Linearly dependent and independent vectors

$\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\}$

$\sum_{i=1}^{n} c_i \underline{x}_i = \underline{0} \Rightarrow c_i = 0 \ \forall i = 1(1)n$

→ Linearly independent

$\underline{0}$ → Null vector is a dependent vector

$\boxed{\underset{\sim}{x}' \underset{\sim}{y} = 0}$ → Orthogonal vector

$\boxed{\text{Orthogonality} \Rightarrow \text{Independence}}$

$$\boxed{\begin{array}{c} \underset{\sim}{y}_i' \underset{\sim}{y}_j = 0 \quad i \neq j \\ \sqrt{\sum y_i^2} = 1, \ ||y_i|| = 1 \end{array}}$$

→ Condⁿ for orthonormal

$\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n$

$\downarrow$

$\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_n$

$\underset{\sim}{y}_1 = \underset{\sim}{x}_1$

$\underset{\sim}{y}_2 = \underset{\sim}{x}_2 + b_{21} \underset{\sim}{x}_1$

$\underset{\sim}{y}_2' \underset{\sim}{y}_1 = 0$

$\Rightarrow \underset{\sim}{x}_2' \underset{\sim}{x}_1 + b_{21} \underset{\sim}{x}_1' \underset{\sim}{x}_1 = 0$

$\Rightarrow \boxed{b_{21} = -\frac{\underset{\sim}{x}_2' \underset{\sim}{x}_1}{\underset{\sim}{x}_1' \underset{\sim}{x}_1}.}$

$\Rightarrow b_{31} = -\frac{\underset{\sim}{x}_3' \underset{\sim}{x}_1}{\underset{\sim}{x}_1' \underset{\sim}{x}_1}$

$\Rightarrow x_3' x_1 + b_{31} \underset{\sim}{x}_1' \underset{\sim}{x}_1 = 0 \ \left[\because \underset{\sim}{x}_2' \underset{\sim}{x}_1 \Rightarrow \underset{\sim}{x}_2' \underset{\sim}{y}_1 = 0\right]$

$\underset{\sim}{y}_3 = \underset{\sim}{x}_3 + b_{31} \underset{\sim}{x}_1 + b_{32} \underset{\sim}{x}_2$

$\underset{\sim}{y}_3' \underset{\sim}{y}_1 = 0$

$\underset{\sim}{y}_3' \underset{\sim}{y}_2 = 0$

$\Rightarrow \underset{\sim}{x}_3' \underset{\sim}{x}_1 + b_{31} \underset{\sim}{x}_1' \underset{\sim}{x}_1 + b_{32} \underset{\sim}{x}_2' \underset{\sim}{x}_1 = 0 \text{ -- (i)}$

$\Rightarrow (\underset{\sim}{x}_3' + b_{31} \underset{\sim}{x}_1' + b_{32} \underset{\sim}{x}_2')(\underset{\sim}{x}_2 + b_{21} \underset{\sim}{x}_1) = 0$

$\Rightarrow \underset{\sim}{x}_3' \underset{\sim}{x}_2 + \cancel{b_{21} \underset{\sim}{x}_3' \underset{\sim}{x}_1}^{0} + \cancel{b_{31} \underset{\sim}{x}_1' \underset{\sim}{x}_2}^{0} + b_{31} b_{21} \underset{\sim}{x}_1' \underset{\sim}{x}_1$

$+ b_{32} \underset{\sim}{x}_2' \underset{\sim}{x}_2 + \cancel{b_{32} b_{21} \underset{\sim}{x}_2' \underset{\sim}{x}_1}^{0} = 0 \text{ -- (ii)}$

$\left[\because \underset{\sim}{y}_1 = \underset{\sim}{x}_1 \ \& \ \underset{\sim}{y}_1 \underset{\sim}{x}_2 = 0\right]$

$\Rightarrow b_{32} = -\frac{\underset{\sim}{x}_3' \underset{\sim}{x}_2}{\underset{\sim}{x}_2' \underset{\sim}{x}_2}$

In general, $b_{ij} = -\frac{\underset{\sim}{x}_i' \underset{\sim}{x}_j}{\underset{\sim}{x}_j' \underset{\sim}{x}_j}$

$$\boxed{\begin{array}{l} z_1, z_2, \ldots, z_n \\ \underset{\sim}{z}_i = \dfrac{\underset{\sim}{y}_i}{\|\underset{\sim}{y}_i\|} \end{array}}$$

$$\boxed{\begin{array}{l} A^{m\times n} \\ r(A) \le \min(m,n) \end{array}}$$

$r(AB) \le r(A)$ or $r(B)$

$= r(A)$ if B is non singular

$= r(B)$ if A " "

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D - CA^{-1}B| = |D|\,|A - BD^{-1}C|$$

$$\boxed{|A| = \sum_{\underline{j} \in N(\underline{j})} (-1)^{N(\underline{j})} \prod_{i=1}^{n} a_{ij_i}}$$

$N(\underline{j})$: No. of inversion needed to get natural ordering

$$\boxed{\frac{dF}{d\underset{\sim}{x}} = \begin{pmatrix} \frac{dF}{dx_1} \\ \vdots \\ \frac{dF}{dx_n} \end{pmatrix}}$$

$$\boxed{\frac{d}{d\underset{\sim}{x}}(\underset{\sim}{a}'\underset{\sim}{x}) = \frac{d}{d\underset{\sim}{x}}(\underset{\sim}{x}'\underset{\sim}{a}) = \underset{\sim}{a}}$$

$$\boxed{\frac{d}{d\underset{\sim}{x}}\left(\underset{\sim}{x}' A \underset{\sim}{x}\right) = 2A\underset{\sim}{x}}$$

Symmetric Matrix

$$\boxed{\begin{array}{l} R(A) = \text{Trace}(A) \\ \quad\quad \underline{\text{if}} \\ \text{Idempotent Matrix} \end{array}}$$

$$\boxed{A = \lambda_1 \underset{\sim}{l_1} \underset{\sim}{l_1}' + \lambda_2 \underset{\sim}{l_2} \underset{\sim}{l_2}' + \dots + \lambda_n \underset{\sim}{l_n} \underset{\sim}{l_n}'}$$

λ_i = Eigen values

l_i = Eigen vectors

$$\boxed{\begin{array}{l} V(x) = \Sigma \\ V(A\underset{\sim}{x}) = A \Sigma A' \end{array}}$$

Linear Models

$$\underset{\sim}{y}^{n\times 1} = X^{n\times p} \underset{\sim}{\beta}^{p\times 1} + \underset{\sim}{\varepsilon}^{n\times 1}$$

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\text{known}} = \underbrace{\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & & & \\ x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix}}_{\text{known}} \underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix}}_{\text{Unknown and to be estimated}} + \underbrace{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}}_{\text{random}}$$

$$\boxed{\begin{array}{l} E(\underset{\sim}{\varepsilon}) = \underset{\sim}{0} \\ D(\underset{\sim}{\varepsilon}) = \sigma^2 I \end{array}}$$

$$\boxed{\begin{array}{l} \text{Col}^{n} \text{ of } X = \underset{\sim}{x}_1, \dots, \underset{\sim}{x}_p \\ \text{Rows of } X = \underset{\sim}{x}'_{(1)}, \underset{\sim}{x}'_{(2)}, \dots, \underset{\sim}{x}'_{(n)} \end{array}}$$

Linear comb^n of rows of X

$$\boxed{\underset{\sim}{b}' = a_1 \underset{\sim}{x}'_{(1)} + a_2 \underset{\sim}{x}'_{(2)} + \dots + a_n \underset{\sim}{x}'_{(n)} = \underset{\sim}{a}' X}$$

Linear comb^n of col^n of X

$$\boxed{\underset{\sim}{m} = l_1 \underset{\sim}{x}_1 + l_2 \underset{\sim}{x}_2 + \dots + l_p \underset{\sim}{x}_p = X \underset{\sim}{l}}$$

$$\boxed{R(X) = r \le \min(n, p)}$$

$\langle r = p \,(\le n)$: Full rank model $\rangle$

$$\underset{\sim}{y} - X\underset{\sim}{\hat{\beta}} = \underset{\sim}{e} \rightarrow \text{Residuals}$$

$$\underset{\sim}{y} - X\underset{\sim}{\beta} = \underset{\sim}{\varepsilon} \rightarrow \text{error}$$

$$\underset{\sim}{\varepsilon}'\underset{\sim}{\varepsilon} = (\underset{\sim}{y} - X\underset{\sim}{\beta})'(\underset{\sim}{y} - X\underset{\sim}{\beta}) ,$$

$$\underset{\sim}{e}'\underset{\sim}{e} = (\underset{\sim}{y} - X\underset{\sim}{\hat{\beta}})'(\underset{\sim}{y} - X\underset{\sim}{\hat{\beta}})$$

$$= \underset{\sim}{y}'\underset{\sim}{y} - 2\underset{\sim}{\hat{\beta}}'X'\underset{\sim}{y} + \underset{\sim}{\hat{\beta}}'X'X\underset{\sim}{\hat{\beta}}$$

$$\frac{d}{d\underset{\sim}{\hat{\beta}}}(\underset{\sim}{e}'\underset{\sim}{e}) = -2X'\underset{\sim}{y} + 2(X'X)\underset{\sim}{\hat{\beta}} = 0$$

$$\Rightarrow \boxed{\underbrace{X'\underset{\sim}{y}}_{\underset{\sim}{q}^{p\times 1}} = \underbrace{(X'X)}_{S^{p\times p}}\underset{\sim}{\hat{\beta}}} \rightarrow \text{Normal eq}^{n}$$

$$\boxed{\begin{array}{l} S = (X'X) \ : \text{symmetric} \\ \text{Rank}(S) = \text{Rank}(X) \end{array}} \longrightarrow \boxed{\begin{array}{l} p - \dim N(S) \\ = p - \dim N(X) \end{array}}$$

$$\boxed{r(A^{m\times n}) = n - \dim(N(A))}$$

$$\boxed{\begin{array}{l} X\underset{\sim}{\alpha} = \underset{\sim}{0} \Rightarrow \alpha \perp \text{rows of } X \\ (X'X)\underset{\sim}{\alpha} = \underset{\sim}{0} \Rightarrow \alpha \perp \text{rows of } (X'X) \end{array}}$$

$$\boxed{\begin{array}{l} X'X\underset{\sim}{\alpha} = \underset{\sim}{0} \\ \Rightarrow \underset{\sim}{\alpha}'X'X\underset{\sim}{\alpha} = \underset{\sim}{0} \\ \Rightarrow (X\underset{\sim}{\alpha})'(X\underset{\sim}{\alpha}) = 0 \Rightarrow X\underset{\sim}{\alpha}' = 0 \end{array}}$$

$$\boxed{\begin{array}{l} A\underset{\sim}{x} = \underset{\sim}{b} \\ \Rightarrow R(A : b) = R(A) \end{array}}$$

→ For consistency

$$\boxed{\begin{array}{l} \therefore X'\underset{\sim}{y} = (X'X)\hat{\underset{\sim}{\beta}} \\ R\left((X'X) : X'\underset{\sim}{y}\right) = R(X'X) \end{array}}$$

$$\boxed{R\left((X'X) : X'\underset{\sim}{y}\right) \geq R(X'X)} \quad \text{--(i)}$$

$$\boxed{\begin{array}{l} R\left(X'X : X'\underset{\sim}{y}\right) \\ = R\left[X'(X : \underset{\sim}{y})\right] \leq R(X') = R(X'X) \end{array}} \quad \text{--(ii)}$$

Combining (i) & (ii) we will get,

Proof of the fact that $\hat{\underset{\sim}{\beta}}$ minimize the SSE.

$$\boxed{R\left((X'X) : X'\underset{\sim}{y}\right) = R(X'X)}$$

$$\boxed{SSE = (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})}$$

$$(\underset{\sim}{y} - X\underset{\sim}{\beta_0})'(\underset{\sim}{y} - X\underset{\sim}{\beta_0})$$

$$= (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}} + X\hat{\underset{\sim}{\beta}} - X\underset{\sim}{\beta_0})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}} + X\hat{\underset{\sim}{\beta}} - X\underset{\sim}{\beta_0})$$

$$= (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}) + \underbrace{(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'X(\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta_0})}_{0}$$

$$+ \underbrace{(\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta_0})'X'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})}_{0}$$

$$+ (\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta_0})'X'X(\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta_0})$$

[Both terms are zero due to the normal eqns]

$$= SSE + \underset{\sim}{l}'\underset{\sim}{l}, \quad \underset{\sim}{l} = X(\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta_0})$$

$$\geq SSE \quad \left[\because \underset{\sim}{l}'\underset{\sim}{l} = \sum_i l_i^2 \geq 0\right]$$

G- inverse

Def^n 1 >> An $n \times m$ matrix ~~x~~ A^- is defined to be a generalized inverse of the $m \times n$ matrix A if for every vector $\underset{\sim}{u}$ satisfying (*) $\left[\begin{matrix} A\underset{\sim}{x} = u \\ r(A) = r(A/\underset{\sim}{u}) \ldots (*) \end{matrix}\right]$, $A^-\underset{\sim}{u}$ is a sol^n of the eq^n $\boxed{A\underset{\sim}{x} = \underset{\sim}{u}}$

One method of ~~A~~ obtaining A^- is therefore to take an algebric vector $\underset{\sim}{u}$ with elements $u_1, u_2, \ldots, u_m$ assuming (*) holds and try to solve $A\underset{\sim}{x} = \underset{\sim}{u}$, though $A\underset{\sim}{x} = \underset{\sim}{u}$ appears to be m eq^{n}s in n unknowns actually they may have fewer eq^{n}s. Suppose there are really only k eq^{n}s then use any "suitable", "consistent" additional $n-k$ eq^{n}s. Since def^n 1 needs only a sol^n of $A\underset{\sim}{x} = \underset{\sim}{u}$, it is immaterial what addi^{n}al eq^{n}s we take

$$\boxed{\begin{gathered} x_1 = a^{11}u_1 + a^{12}u_2 + \ldots + a^{1m}u_m \\ \vdots \\ x_n = a^{n1}u_1 + \ldots + a^{nm}u_m \\ \underset{\sim}{x} = \underbrace{\left(\left(a^{ij}\right)\right)}_{\to \text{ g-inverse}} \underset{\sim}{u} \end{gathered}}$$

(*) $A = \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix}$ $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \begin{aligned} 3x_1 + 5x_2 &= u_1 \\ 6x_1 + 10x_2 &= u_2 \\ 9x_1 + 15x_2 &= u_3 \end{aligned}$$

Let, $x_2 = 0$ as add^{n}al eq^n.

$\therefore x_1 = \frac{1}{3} u_1 + 0.u_2 + 0.u_3$

$x_2 = 0.u_1 + 0.u_2 + 0.u_3$

$$\underset{\sim}{x} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underset{\sim}{u}$$

Let, take, $x_2 = u_2$

$\therefore 3x_1 = u_1 - 3u_2$

$\Rightarrow x_1 = 1/3\, u_1 - 5/3\, u_2 + 0.u_3$

$x_2 = 0.u_1 + 1.u_2 + 0.u_3$

$$\underset{\sim}{x} = \begin{pmatrix} 1/3 & -5/3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \underset{\sim}{u}$$

$$A^- = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix}$$

$$\boxed{\begin{aligned} & AA^-A \\ &= \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} \\ & \cancel{\begin{pmatrix} 3 & 5 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} = A \end{aligned}}$$

$$A^- = \begin{pmatrix} 1/3 & -5/3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix}$$

$$AA^-A = \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} \begin{pmatrix} 1/3 & -5/3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} = A$$

Def^n 2⟩⟩ Any n×m matrix A^- satisfying the rela^n $\boxed{AA^-A = A}$ is defined as a generalized inverse of m×n matrix A.

⊛ Show that both the def^ns are equivalent.

⇒ Suppose def^n (2) holds, then,

$$AA^-A = A$$

$$\Rightarrow AA^-A\underset{\sim}{x} = A\underset{\sim}{x}$$

$$\Rightarrow A\underbrace{A^-\underset{\sim}{y}} = \underset{\sim}{y}$$

Showing that $A^-\underset{\sim}{y}$ is a sol^n of $A\underset{\sim}{x} = \underset{\sim}{y}$ for every vector $\underset{\sim}{y}$ for which $A\underset{\sim}{x} = \underset{\sim}{y}$ is consistent. This shows that def^n (1) holds.

Now, suppose def^n (1) holds,

Let, $\underset{\sim}{a_i}$ be a the i^{th} column $[\forall i=1(1)n]$ vector of A, we know that rank of A = no. of independent columns of A

$$= r(A) = \text{rank}[A, \underset{\sim}{a_i}]$$

$A\underset{\sim}{x} = \underset{\sim}{a_i}$ ~~is a~~ is obviously consistent and by def^n (1) $A^-\underset{\sim}{a_i}$ is a sol^n.

Hence, $AA^-\underset{\sim}{a_i} = \underset{\sim}{a_i} \ \forall i = 1(1)n$

$$\Rightarrow \boxed{AA^-A = A}$$

⊛ $\boxed{A^-A = H^{n\times n}}$

Property 1⟩⟩ $\boxed{AH = A}$

Property 2⟩⟩ $\boxed{H^2 = A^-AA^-A = A^-A = H}$

property 3 ≫ $\boxed{r(H) = r(A) = tr(H)}$

$\Rightarrow \quad r(H) \leq rank(A) \; .. (i)$

$r(A) \leq rank(H) \; .. (ii)$

Combining (i) and (ii) we will get

~~r(H)~~ $\boxed{r(H) = r(A)}$

(*) [The general solⁿ of the system of homogeneous eqⁿ $A\underset{\sim}{x} = \underset{\sim}{0}$ can be expressed as $\hat{\underset{\sim}{x}} = (I-H)\underset{\sim}{z}$ where $\underset{\sim}{z}$ is any arbitrary vector.]

$$\Rightarrow \quad A(I-H)$$
$$= A - AH = A - A = 0$$

$\Rightarrow$ Colⁿ of $(I-H)$ $(\underset{\sim}{b_1}, \underset{\sim}{b_2}, \ldots, \underset{\sim}{b_n})$ are orthogonal to the rows of A.

$$(I-H)^2 = (I-H)(I-H)$$
$$= I - H - H + H^2 = I - H \; [\because H^2 = H]$$

$$\boxed{r(I-H) = tr(I-H)} = \boxed{n - r} \quad \left[\text{Let, } r(A) = r\right]$$

Only $(n-r)$ of the colⁿ vectors $(\underset{\sim}{b_1}, \underset{\sim}{b_2}, \ldots, \underset{\sim}{b_n})$ are linearly independent without loss of generality assume $(\underset{\sim}{b_1}, \underset{\sim}{b_2}, \ldots, \underset{\sim}{b}_{n-r})$ are linearly independent.

Since A is an $m \times n$ matrix of rank r, it's rows are n vectors and therefore we can find at most $(n-r)$ linearly independent vectors orthogonal to them.

Let, $(\underset{\sim}{b_1}, \underset{\sim}{b_2}, \ldots, \underset{\sim}{b}_{n-r})$ is one such set.

If there is any other vector orthogonal to the rows of A it must be a linear combina^n of $(\underset{\sim}{b}_1, \underset{\sim}{b}_2, \dots, \underset{\sim}{b}_{n-r})$ But this is also equivalent to saying that $\underset{\sim}{x}$ will be a linear combina^n of $(\underset{\sim}{b}_1, \underset{\sim}{b}_2, \dots, \underset{\sim}{b}_n)$ because $\underset{\sim}{b}_{n-r+1}, \dots, \underset{\sim}{b}_n$ are linear combina^{n}s of $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \dots, \underset{\sim}{b}_{n-r}$. Hence $\underset{\sim}{x}$ must be of the form -

$$\boxed{\underset{\sim}{x} = z_1\underset{\sim}{b}_1 + z_2\underset{\sim}{b}_2 + \dots + z_n\underset{\sim}{b}_n}$$

$$= (\underset{\sim}{b}_1 \ \underset{\sim}{b}_2 \dots \underset{\sim}{b}_n)\underset{\sim}{z}$$

$$= \boxed{(I-H)\underset{\sim}{z}}$$

Conversely, if $\underset{\sim}{x} = (I-H)\underset{\sim}{z}$

$$\Rightarrow A\underset{\sim}{x} = A(I-H)\underset{\sim}{z}$$

$$\Rightarrow A\underset{\sim}{x} = (A-AH)\underset{\sim}{z}$$

$$\Rightarrow A\underset{\sim}{x} = (A-A)\underset{\sim}{z}$$

$$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$$

System of non-homogeneous eq^n :-

$$\boxed{A\underset{\sim}{x} = \underset{\sim}{u}}$$

$A^{-1}\underset{\sim}{u}$ is a sol^n

$$\therefore AA^{-1}\underset{\sim}{u} = \underset{\sim}{u}$$

$$\therefore A\underset{\sim}{x} - AA^{-1}\underset{\sim}{u} = \underset{\sim}{u} - \underset{\sim}{u} = 0.$$

$$\Rightarrow A(\underset{\sim}{x} - A^{-1}\underset{\sim}{u}) = 0$$

$$\Rightarrow A\,\underset{\sim}{y} = 0$$

$$\therefore \underset{\sim}{\tilde{x}} - A^{-}\underset{\sim}{u} = (I-H)\underset{\sim}{z} \rightarrow$$ We get that sol^n from the system of homogeneous eq^n.

$$\Rightarrow \boxed{\underset{\sim}{\tilde{x}} = A^{-}\underset{\sim}{u} + (I-H)\underset{\sim}{z}}$$

Solⁿ of normal eqⁿ:-

$$\boxed{\underset{\sim}{x}'\underset{\sim}{y} = (x'x)\hat{\underset{\sim}{\beta}}}$$

$$\Rightarrow \hat{\underset{\sim}{\beta}} = (x'x)^{-}\underset{\sim}{x}'\underset{\sim}{y}$$

$$= S^{-}\underset{\sim}{x}'\underset{\sim}{y} = S^{-}\underset{\sim}{q}$$

$$\therefore \boxed{\hat{\beta} = S^{-}\underset{\sim}{q} + (I-H)\underset{\sim}{z}} \quad \left[H = S^{-}S\right]$$

Results:-

i) If S^{-} is a g-inverse of $\boxed{x'x = S}$, its transpose $(S^{-})'$ is also a g-inverse.

$$SS^{-}S = S$$

$$\Rightarrow S'(S^{-})'S' = S' \qquad \boxed{\begin{array}{l} S = x'x \\ \Rightarrow S' = x'x = S \end{array}}$$

$$\Rightarrow S(S^{-})'S = S$$

ii) $\boxed{X = XH}$

$$S = x'x \quad , H = S^{-}S$$

$$SH = SS^{-}S = S$$

$$\Rightarrow S - SH = 0$$

$$\Rightarrow (I-H)'(S-SH) = 0$$

$$\Rightarrow (I-H)'S(I-H) = 0$$

$$\Rightarrow (I-H)'x'x(I-H) = 0$$

$$\Rightarrow \left(X(I-H)\right)'\left(X(I-H)\right) = 0$$

$$\Rightarrow X - XH = 0$$

$$\Rightarrow \boxed{X = XH}$$

iii) If S_a^- and S_b^- are two g-inverses of $(X'X)$ then $\boxed{XS_a^-X' = XS_b^-X'}$

$$\therefore \boxed{H_a = S_a^- S_a \,\Big|\, H_b = S_b^- S_b}$$

$$X = XH_a \qquad X = XH_b$$
$$= XS_a^- S_a \qquad = XS_b^- S_b$$
$$= \boxed{XS_a^- X'X} \qquad = \boxed{XS_b^- X'X}$$

$$\therefore \boxed{XS_a^- X'X = XS_b^- X'X}$$

$$\Rightarrow XS_a^- X'X - XS_b^- X'X = 0$$

$$\Rightarrow (XS_a^- X'X - XS_b^- X'X)(XS_a^- - XS_b^-)' = 0$$

$$\Rightarrow (XS_a^- X' - XS_b^- X')X(XS_a^- - XS_b^-)' = 0$$

$$\Rightarrow (XS_a^- X' - XS_b^- X')(XS_a^- X' - XS_b^- X')' = 0$$

$$\Rightarrow XS_a^- X' - XS_b^- X' = 0$$

$$\Rightarrow \boxed{XS_a^- X' = XS_b^- X'}$$

iv) A solⁿ of the normal eqⁿ is unique iff $R(X) = R(X'X) = p$

From the solⁿ of the non-homogeneous eqⁿ we get that,

$$\tilde{x} = A^- \underset{\sim}{y} + (I - H)\underset{\sim}{z}$$

We will get the unique solⁿ when $I - H = 0$ [$\because \underset{\sim}{z}$ is arbitrary]

$$\Rightarrow I = H \qquad \therefore [S^- \text{ will be the inverse of } S] \text{ i.e } \boxed{R(X) = R(X'X) = p}$$
$$\Rightarrow S^- S = I$$

$$\therefore \boxed{S^{p\times p} = (X'X)_{p\times p}}$$

- A necessary and sufficient cond^{n} for the expression $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$ where $\hat{\underset{\sim}{\beta}}$ is any sol^{n} of the normal eq^{n}s $\underset{\sim}{X}'\underset{\sim}{y} = (X'X)\hat{\beta}$ to have a unique value is $\underset{\sim}{\lambda}' = \underset{\sim}{\lambda}'H$ where $\hat{\underset{\sim}{\beta}} = S^{-}\underset{\sim}{q}$, $H = S^{-}S$ and S^{-} is the g-inverse of S.

$$y_1 = \beta_1 + \beta_2 + \varepsilon_1$$
$$y_2 = \beta_1 + \beta_3 + \varepsilon_2$$
$$y_3 = \beta_1 + \beta_2 + \varepsilon_3$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

$$\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$$

$$\boxed{\underset{\sim}{q} = \boxed{X'\underset{\sim}{y} = (X'X)\underset{\sim}{\beta}}} \rightarrow \hat{\underset{\sim}{\beta}} = (X'X)^{-}\underset{\sim}{q}$$

$$\rightarrow \text{Normal eq}^{n}$$

$$X'X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} q_1 &= \boxed{y_1 + y_2 + y_3 = 3\beta_1 + 2\beta_2 + \beta_3} \\ q_2 &= \boxed{y_1 + y_3 = 2\beta_1 + 2\beta_2} \\ q_3 &= \boxed{y_2 = \beta_1 + \beta_3} \end{aligned}$$

Let, $\hat{\beta}_2 = 0$

$$\hat{\beta}_1 = \frac{q_2}{2}$$

$$\hat{\beta}_3 = q_1 - \frac{3q_2}{2}$$

$$(X'X)^{-} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 1 & -3/2 & 0 \end{pmatrix}$$

$$\boxed{H = S^{-}S} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 1 & -3/2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\boxed{\underset{\sim}{\lambda}' = \underset{\sim}{\lambda}' H}$$

~~λ_1 λ_2 λ_1 λ_2 λ_3~~

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & \lambda_1 - \lambda_3 & \lambda_3 \end{pmatrix}$$

$$\therefore \boxed{\lambda_2 = \lambda_1 - \lambda_3}$$

$$\Rightarrow \boxed{\lambda_1 = \lambda_2 + \lambda_3}$$

Q.

$y_1 = \mu + \alpha_1 + \beta_1 + \varepsilon_1$

$y_2 = \mu + \alpha_1 + \beta_2 + \varepsilon_2$

$y_3 = \mu + \alpha_2 + \beta_1 + \varepsilon_3$

$y_4 = \mu + \alpha_2 + \beta_2 + \varepsilon_4$

$y_5 = \mu + \alpha_3 + \beta_1 + \varepsilon_5$

HW $y_6 = \mu + \alpha_3 + \beta_2 + \varepsilon_6$

i) When is $\lambda_0 \mu + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3 + \lambda_4 \beta_1 + \lambda_5 \beta_2$ estimable.

ii) Is $\alpha_1 + \alpha_2$ estimable?

〃 $\beta_1 - \beta_2$ 〃 ?

iii) Is $\mu + \alpha_1$ estimable?

iv) Is $6\mu + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\beta_1 + 3\beta_2$ 〃 ?

v) Is $\alpha_1 - 2\alpha_2 + \alpha_3$ estimable?

$\Rightarrow$

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$X'X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 0 & 0 & 1 & 1 \\ 2 & 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 & 3 & 0 \\ 3 & 1 & 1 & 1 & 0 & 3 \end{pmatrix}$$

$$\boxed{\begin{aligned} q_1 &= \sum_{i=1}^{6} y_i \\ q_2 &= y_1 + y_2 \\ q_3 &= y_3 + y_4 \\ q_4 &= y_5 + y_6 \\ q_5 &= y_1 + y_3 + y_5 \\ q_6 &= y_2 + y_4 + y_6 \end{aligned}} \quad \begin{aligned} &= 6\mu + 2(\alpha_1+\alpha_2+\alpha_3) + 3(\beta_1+\beta_2 \\ &= 2\mu + 2\alpha_1 + \beta_1 + \beta_2 \\ &= 2\mu + 2\alpha_2 + \beta_1 + \beta_2 \\ &= 2\mu + 2\alpha_3 + \beta_1 + \beta_2 \\ &= 3\mu + \alpha_1 + \alpha_2 + \alpha_3 + 3\beta_1 \\ &= 3\mu + \alpha_1 + \alpha_2 + \alpha_3 + 3\beta_2 \end{aligned}$$

Let, $\hat{\beta}_1 + \hat{\beta}_2 = 0$

$\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 = 0$

$$\therefore \hat{\mu} = \cancel{\boxed{\frac{q_2 + q_3 + q_4}{2}}}\ \frac{q_1}{6}$$

$$\hat{\alpha}_1 = \frac{q_2}{2} - \frac{q_1}{6}$$

$$\hat{\alpha}_2 = \frac{q_3}{2} - \frac{q_1}{6}$$

$$\hat{\alpha}_3 = \frac{q_4}{2} - \frac{q_1}{6}$$

$$\hat{\beta}_1 = \frac{q_5}{3} - \frac{q_1}{6}$$

$$\hat{\beta}_2 = \frac{q_6}{3} - \frac{q_1}{6}$$

$$\therefore (X'X)^- = \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -1/6 & 0 & 0 & 0 & 1/3 & 0 \\ -1/6 & 0 & 0 & 0 & 0 & 1/3 \end{pmatrix}$$

$$H = (X'X)^-(X'X) = \begin{pmatrix} 1/6 & 0 & 0 & 0 & 0 & 0 \\ -1/6 & 1/2 & 0 & 0 & 0 & 0 \\ -1/6 & 0 & 1/2 & 0 & 0 & 0 \\ -1/6 & 0 & 0 & 1/2 & 0 & 0 \\ -1/6 & 0 & 0 & 0 & 1/3 & 0 \\ -1/6 & 0 & 0 & 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 6 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 0 & 0 & 1 & 1 \\ 2 & 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 & 3 & 0 \\ 3 & 1 & 1 & 1 & 0 & 3 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} 1 & 3 & 3 & 3 & 2 & 2 \\ 0 & -2 & -3 & -3 & 0 & -2 \\ 0 & -3 & -2 & -3 & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}} = \begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 & 1/2 & 1/2 \\ 0 & 2/3 & -1/3 & -1/3 & 0 & 0 \\ 0 & -1/3 & 2/3 & -1/3 & 0 & 0 \\ 0 & -1/3 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & 1/2 \end{pmatrix}$$

Estimable if $\underset{\sim}{\lambda}' = \underset{\sim}{\lambda}' H$

$$\boxed{(\cancel{\mu}\ \cancel{\alpha_1}\ \alpha_2\ \cancel{\alpha_3}\ \beta_1\ \cancel{\beta_2}) = (\mu)}$$

$$\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{pmatrix} = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{pmatrix} \begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 & 1/2 & 1/2 \\ 0 & 2/3 & -1/3 & -1/3 & 0 & 0 \\ 0 & -1/3 & 2/3 & -1/3 & 0 & 0 \\ 0 & -1/3 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & 1/2 \end{pmatrix}$$

$$= \Big(\lambda_0 \quad \frac{\lambda_0}{3} + \frac{2\lambda_1}{3} - \frac{(\lambda_2+\lambda_3)}{3} \quad \frac{\lambda_0}{3} + \frac{2\lambda_2}{3} - \frac{(\lambda_1+\lambda_3)}{3}$$

$$\frac{\lambda_0}{3} + \frac{2\lambda_3}{3} - \frac{(\lambda_1+\lambda_2)}{3} \quad \frac{1}{2}(\lambda_0 + \lambda_4 - \lambda_5)$$

$$\frac{1}{2}(\lambda_0 + \lambda_4 + \lambda_5) \Big)$$

$$\therefore \lambda_1 = \frac{\lambda_0}{3} + \frac{2\lambda_1}{3} - \frac{(\lambda_2+\lambda_3)}{3}$$

$$\lambda_2 = \frac{\lambda_0}{3} + \frac{2\lambda_2}{3} - \frac{(\lambda_1+\lambda_3)}{3} \qquad \boxed{\lambda_1 + \lambda_2 + \lambda_3 = \lambda_0}$$

$$\lambda_3 = \frac{\lambda_0}{3} + \frac{2\lambda_3}{3} - \frac{(\lambda_1+\lambda_2)}{3}$$

$$\lambda_4 = \frac{1}{2}(\lambda_0 + \lambda_4 - \lambda_5) \Rightarrow \boxed{\lambda_4 = \text{[illegible]}\lambda_0 - \text{[illegible]}\lambda_5}$$

$$\lambda_5 = \frac{1}{2}(\lambda_0 \pm \lambda_4 + \lambda_5) \Rightarrow \boxed{\lambda_5 = \text{[illegible]}\lambda_0 - \text{[illegible]}\lambda_4}$$

$$\Rightarrow \boxed{\lambda_4 = \lambda_5} \quad \cancel{\Rightarrow \lambda_5 = 2\lambda_0 - 4\lambda_0 + 4\lambda_5}$$

$$\Rightarrow \cancel{\lambda_5} = \cancel{\frac{2\lambda_0}{3}}$$

$$\boxed{\cancel{\lambda_2 \sim \lambda_3}}$$

$2\lambda_0$

~~(ii) (0 1 1 0 0 0) = (0 1 1 0 0 0)~~

ii) $\lambda_0 = 0$
$\lambda_1 = 1$
$\lambda_2 = 1$
$\lambda_3 = 0$
$\lambda_4 = 0$
$\lambda_5 = 0$
$\hookrightarrow$ not

0
0
0
0
1
−1
$\hookrightarrow$ not

iii) 1
1
0
0
0
0
$\hookrightarrow$ estimable

iv) $\lambda_0 = 6$
$\lambda_1 = 2$
$\lambda_2 = 2$
$\lambda_3 = 2$
$\lambda_4 = 3$
$\lambda_5 = 3$ $\hookrightarrow$ estimable

v) $\lambda_0 = 0$
$\lambda_1 = 1$
$\lambda_2 = -2$
$\lambda_3 = 1$
$\lambda_4 = 0$
$\lambda_5 = 0$
$\rightarrow$ estimable.

Gauss Markov Theorem:-

For the model $\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$, $E(\underset{\sim}{\varepsilon}) = 0$, $V(\underset{\sim}{\varepsilon}) = \sigma^2 I$ where Y is observed, X is known, $\underset{\sim}{\beta}$, $\hat{\sigma}$ unknown, the best linear unbiased estimate (BLUE) of an <u>estimable</u> linear parametric f^n $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$ (where λ is known) is $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$, $\hat{\underset{\sim}{\beta}}$ being <u>any sol^n</u> of the normal eq^n $X'\underset{\sim}{y} = (X'X)\hat{\underset{\sim}{\beta}}$ which is obtained by minimizing $(\underset{\sim}{y} - X\underset{\sim}{\beta})'(\underset{\sim}{y} - X\underset{\sim}{\beta})$ w.r.t $\underset{\sim}{\beta}$.

$\underline{\text{proof}}$ $\Rightarrow$ First observed that $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$ is unbiased for $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$ and is thus eligble for BLUE.

$$E(\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}})$$
$$= E(\underset{\sim}{\lambda}' S^- X' \underset{\sim}{y})$$
$$= \underset{\sim}{\lambda}' S^- X' (E(\underset{\sim}{y}))$$
$$= \underset{\sim}{\lambda}' S^- S \underset{\sim}{\beta}$$
$$= \underset{\sim}{\lambda}' H \underset{\sim}{\beta}$$
$$= \underset{\sim}{\lambda}'\underset{\sim}{\beta} \quad [\because \underset{\sim}{\lambda}' H = \underset{\sim}{\lambda}']$$

It remains to prove that the variance of $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$ is not larger than that of any other unbiased estimator of $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$.

Let, $\underset{\sim}{u}'\underset{\sim}{y}$ be any other u.e of $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$.

$$E(\underset{\sim}{u}'\underset{\sim}{y}) = \underset{\sim}{\lambda}'\underset{\sim}{\beta}$$
$$\Rightarrow \underset{\sim}{u}' X \underset{\sim}{\beta} = \underset{\sim}{\lambda}'\underset{\sim}{\beta}$$
$$\Rightarrow \underset{\sim}{u}' X = \underset{\sim}{\lambda}'$$

$$\underset{\sim}{u}'\underset{\sim}{y} = \underset{\sim}{u}'\underset{\sim}{y} - \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}} + \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$$
$$\Rightarrow V(\underset{\sim}{u}'\underset{\sim}{y}) = V(\underset{\sim}{u}'\underset{\sim}{y} - \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}) + V(\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}) + 2\,\text{cov}(\underset{\sim}{u}'\underset{\sim}{y} - \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}, \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}})$$

$$\text{Cov}(\underset{\sim}{u}'\underset{\sim}{y} - \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}, \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}})$$
$$= \text{cov}(\underset{\sim}{u}'\underset{\sim}{y} - \underset{\sim}{\lambda}' S^- X' \underset{\sim}{y}, \underset{\sim}{\lambda}' S^- X' \underset{\sim}{y}]$$
$$= \text{cov}((\underset{\sim}{u}' - \underset{\sim}{\lambda}' S^- X')\underset{\sim}{y}, \underset{\sim}{\lambda}' S^- X' \underset{\sim}{y})$$
$$= (\underset{\sim}{u}' - \underset{\sim}{\lambda}' S^- X') V(\underset{\sim}{y}) (\underset{\sim}{\lambda}' S^- X')'$$
$$= \sigma^2 (\underset{\sim}{u}' - \underset{\sim}{\lambda}' S^- X')(X S^- \underset{\sim}{\lambda})$$

$$= \sigma^2 (u'X - \lambda' S^- X'X)(S^- \lambda)$$

$$= \sigma^2 (\lambda' - \underbrace{\lambda' H}_{\lambda'})(S^- \lambda)$$

$$= 0$$

$$V(u'y) = V(u'y - \lambda'\hat{\beta}) + V(\lambda'\hat{\beta}) + \cancelto{0}{2\,\mathrm{cov}(u'y - \lambda'\hat{\beta}, \lambda'\hat{\beta})}$$

$$\Rightarrow V(u'y) \geq V(\lambda'\hat{\beta}) \quad \left[\because V(u'y - \lambda'\hat{\beta}) \geq 0\right]$$

For equality

$$V(u'y - \lambda'\hat{\beta}) = 0$$

$$\Rightarrow E\left[(u'y - \lambda'\hat{\beta})^2\right] = 0 \quad \left[\because V(x) = E(x^2) - [E(x)]^2\right]$$

$$\Rightarrow u'y = \lambda'\hat{\beta}$$

Equality holds if, $u'y = \lambda'\hat{\beta}$ with prob. 1.

- In other words if $\lambda'\beta$ is estimable, $\lambda'\hat{\beta}$ is its BLUE and if any other unbiased estimate of $\lambda'\beta$ has the same variance as $\lambda'\hat{\beta}$, it cannot be different from $\lambda'\hat{\beta}$. We therefore conclude that the BLUE of an estimable fun is unique.

- The Gauss-Markov thm. thus provide a convenient method of obtaining the BLUE of an estimable parametric fun $\lambda'\beta$. Obtain any soln $\hat{\beta}$ of the normal eqn and substitute $\hat{\beta}$ for β in the linear parametric fun to get its BLUE.

Suppose $\hat{\underset{\sim}{\beta}}_{(1)}$ and $\hat{\underset{\sim}{\beta}}_{(2)}$ are two different solⁿs of the normal eqⁿ, if they are substituted in an estimable parametric fuⁿ $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$, apparently it looks as if we have two different BLUEs, namely $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}_{(1)}$ and $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}_{(2)}$, but it is not so; they are the same. However if $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$ is not estimable, ~~but~~ substituting two different solⁿs may result in different expression.

Variance & Covariance of BLUEs

$$\hat{\underset{\sim}{\beta}} = S^- X'\underset{\sim}{y}$$

$$\Rightarrow V(\hat{\underset{\sim}{\beta}}) = V(S^- X'\underset{\sim}{y})$$

$$= S^- X' V(\underset{\sim}{y}) X (S^-)'$$

$$= \sigma^2 S^-(X'X)(S^-)'$$

$$= \sigma^2 S^- S (S^-)'$$

$$\neq \sigma^2 S^-$$

$$\boxed{\text{In general } V(\hat{\underset{\sim}{\beta}}) \neq \sigma^2 S^-}$$

$$V(\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}})$$

$$= \underset{\sim}{\lambda}' V(\hat{\underset{\sim}{\beta}}) \underset{\sim}{\lambda}$$

$$= \sigma^2 \underset{\sim}{\lambda}' \underbrace{S^- S}_{H} \underbrace{(S^-)'}_{H'} \underset{\sim}{\lambda}$$

$$= \sigma^2 \underset{\sim}{\lambda}' S^- H' \lambda$$

$$\left[\begin{aligned} H &= S^- S \\ \Rightarrow H' &= S'(S^-)' \\ &= S(S^-)' \end{aligned}\right]$$

[S is symmetric]

$$= \sigma^2 \underset{\sim}{\lambda}' H (S^-)' \underset{\sim}{\lambda}$$

$$= \boxed{\sigma^2 \underset{\sim}{\lambda}'(S^-)'\underset{\sim}{\lambda}}$$

$$= \sigma^2 \underset{\sim}{\lambda}' S^- (\underset{\sim}{\lambda}' H)'$$

$$= \boxed{\sigma^2 \underset{\sim}{\lambda}' S^- \underset{\sim}{\lambda}}$$

$\text{Cov}\left(\underset{\sim}{\lambda}'_{(1)}\hat{\underset{\sim}{\beta}},\ \underset{\sim}{\lambda}_{(2)}\hat{\underset{\sim}{\beta}}\right)$

$= \underset{\sim}{\lambda}'_{(1)} V(\hat{\underset{\sim}{\beta}})\, \underset{\sim}{\lambda}_{(2)}$

$= \sigma^2 \underset{\sim}{\lambda}'_{(1)} \underbrace{S^- S}_{H}\underbrace{(S^-)'}_{H'} \underset{\sim}{\lambda}_{(2)} \longrightarrow = \sigma^2 \underset{\sim}{\lambda}'_{(1)} S^- (\underset{\sim}{\lambda}'_{(2)} H)'$

$= \boxed{\sigma^2 \underset{\sim}{\lambda}'_{(1)} S^- \underset{\sim}{\lambda}_{(2)}}$

$= \sigma^2 \underset{\sim}{\lambda}'_{(1)} H (S^-)' \underset{\sim}{\lambda}_{(2)}$

$= \boxed{\sigma^2 \underset{\sim}{\lambda}'_{(1)} (S^-)' \underset{\sim}{\lambda}_{(2)}}$

Q. Let there be m estimable parametric fu^n, $\underset{\sim}{\lambda}'_{(i)} \underset{\sim}{\beta}$ $\forall i = 1(1)m$

Denote $\Lambda = \begin{pmatrix} \underset{\sim}{\lambda}'_{(1)} \\ \underset{\sim}{\lambda}'_{(2)} \\ \vdots \\ \underset{\sim}{\lambda}'_{(m)} \end{pmatrix}$, then all linear parametric fu^n may be expressed as $\Lambda \underset{\sim}{\beta}$ and $\Lambda H = \Lambda$ (due to estimibility)

$$V(\Lambda \hat{\underset{\sim}{\beta}}) = \Lambda V(\hat{\underset{\sim}{\beta}}) \Lambda'$$
$$= \Lambda S^- S (S^-)' \Lambda' \sigma^2$$
$$= \Lambda H (S^-)' \Lambda' \sigma^2$$
$$= \Lambda (S^-)' \Lambda' \sigma^2$$

If the m-parametric fu^n $\Lambda \underset{\sim}{\beta}$ are linearly independent i.e $R(\Lambda) = m$, then show that Variance Covariance matrix, i.e $\Lambda (S^-)' \Lambda' \sigma^2 = \Lambda S^- \Lambda' \sigma^2$ is non-singular.

$\Rightarrow$

$$\Lambda = \Lambda H$$
$$= \Lambda S^- S$$
$$= \Lambda S^- X' X$$

$$m = r(\Lambda) = r(\Lambda S^- X' X) \leq r(\Lambda S^- X') \leq r(\Lambda) = m$$

$$\therefore \boxed{r(\Lambda S^- X') = m}$$

$$m = r(\Lambda S^{-} X')$$

$$= r(\Lambda S^{-} X' X (S^{-})' \Lambda')$$

$$= r(\Lambda S^{-} S (S^{-})' \Lambda')$$

$$= r(\Lambda H (S^{-})' \Lambda')$$

$$= r(\Lambda (S^{-})' \Lambda')$$

$$= m$$

$\therefore$ [$\Rightarrow \Lambda (S^{-})' \Lambda'$ is a full rank matrix

$\Rightarrow \therefore |\Lambda (S^{-})' \Lambda'| \neq 0$ $\therefore$]

$|\Lambda| \neq 0$ $\therefore$

Estimaⁿ Space

$$\underset{\sim}{\lambda}' \hat{\underset{\sim}{\beta}} = \underset{\sim}{\lambda}' S^{-} X' \underset{\sim}{y}$$

$$= \underset{\sim}{l}' \underset{\sim}{q} \;, \quad \boxed{\begin{array}{l} \underset{\sim}{q} = X' \underset{\sim}{y} \\ \underset{\sim}{l} = (S^{-})' \underset{\sim}{\lambda} \end{array}}$$

The BLUE $\underset{\sim}{\lambda}' \hat{\underset{\sim}{\beta}}$ is thus a linear combinaⁿ of the left hand side $q_1, q_2, \dots q_p$. of the normal eqⁿ,

$$\boxed{X' \underset{\sim}{y} = (X'X) \underset{\sim}{\beta}}$$

Conversely if we consider a linear combinaⁿ $\underset{\sim}{l}' \underset{\sim}{q} = \sum_{i=1}^{p} l_i q_i$ of the left hand sides of the normal eqⁿs, it is the BLUE of its expected value because, $E(\underset{\sim}{l}' \underset{\sim}{q}) = \underset{\sim}{l}' X' X \underset{\sim}{\beta}$

$$= \boxed{\underset{\sim}{l}' X' \underset{\sim}{y}}$$

By Gauss-Markov thm.

$$\boxed{\text{BLUE of } \underset{\sim}{l}' X' X \underset{\sim}{\beta} \text{ is } \underset{\sim}{l}' X' X \hat{\underset{\sim}{\beta}}}$$

$$= \underset{\sim}{l}' X' \underset{\sim}{y}$$

$$= \underset{\sim}{l}' \underset{\sim}{q}$$

So we have the following theorem:-

Theorem 5 :-

For the model $\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$, the BLUE of every estimable parametric fu^{n} is a linear combina^{n} of the left hand sides $X'\underset{\sim}{y} = \underset{\sim}{q}$ of the normal eq^{n} and conversely any linear combina^{n} of the left hand sides $\underset{\sim}{q}$ of the normal eq^{n} is the BLUE of its expected values.

Corollary :-

A NAS (necessary & sufficient) cond^{n} for a linear parametric fu^{n} $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$ to be estimable is that $\underset{\sim}{\lambda}'$ is a linear combina^{n} of rows of $X'X$

$$r(X) = r(X'X)$$

$\Rightarrow$ rows of X & rows of $X'X$ $\boxed{\text{span}}$ same vector space.

Theorem 6 :-

The BLUE of any linear combina^{n}s of estimable parametric fu^{n}s is the same linear combina^{n} of ~~there~~ their BLUEs. In other words if $\underset{\sim}{\lambda}'_{(i)}\underset{\sim}{\beta}$ $\langle i = 1(1)m \rangle$ are all estimable the BLUE of $\underset{\sim}{\lambda}'\underset{\sim}{\beta} = k_1\underset{\sim}{\lambda}'_1\underset{\sim}{\beta} + k_2\underset{\sim}{\lambda}'_2\underset{\sim}{\beta} + \cdots + k_m\underset{\sim}{\lambda}'_m\underset{\sim}{\beta}$ is $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$

$$= k_1\underset{\sim}{\lambda}'_1\hat{\underset{\sim}{\beta}} + \cdots + k_m\underset{\sim}{\lambda}'_m\hat{\underset{\sim}{\beta}}$$

The proof follows from the fact that $\underset{\sim}{\lambda}' = \underset{\sim}{\lambda}'H$ and each $\underset{\sim}{\lambda}'_{(i)}$ satisfies $\underset{\sim}{\lambda}'_{(i)} = \underset{\sim}{\lambda}'_{(i)}H$ and by the Gauss-Markov theorem $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$ is the BLUE of $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$

Theorem 7 :-

If every BLUE is expressed in terms of the obs^{n}s $\underset{\sim}{y}$ as $\underset{\sim}{a}'\underset{\sim}{y}$, the coefficient vector $\underset{\sim}{a}$ is a linear combina^{n} of the columns of X and conversely every linear fu^{n} $\underset{\sim}{a}'\underset{\sim}{y}$ of the obs^{n} such that the coefficient vector $\underset{\sim}{a}$ is a linear combina^{n} of the columns of X, is the BLUE of its expected value.

proof :-

If $\utilde{\lambda}'\utilde{\beta}$ is estimable, its BLUE is $\utilde{\lambda}'\hat{\utilde{\beta}} = \utilde{\lambda}'S^{-}X'\utilde{y}$

$= \utilde{a}'\utilde{y}$

$\therefore \utilde{a}' = \utilde{\lambda}'S^{-}X'$

$\Rightarrow \utilde{a} = X(S^{-})'\utilde{\lambda}$

linear combina^n of the columns of X

$= X\utilde{l}$

So conversely,

$\utilde{a} = X\utilde{l}$

$\therefore E(\utilde{a}'\utilde{y}) = E(\utilde{l}'X'\utilde{y}) = \utilde{l}'X'X\utilde{\beta}$

BLUE of $\utilde{l}'X'X\utilde{\beta}$ is $\utilde{l}'X'X\hat{\utilde{\beta}}$

$= \utilde{l}'X'\utilde{y}$

$= \utilde{a}'\utilde{y}$ (proved)

Note:- [We must check the estimibility of $\utilde{l}'X'X\utilde{\beta}$]

<As $\utilde{a}'\utilde{y}$ is a linear fn such that its expected value is $\utilde{l}'X'X\utilde{\beta}$, by def^n it is estimable.>

Error Space :-

A linear fn of the obs^ns is said to belong to the error space iff its expected value is identically equal to 0, irrespective of the value of $\utilde{\beta}$.

Thus if $\utilde{b}'\utilde{y}$ belongs to the error space, by def^n $E(\utilde{b}'\utilde{y}) = 0$, i.e $\boxed{\utilde{b}'X\utilde{\beta} = 0}$

$\Rightarrow \boxed{\utilde{b}'X = 0 \text{ or } X'\utilde{b} = 0}$

[$\utilde{b}$ is orthogonal to the columns of X]

Theorem8:-

A linear fu^{n} of obs^{n}s belongs to the error space iff x coeff. vector is orthogonal to the columns of X.

$$\circledast\ E(\underset{\sim}{u}'\underset{\sim}{y}) = \underset{\sim}{\lambda}'\beta$$

$$\circledast\ E(\underbrace{\underset{\sim}{u}'\underset{\sim}{y} - \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}})$$

$$= E(\underset{\sim}{u}'\underset{\sim}{y}) - \underset{\sim}{\lambda}'\underset{\sim}{\beta}$$

It's a error fu^{n}

$$= \underset{\sim}{\lambda}'\underset{\sim}{\beta} - \underset{\sim}{\lambda}'\underset{\sim}{\beta}$$

$$= 0$$

$$E(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})$$

$$= X\underset{\sim}{\beta} - E(XS^{-}X'\underset{\sim}{y})$$

$$= X\underset{\sim}{\beta} - XS^{-}X'X\underset{\sim}{\beta}$$

$$= X\underset{\sim}{\beta} - XS^{-}S\underset{\sim}{\beta}$$

$$= X\underset{\sim}{\beta} - \underset{\sim}{XH}\underset{\sim}{\beta} \longrightarrow X$$

$$= 0$$

$$X'\underset{\sim}{b}_{(i)} = 0 \quad \forall i;\ i = 1(1)k$$

$$X'\underbrace{(c_1\underset{\sim}{b}_{(1)} + c_2\underset{\sim}{b}_{(2)} + \dots + c_k\underset{\sim}{b}_{(k)})}_{\text{belong to the error space}} = 0$$

Theorem 9:-

The coeff. vector of any BLUE, when expressed in terms of the obs^{n} is orthogonal to the coeff. vector of any linear fu^{n} ~~of~~ of the obs^{n} belonging to the error space.

proof: The proof of the thm is obvious from the fact that if $\underset{\sim}{b}'\underset{\sim}{y} \in$ error space, $\underset{\sim}{b}$ is orthogonal to the columns of X and by thm.7 the coeff. vector of any BLUE is a linear combina^{n} of columns of X. Thus any vector in the estima^{n} space is orthogonal to any vector in the error space and so we say that the error space is orthogonal to the estimation space. Since the estimation space generated by columns of X has rank p and since we can find at most n-r independent vectors orthogonal to clm: of X, the R(error space) = n-r

Example:-

$$\underset{\text{It belongs to the error space.}}{\underbrace{\underset{\sim}{y}'\underset{\sim}{y} - \underset{\text{BLUE of } \underset{\sim}{\lambda}'\underset{\sim}{\beta}}{\underbrace{\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}}}}}$$

$$\therefore E(\underset{\sim}{y}'\underset{\sim}{y} - \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}})$$
$$= \underset{\sim}{y}'X\underset{\sim}{\beta} - \underset{\sim}{\lambda}'\underset{\sim}{\beta}$$
$$= \underset{\sim}{\lambda}'\underset{\sim}{\beta} - \underset{\sim}{\lambda}'\underset{\sim}{\beta}$$
$$= 0$$

Theorem 10:-

The covariance b/w any linear fn: belonging to the error space and any BLUE is 0.

proof:»

$$\text{Cov}(\underset{\sim}{b}'\underset{\sim}{y}, \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}})$$
$$= \text{Cov}(\underset{\sim}{b}'\underset{\sim}{y}, \underset{\sim}{\lambda}'S^{-}X'\underset{\sim}{y})$$
$$= \underset{\sim}{b}'V(\underset{\sim}{y})X(S^{-})'\underset{\sim}{\lambda}$$
$$= \underset{\sim}{b}'X(S^{-})'\underset{\sim}{\lambda}\sigma^2$$
$$= \underset{\sim}{0} \quad [\because \underset{\sim}{b}'X = 0] \quad \left[\begin{array}{l} E(\underset{\sim}{b}'\underset{\sim}{y}) = 0 \\ \Rightarrow \underbrace{\underset{\sim}{b}'X\underset{\sim}{\beta}}_{\ne 0} = 0 \end{array}\right]$$

Role of Error Space:-

If $\underset{\sim}{b}'\underset{\sim}{y} \in$ error space and $\underset{\sim}{b}'\underset{\sim}{b} = 1$

$$\left[\begin{array}{l} E(\underset{\sim}{b}'\underset{\sim}{y}) = 0 \\ \therefore E(\underset{\sim}{b}'\underset{\sim}{y})^2 \\ = V(\underset{\sim}{b}'\underset{\sim}{y}) \\ = \underset{\sim}{b}'V(\underset{\sim}{y})\underset{\sim}{b} \\ = \underset{\sim}{b}'\underset{\sim}{b}\sigma^2 \\ = \sigma^2 \; [\because \underset{\sim}{b}'\underset{\sim}{b} = 1] \end{array}\right]$$

$\therefore \sigma^2$ unbiasedly estimated by $(\underset{\sim}{b}'\underset{\sim}{y})^2$

$$B_1 = \begin{pmatrix} \underset{\sim}{b}_{(1)} \\ \vdots \\ \underset{\sim}{b}_{(n-r)} \end{pmatrix}$$

$$\boxed{\begin{array}{l} \underset{\sim}{b}'_{(i)} \underset{\sim}{b}_{(j)} = 0 \\ \hline \underset{\sim}{b}'_{(i)} X = 0 \\ \hline \underset{\sim}{b}'_{(i)} \underset{\sim}{b}_{(i)} = 1 \end{array}}$$

- $B_1 X = 0$
- $B_1 B_1' = I_{n-r}$

$$\left\langle (\underset{\sim}{b}_1' \underset{\sim}{y})^2 + (\underset{\sim}{b}_2' \underset{\sim}{y})^2 + \dots + (\underset{\sim}{b}_{n-r}' \underset{\sim}{y})^2 = \underset{\sim}{y}' B_1' B_1 \underset{\sim}{y} \right\rangle$$

$\Downarrow$

This is the sum of squares of a complete set of $(n-r)$ unit, mutually orthogonal linear fu^{n}s belonging to the error space. This is why we called it as SSE.

$$\boxed{E(\underset{\sim}{y}' B_1' B_1 \underset{\sim}{y}) = (n-r)\sigma^2}$$

Thus by pulling together all the linearly independent fu^{n}s belonging to the error space, we can obtain the estimate ~~as~~ $\text{SSE}/n-r = \dfrac{\underset{\sim}{y}' B_1' B_1 \underset{\sim}{y}}{n-r}$ of σ^2

We know $\text{SSE} = (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})$ where $\hat{\underset{\sim}{\beta}}$ is any sol^{n} of the normal eq^{n}. To establish equivalency of the def^{n}, let us consider r mutually orthogonal rows such that $B = \left(\dfrac{B_1\,(n-r)}{B_2\,(r)}\right)$ becomes a $n\times n$ orthogonal matrix. By def^{n} $B_1 B_2' = 0$ again $B_1 X = 0$ implies rows of B_1 are orthogonal to the columns of X also rows of B_2 are orthogonal to the rows of B_1 but there can not be more than $(n-r)$ linearly independent vectors orthogonal to the rows of B_1 and so rows of B_2 must be a linear combination of columns of X.

$$\therefore \boxed{B_2 = \overset{\overline{n-r\times n}}{C} X'} \Rightarrow \left\langle B_2 X \hat{\underset{\sim}{\beta}} = C X' X \hat{\underset{\sim}{\beta}} = C X' \underset{\sim}{y} = B_2 \underset{\sim}{y} \right\rangle$$

$$I = B'B$$
$$= (B_1' : B_2')\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$
$$= B_1'B_1 + B_2'B_2$$

$$\therefore \mathbf{SSE} = (y - X\hat{\beta})'(y - X\hat{\beta})$$
$$= (y - X\hat{\beta})'(B_1'B_1 + B_2'B_2)(y - X\hat{\beta})$$
$$= (y - X\hat{\beta})'B_1'B_1(y - X\hat{\beta})$$
$$+ (y - X\hat{\beta})' \cdot B_2'B_2(y - X\hat{\beta})$$

$$= (B_1 y - B_1 X\hat{\beta})'(B_1 y - B_1 X\hat{\beta})$$
$$+ (B_2 y - B_2 X\hat{\beta})'(B_2 y - B_2 X\hat{\beta})$$

$$= (B_1 y)'(B_1 y) \qquad \left[\because B_2 X\hat{\beta} = B_2 y \text{ and } B_1 X = 0\right]$$
$$= \boxed{y'B_1'B_1 y}$$

$$\therefore \text{SSE} = (y - X\hat{\beta})'(y - X\hat{\beta})$$
$$= y'y - 2\hat{\beta}X'X\hat{\beta} + (X\hat{\beta})'(X\hat{\beta})$$
$$= \sum_{i=1}^{n} y_i^2 - 2\hat{\beta}'X'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}$$
$$= \sum_{i=1}^{n} y_i^2 - \hat{\beta}\underbrace{X'X\hat{\beta}}_{X'y}$$
$$= \sum_{i=1}^{n} y_i^2 - \hat{\beta}'X'y$$
$$= \sum_{i=1}^{n} y_i^2 - \underbrace{(\hat{\beta}_1 q_1 + \hat{\beta}_2 q_2 + \dots + \hat{\beta}_p q_p)}_{\text{SSR}}$$

$$\therefore E(SSR) = E\left[\sum y_i^2\right] - E(SSE)$$
$$= \sum\left[V(Y_i) + (E(Y_i))^2\right] - (n-n)\sigma^2$$
$$= n\sigma^2 + \beta X'X\beta - n\sigma^2 + r\sigma^2$$

$= r\sigma^2 + \underset{\sim}{\beta}'X'X\underset{\sim}{\beta}$

Source	d.f	SS	E(ms)
Regression	r	$\hat{\underset{\sim}{\beta}}'\underset{\sim}{q}$	$\sigma^2 + \frac{1}{r}\underset{\sim}{\beta}'X'X\underset{\sim}{\beta}$
Error	$n-r$	$\underset{\sim}{y}'\underset{\sim}{y} - \hat{\underset{\sim}{\beta}}'\underset{\sim}{q}$	σ^2
Total	n	$\underset{\sim}{y}'\underset{\sim}{y}$	

$$\boxed{E(MSR) \geqslant E(MSE)}$$

$\downarrow$ Equality holds if $\boxed{X\underset{\sim}{\beta} = 0}$

■ Interval Estimate & Testing of hypothesis

$\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$ $\quad \boxed{E(\underset{\sim}{y}) = X\underset{\sim}{\beta}}$

$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

$(\underset{\sim}{b}'_{(1)}\underset{\sim}{y}, \ldots, \underset{\sim}{b}'_{(n-r)}\underset{\sim}{y})$

$$Cov(\underset{\sim}{b}'_{(i)}\underset{\sim}{y}, \underset{\sim}{b}'_{(j)}\underset{\sim}{y}) = \underset{\sim}{b}'_{(i)}(\sigma^2 I)\underset{\sim}{b}_{(j)} = 0$$

Theorem 1:

Linear fu^n s of normal variables have a joint multivariate normal dist^n, the parameters of these multivariate dist^n are the means of these linear fu^n s and their variances and covariances.

Acc^n ly $\underset{\sim}{b}'_{(i)}\underset{\sim}{y}$, $i = 1(1)(n-r)$ are $(n-r)$ linear fu^n s of the normal variable $\underset{\sim}{y}$. There also, they are $(n-r)$ independent " " s with 0 mean and variance σ^2.

$$E(\underset{\sim}{b}_{(i)}'\underset{\sim}{y}) = 0, \quad V(\underset{\sim}{b}'_{(i)}\underset{\sim}{y}) = \sigma^2$$

$$\sum_{i=1}^{n-r} \frac{(b'_{(i)}\underset{\sim}{y})^2}{\sigma^2} \sim \chi^2_{n-r} \Rightarrow = \frac{SSE}{\sigma^2}$$

Theorem 2:

If $\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$ where $\underset{\sim}{\varepsilon} \sim N(0, \sigma^2)$ iid, the dist^n of $\frac{1}{\sigma^2} \text{Min} \underbrace{(\underset{\sim}{y} - X\underset{\sim}{\beta})'(\underset{\sim}{y} - X\underset{\sim}{\beta})}_{SSE}$ is a chi-square with $(n-r)$ degrees of freedom. Where r is the rank of X.

Theorem 3:

The error sum of square is ~~is independently~~ distributed independently of the BLUE of any estimable fu^n.

proof:>> By theorem 1, the joint dist^n of $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$, the BLUE of an estimable parametric fu^n $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$ and the $(n-r)$ linear fu^{n}s $\underset{\sim}{b}'_{(i)}\underset{\sim}{y}$ belonging to the error space is multivariate normal.

By theorem 10, $\text{Cov}(\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}, \text{every } \underset{\sim}{b}'_{(i)}\underset{\sim}{y}) = 0$, hence $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$ is independently distributed of $\underset{\sim}{b}'_{(i)}\underset{\sim}{y}$.

∴ It is also independently distributed of $SSE = \sum_{i=1}^{n-r} (b'_{(i)}\underset{\sim}{y})^2$.

Theorem 4:

The joint dist^n of the BLUEs of any m linearly independent estimable parametric fu^n $\Lambda\underset{\sim}{\beta}$, where Λ is $(m \times p)$ matrix of rank m, is multivariate normal with mean $\Lambda\underset{\sim}{\beta}$ and variance covariance matrix $\boxed{(\Lambda S^{-}\Lambda')\sigma^2}$. Further this dist^n is independent of the proof:》

The BLUE of $\Lambda\beta$ are m linear fn's of normal variables as $\hat{\underset{\sim}{\beta}} = \cancel{(X'X)^{-}\underset{\sim}{y}}$ $\boxed{S^{-}X'\underset{\sim}{y}}$ with mean $\Lambda\beta$ and variance covariance matrix $(\Lambda S^{-}\Lambda')\sigma^2$

$$V(\Lambda\hat{\underset{\sim}{\beta}})$$
$$= V(\Lambda S^{-}X'\underset{\sim}{y})$$
$$= (\Lambda S^{-}X'X S^{-}\Lambda')\sigma^2$$
$$= (\Lambda S^{-}S S^{-}\Lambda')\sigma^2$$
$$= (\Lambda H S^{-}\Lambda')\sigma^2$$
$$= \boxed{(\Lambda S^{-}\Lambda')\sigma^2}$$

〈By theorem 3 they are independent.〉

Theorem 5: The dist^n of $(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta})'(\Lambda S^{-}\Lambda'\sigma^2)^{-1}(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta}) \sim \chi^2_m$

$$\boxed{\frac{SSE}{\sigma^2} \sim \chi^2_{m-r}}$$

$$\boxed{\frac{(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta})'(\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta})/m}{SSE/n-r} \sim F_{m,n-r}}$$

Let, $\Lambda^{1\times p}\underset{\sim}{\beta} = \lambda'\underset{\sim}{\beta}$

$$\therefore \frac{(\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}} - \underset{\sim}{\lambda}'\underset{\sim}{\beta})'(\underset{\sim}{\lambda}'S^{-}\underset{\sim}{\lambda})^{-1}(\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}} - \underset{\sim}{\lambda}'\underset{\sim}{\beta})}{SSE/n-r}$$

$$= \left[\frac{(\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}} - \underset{\sim}{\lambda}'\underset{\sim}{\beta})(\lambda'S^{-}\lambda)^{-1/2}}{\sqrt{SSE/n-r}}\right]^2$$

↳ It follows t_{n-r}