the blocks B_p'' are obtained from B_p' by cutting out the adjoined variety α, are symmetrically repeated, each occurring λ times. If θ be any element of M, we can then from each block B_l (or B_p'), we can form another block $B_{l,\theta}$ (or $B_{p,\theta}'$) by taking corresponding to every variety $x_i^{(n)}$ in B_l (or B_p'), a variety $x_i^{(\theta)}$ in $B_{l,\theta}$ (or $B_{p,\theta}'$), where

$x_i^{(\theta)} = x_i^{(n)} + \theta$ (and in case of $B_{p,\theta}'$ completing it by the adjunction of α). Then the $(S+t)$ blocks $B_{l,\theta}$ & $B_{p,\theta}'$ $(l = 1(1)t, p = 1(1)s$; $\theta = x^{(0)}, x^{(1)}, \ldots, x^{(n-1)})$ provide us a BIBD with parameters

$$v = mn+1$$
$$b = n(s+t)$$
$$r = ns$$
$$K, \lambda$$

BIBD Construction of $v = 15$, $b = 35$

	[1, 1]	[2, 2]	[3, 3]	[1, 2]	[1, 3]	[2, 3]	[2, 1]	[3, 1]	[3, 2]
$(0_1, 1_1, 0_2)$	4, 1	–	–	0, 1	–	–	0, 4	–	–
$(0_2, 1_2, 2_3)$	–	4, 1	–	–	–	3, 4	–	–	2, 1
$(0_3, 1_3, 2_1)$	–	–	4, 1	–	2, 1	–	–	3, 4	–
$(0_1, 2_1, 3_2)$	2, 3	–	–	2, 4	–	–	1, 3	–	–
$(0_2, 2_2, 3_2)$	–	3, 2	–	–	–	0, 2	–	–	0, 3
$(0_3, 2_3, 0_1)$	–	–	2, 3	–	0, 3	–	–	0, 2	–
$(0_1, 2_2, 1_3)$	–	–	–	3	4	1	2	1	4

Here, $n = 5 \rightarrow$ number of modul elements

$m = 3 \rightarrow$ Variety corresponding to each element

$\lambda = 1$

$k = 3$

$t = 7 \rightarrow$ no. of blocks

$v = mn = 15$

$b = nt = 35$

$kt = mr \Rightarrow r = \frac{kt}{m} = \frac{3 \times 7}{3} = 7$

From the table we see that o the differences arising out of the blocks, each difference is repeated symmetrically. It must be noted that 0 cannot occur as a pure difference since the block constants are distinct. Here $\lambda = 1$. Now to find the BIBD, construction with parameters $v = 15$, $r = 7$, $\lambda = 1$, $b = 35$, $k = 3$.

$(0_1, 1_1, 0_2)$, $(1_1, 2_1, 1_2)$, $(2_1, 3_1, 2_2)$, $(3_1, 4_1, 3_2)$, $(4_1, 0_1, 4_2)$

$(0_2, 1_2, 2_3)$, $(1_2, 2_2, 3_3)$, $(2_2, 3_2, 4_3)$, $(3_2, 4_2, 0_3)$, $(4_2, 0_2, 1_3)$

$(0_3, 1_3, 2_1)$, $(1_3, 2_3, 3_1)$, $(2_3, 3_3, 4_1)$, $(3_3, 4_3, 0_1)$, $(4_3, 0_3, 1_1)$

$(0_1, 2_2, 0_3)$, $(1_2, 3_2, 1_3)$, $(2_2, 4_2, 2_3)$, $(3_2, 0_2, 3_3)$, $(4_2, 1_2, 4_3)$

$(0_3, 2_3, 0_1)$, $(1_3, 3_3, 1_1)$, $(2_3, 4_3, 2_1)$, $(3_3, 0_3, 3_1)$, $(4_3, 1_3, 4_1)$

$(0_1, 2_2, 1_3)$, $(1_1, 3_2, 2_3)$, $(2_1, 4_2, 3_3)$, $(3_1, 0_2, 4_3)$, $(4_1, 1_2, 0_3)$

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FRACTIONAL FACTORIAL

In factorial experiment when the no. of factors & / or the level of the factors are large, the total number of treatment combinations becomes so large that it is very difficult to organize an experiment involving these treatments each in a single replication as it is beyond the resources of the investigator to experiment with all of them. Economy of space & material may be attained by observing the response only on a fraction of all possible treatment combinations.

Obviously, by considering only a fraction, there will be some loss of information i.e. available from a complete replicate. In this experiment also confounding is necessary to reduce the block size. Preferably higher order interaction alone are confounded.

In confounding we ~~have~~ make the groups & we take the groups in different blocks. In fractional replication, we do the same but only take the block containing the control, & reject other groups or blocks.

The interaction or interactions by means of which the fraction is defined is known as defining contrast or the identity group of interaction.

Consider a 2^4 factorial experiment, we need 16 experimental units to get information of all the factorial effects. If this no. is too much for the experimenter we will try to get as much information as possible from half of the 16 observations, i.e $1/2\ (2^4)$.

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Suppose, the 4 factors are A, B, C, D. Each at two levels. Let us confound $ABCD$ to get two blocks of size 8. The key block is given by -

	A	B	BCD	ABCD
1001	+	−	+	−
0101	−	+	−	−
0011	−	−	−	−
1100	+	+	+	−
1010	+	−	−	−
0110	−	+	+	−
1111	+	+	+	−
0000	−	−	−	−

$$A = (a-1)(b+1)(c+1)(d+1)$$
$$= (ab-b+a-1)(cd+d+c+1)$$

$$BCD = (a+1)(b-1)(c-1)(d-1)$$
$$= (ab+b-a-1)(cd-c-d+1)$$

This way we can get relation between the responses from these 8 treatment combinations & the main effect & interactions.

Suppose, $y_1, y_2, \ldots, y_8$ are the responses of the 8 treatment combinations. Then,

Main effect of $A = \frac{1}{8}(y_1 - y_2 - y_3 + y_4 + y_5 - y_6 + y_7 - y_8)$

$\equiv$ Interaction effect BCD

Then, the main effect A & the interaction BCD is estimated by the same contrast in the $1/2\,(2^4)$, viz. only 8 combinations are used.

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A are two factorial effects which are represented by the same comparisons. Thus, A & BCD are . This is represented by $A \equiv BD$, $AD \equiv BC$, $\Sigma = ABCD$, $B \equiv ACD$, $C \equiv ABD$, $D \equiv ABC$. The four factor interaction ABCD in the fraction cannot be estimated at all. If all the 16 treatment combinations were available ABCD may be computed. The 8 treatment combinations which carry a negative sign in the above contrast are chosen. The 8 treatment combinations chosen for the fraction are solution of the equation $x_1 + x_2 + x_3 + x_4 = 0 \pmod 2$ where x_i denotes the levels of the ith factor (1, 2, 3, 4), thus the 4 factor interaction ABCD is confounded for obtaining the fraction. Infact the fraction consider is nothing but one of the possible two blocks of size 8 each, obtained by confounding ABCD with the blocks.