Proof :-

If $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$ is estimable, its BLUE is $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}} = \underset{\sim}{\lambda}'S^{-}X'\underset{\sim}{y}$

$$= \underset{\sim}{a}'\underset{\sim}{y}$$

$$\therefore \underset{\sim}{a}' = \underset{\sim}{\lambda}'S^{-}X'$$

$$\Rightarrow \underset{\sim}{a} = X(S^{-})'\underset{\sim}{\lambda}$$

$$= X\underset{\sim}{l}$$

linear combin^n of the columns of X

So conversely,

$$\underset{\sim}{a} = X\underset{\sim}{l}$$

$$\therefore E(\underset{\sim}{a}'\underset{\sim}{y}) = E(\underset{\sim}{l}'X'\underset{\sim}{y}) = \underset{\sim}{l}'X'X\underset{\sim}{\beta}$$

BLUE of $\underset{\sim}{l}'X'X\underset{\sim}{\beta}$ is $\underset{\sim}{l}'X'X\hat{\underset{\sim}{\beta}}$

$$= \underset{\sim}{l}'X'\underset{\sim}{y}$$

$$= \underset{\sim}{a}'\underset{\sim}{y} \text{ (proved)}$$

Note:- • [We must check the estimibility of $\underset{\sim}{l}'X'X\underset{\sim}{\beta}$]

<As $\underset{\sim}{a}'\underset{\sim}{y}$ is a linear fu^n such that its expected value is $\underset{\sim}{l}'X'X\underset{\sim}{\beta}$, by def^n it is estimable.>

Error Space :-

A linear fu^n of the obs^{n}s is said to belong to the error space iff its expected value is identically equal to 0, irrespective of the value of $\underset{\sim}{\beta}$.

Thus if $\underset{\sim}{b}'\underset{\sim}{y}$ belongs to the error space, by def^n $E(\underset{\sim}{b}'\underset{\sim}{y}) = 0$, i.e $\boxed{\underset{\sim}{b}'X\underset{\sim}{\beta} = 0}$

$$\Rightarrow \boxed{\underset{\sim}{b}'X = 0 \text{ or } X'\underset{\sim}{b} = 0}$$

[$\underset{\sim}{b}$ is orthogonal to the columns of X]

Theorem 8:-

A linear fn of obs^{n}s belongs to the error space iff x coeff. vector is orthogonal to the columns of X.

$$\circledast \; E(\underset{\sim}{u}'\underset{\sim}{y}) = \underset{\sim}{\lambda}'\underset{\sim}{\beta}$$

$$E(\underset{\sim}{u}'\underset{\sim}{y} - \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}})$$

$$= E(\underset{\sim}{u}'\underset{\sim}{y}) - \underset{\sim}{\lambda}'\underset{\sim}{\beta}$$

It's a error fn

$$= \underset{\sim}{\lambda}'\underset{\sim}{\beta} - \underset{\sim}{\lambda}'\underset{\sim}{\beta}$$

$$= 0$$

$$E(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})$$

$$= X\underset{\sim}{\beta} - E(XS^{-}X'\underset{\sim}{y})$$

$$= X\underset{\sim}{\beta} - XS^{-}X'X\underset{\sim}{\beta}$$

$$= X\underset{\sim}{\beta} - XS^{-}S\underset{\sim}{\beta}$$

$$= X\underset{\sim}{\beta} - \underset{\sim}{XH}\underset{\sim}{\beta} \to X$$

$$= 0$$

$$X'\underset{\sim}{b}_{(i)} = 0 \quad \forall i, i = 1(1)k$$

$$X'\underbrace{(c_1\underset{\sim}{b}_{(1)} + c_2\underset{\sim}{b}_{(2)} + \dots + c_k\underset{\sim}{b}_{(k)})}_{\text{belong to the error space}} = 0$$

Theorem 9:-

The coeff. vector of any BLUE, when expressed in terms of the obs^n is orthogonal to the coeff. vector of any linear fn ~~of~~ of the obs^n belonging to the error space.

proof The proof of the thm is obvious from the fact that if $\underset{\sim}{b}'\underset{\sim}{y} \in$ error space, $\underset{\sim}{b}$ is orthogonal to the columns of X and by thm.7 the coeff. vector of any BLUE is a linear combina^n of columns of X. Thus any vector in the estima^n space is orthogonal to any vector in the error space and so we say that the error space is orthogonal to the estimation space. Since the estimation space generated by columns of X has rank p and since we can find at most $n-r$ independent vectors orthogonal to clm. of X, the R(error space) = $n-r$

Example:-

$$\underbrace{\underset{\sim}{y}'\underset{\sim}{y} - \underbrace{\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}}_{\text{BLUE of } \underset{\sim}{\lambda}'\underset{\sim}{\beta}}}_{\text{It belongs to the error space.}}$$

$\therefore E(\underset{\sim}{y}'\underset{\sim}{y} - \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}})$

$= \underset{\sim}{y}'X\underset{\sim}{\beta} - \underset{\sim}{\lambda}'\underset{\sim}{\beta}$

$= \underset{\sim}{\lambda}'\underset{\sim}{\beta} - \underset{\sim}{\lambda}'\underset{\sim}{\beta}$

$= 0$

Theorem 10:-

The covariance b/w any linear fn. belonging to the error space and any BLUE is 0.

Proof>>

$Cov(\underset{\sim}{b}'\underset{\sim}{y}, \underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}})$

$= Cov(\underset{\sim}{b}'\underset{\sim}{y}, \underset{\sim}{\lambda}'S^{-}X'\underset{\sim}{y})$

$= \underset{\sim}{b}'V(\underset{\sim}{y})X(S^{-})'\underset{\sim}{\lambda}$

$= \underset{\sim}{b}'X(S^{-})'\underset{\sim}{\lambda}\sigma^2$

$= \underset{\sim}{0}$ $\quad [\because \underset{\sim}{b}'X = 0]$ $\quad \left[\begin{array}{l} E(\underset{\sim}{b}'\underset{\sim}{y}) = 0 \\ \Rightarrow \underbrace{\underset{\sim}{b}'X}_{0}\underset{\sim}{\beta} = 0 \end{array}\right]$

Role of Error Space:-

If $\underset{\sim}{b}'\underset{\sim}{y} \in$ error space and $\underset{\sim}{b}'\underset{\sim}{b} = 1$

$$\left[\begin{array}{l} E(\underset{\sim}{b}'\underset{\sim}{y}) = 0 \\ \therefore E(\underset{\sim}{b}'\underset{\sim}{y})^2 \\ = V(\underset{\sim}{b}'\underset{\sim}{y}) \\ = \underset{\sim}{b}'V(\underset{\sim}{y})\underset{\sim}{b} \\ = \underset{\sim}{b}'\underset{\sim}{b}\sigma^2 \\ = \sigma^2 \quad [\because \underset{\sim}{b}'\underset{\sim}{b} = 1] \end{array}\right]$$

$\therefore \sigma^2$ unbiasedly estimated by $(\underset{\sim}{b}'\underset{\sim}{y})^2$

$$B_1 = \begin{pmatrix} \underset{\sim}{b}_{(1)} \\ \vdots \\ \underset{\sim}{b}_{(n-r)} \end{pmatrix}$$

$$\boxed{\begin{array}{c} \underset{\sim}{b}'_{(i)} \underset{\sim}{b}_{(j)} = 0 \\ \hline \underset{\sim}{b}'_{(i)} X = 0 \\ \hline \underset{\sim}{b}'_{(i)} \underset{\sim}{b}_{(i)} = 1 \end{array}}$$

- $B_1 X = 0$
- $B_1 B_1' = I_{n-r}$

$$(\underset{\sim}{b}_1' \underset{\sim}{y})^2 + (\underset{\sim}{b}_2' \underset{\sim}{y})^2 + \dots + (\underset{\sim}{b}'_{n-r} \underset{\sim}{y})^2$$
$$= \underset{\sim}{y}' B_1' B_1 \underset{\sim}{y}$$

$\Downarrow$

This is the sum of squares of a complete set of $(n-r)$ unit, mutually orthogonal linear fu^{n}s belonging to the error space. This is why we called it as SSE

$$\boxed{E(\underset{\sim}{y}' B_1' B_1 \underset{\sim}{y}) = (n-r)\sigma^2}$$

Thus by pulling together all the linearly independent fu^{n}s belonging to the error space, we can obtain the estimate ~~as~~ $SSE/n-r = \dfrac{\underset{\sim}{y}' B_1' B_1 \underset{\sim}{y}}{n-r}$ of σ^2

We know $SSE = (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})$ where $\hat{\beta}$ is any sol^n of the normal eq^n. To establish equivalency of the def^n, let us consider r mutually orthogonal rows such that $B = \begin{pmatrix} B_{1(n-r)} \\ \hline B_{2(r)} \end{pmatrix}$ becomes a $n \times n$ orthogonal matrix. By def^n $B_1 B_2' = 0$ again $B_1 X = 0$ implies rows of B_1 are orthogonal to the columns of X also rows of B_2 are orthogonal to the rows of B_1 but there can not be more than $(n-r)$ linearly independent vectors orthogonal to the rows of B_1 and so rows of B_2 must be a linear combination of columns of X.

$$\therefore \boxed{B_2 = \overset{r \times n}{C} X'} \Rightarrow B_2 X \hat{\underset{\sim}{\beta}} = C X' X \hat{\underset{\sim}{\beta}}$$
$$= C X' \underset{\sim}{y} = B_2 \underset{\sim}{y}$$

$$I = B'B$$
$$= (B_1' \vdots B_2') \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$
$$= B_1'B_1 + B_2'B_2$$

$$\therefore SSE = (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})$$
$$= (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(B_1'B_1 + B_2'B_2)(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})$$
$$= (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'B_1'B_1(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}) + (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'B_2'B_2(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})$$

$$= (B_1\underset{\sim}{y} - B_1X\hat{\underset{\sim}{\beta}})'(B_1\underset{\sim}{y} - B_1X\hat{\underset{\sim}{\beta}}) + (B_2\underset{\sim}{y} - B_2X\hat{\underset{\sim}{\beta}})'(B_2\underset{\sim}{y} - B_2X\hat{\underset{\sim}{\beta}})$$

$$= (B_1\underset{\sim}{y})'(B_1\underset{\sim}{y}) \qquad \left[\because B_2X\hat{\underset{\sim}{\beta}} = B_2\underset{\sim}{y} \text{ and } B_1X = 0\right]$$
$$= \boxed{\underset{\sim}{y}'B_1'B_1\underset{\sim}{y}}$$

$$SSE = (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})$$
$$= \underset{\sim}{y}'\underset{\sim}{y} - 2\hat{\underset{\sim}{\beta}}'X'X\hat{\underset{\sim}{\beta}} + (X\hat{\underset{\sim}{\beta}})'(X\hat{\underset{\sim}{\beta}})$$
$$= \sum_{i=1}^{n} y_i^2 - 2\hat{\underset{\sim}{\beta}}'X'X\hat{\underset{\sim}{\beta}} + \hat{\underset{\sim}{\beta}}'X'X\hat{\underset{\sim}{\beta}}$$
$$= \sum_{i=1}^{n} y_i^2 - \hat{\underset{\sim}{\beta}}\underbrace{X'X\hat{\underset{\sim}{\beta}}}_{X'\underset{\sim}{y}}$$
$$= \sum_{i=1}^{n} y_i^2 - \hat{\underset{\sim}{\beta}}'X'\underset{\sim}{y}$$
$$= \sum_{i=1}^{n} y_i^2 - \underbrace{(\hat{\underset{\sim}{\beta}}_1 q_1 + \hat{\underset{\sim}{\beta}}_2 q_2 + \dots + \hat{\beta}_p q_p)}_{SSR}$$

$$\therefore E(SSR) = E\left[\sum y_i^2\right] - E(SSE)$$
$$= \sum\left[V(Y_i) + (E(Y_i))^2\right] - (n-n)\sigma^2$$
$$= n\sigma^2 + \beta'X'X\beta - n\sigma^2 + n\sigma^2$$

$= r\sigma^2 + \underset{\sim}{\beta}' X' X \underset{\sim}{\beta}$

Source	d.f	SS	E(ms)
Regression	r	$\hat{\underset{\sim}{\beta}}' \underset{\sim}{q}$	$\sigma^2 + \frac{1}{r} \underset{\sim}{\beta}' X' X \underset{\sim}{\beta}$
Error	$n-r$	$\underset{\sim}{y}'\underset{\sim}{y} - \hat{\underset{\sim}{\beta}}' \underset{\sim}{q}$	σ^2
Total	n	$\underset{\sim}{y}'\underset{\sim}{y}$	

$\boxed{E(MSR) \geq E(MSE)}$

Equality holds if $\boxed{X\underset{\sim}{\beta} = 0}$

//

Interval Estimate & Testing of hypothesis

$\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$ $\boxed{E(\underset{\sim}{y}) = X\underset{\sim}{\beta}}$

$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

$(\underset{\sim}{b}'_{(1)} \underset{\sim}{y}, \ldots, \underset{\sim}{b}'_{(n-r)} \underset{\sim}{y})$

$$\mathrm{Cov}(\underset{\sim}{b}'_{(i)} \underset{\sim}{y}, \underset{\sim}{b}'_{(j)} \underset{\sim}{y}) = \underset{\sim}{b}'_{(i)} (\sigma^2 I) \underset{\sim}{b}_{(j)} = 0$$

Theorem 1:

Linear fn's of normal variables have a joint multivariate normal distn, the parameters of these multivariate distn are the means of these linear fn's and their variances and covariances.

Accly $\underset{\sim}{b}'_{(i)} \underset{\sim}{y}$, $i = 1(1)(n-r)$ are $(n-r)$ linear fn's of the normal variable $\underset{\sim}{y}$. There also, they are $(n-r)$ independent " " s with 0 mean and variance σ^2.

$$E(\underset{\sim}{b}_{(i)}' \underset{\sim}{y}) = 0, \quad V(\underset{\sim}{b}_{(i)} \underset{\sim}{y}) = \sigma^2$$

$$\sum_{i=1}^{n-r} \frac{(b'_{(i)} \underset{\sim}{y})^2}{\sigma^2} \sim \chi^2_{n-r} = \frac{SSE}{\sigma^2}$$

Theorem 2:

If $\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$ where $\underset{\sim}{\varepsilon} \sim N(0, \sigma^2)$ iid, the distⁿ of $\frac{1}{\sigma^2} \underbrace{\text{Min}\,(\underset{\sim}{y} - X\underset{\sim}{\beta})'(\underset{\sim}{y} - X\underset{\sim}{\beta})}_{SSE}$ is a chi-square with $(n-r)$ degrees of freedom. Where r is the rank of X.

Theorem 3:

The error sum of square is ~~is independently~~ distributed independently of the BLUE of any estimable fuⁿ.

proof>>

By theorem 1, the joint distⁿ of $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$, the BLUE of an estimable parametric fuⁿ $\underset{\sim}{\lambda}'\underset{\sim}{\beta}$ and the $(n-r)$ linear fuⁿs $\underset{\sim}{b}_{(i)}'\underset{\sim}{y}$ belonging to the error space is multivariate normal.

By theorem 10, $\text{Cov}(\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}, \text{every } \underset{\sim}{b}_{(i)}'\underset{\sim}{y}) = 0$, hence $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$ is independently distributed of $\underset{\sim}{b}_{(i)}'\underset{\sim}{y}$.

∴ It is also independently distributed of $SSE = \sum_{i=1}^{n-r} (\underset{\sim}{b}_{(i)}'\underset{\sim}{y})^2$.

Theorem 4:

The joint distⁿ of the BLUEs of any m linearly independent estimable parametric fuⁿ $\Lambda\underset{\sim}{\beta}$, where Λ is $(m \times p)$ matrix of rank m, is multivariate normal with mean $\Lambda\underset{\sim}{\beta}$ and and variance covariance matrix $\boxed{(\Lambda S^{-}\Lambda')\sigma^2}$.

Further this distⁿ is independent of the error SS.

proof:

The BLUE of $\Lambda\beta$ are m linear fn's of normal variables as $\hat{\beta} = \cancel{(X'X)^{-}y}$ $\boxed{S^{-}X'\underset{\sim}{y}}$ with mean $\Lambda\beta$ and variance covariance matrix $(\Lambda S^{-}\Lambda')\sigma^2$

$$
\begin{aligned}
&V(\Lambda\hat{\underset{\sim}{\beta}}) \\
&= V(\Lambda S^{-}X'\underset{\sim}{y}) \\
&= (\Lambda S^{-}X'X S^{-}\Lambda')\sigma^2 \\
&= (\Lambda S^{-}S S^{-}\Lambda')\sigma^2 \\
&= (\Lambda H S^{-}\Lambda')\sigma^2 \\
&= \boxed{(\Lambda S^{-}\Lambda')\sigma^2}
\end{aligned}
$$

⟨By theorem 3 they are independent.⟩

Theorem 5: The distn of $(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta})'(\Lambda S^{-}\Lambda'\sigma^2)^{-1}(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta}) \sim \chi^2_m$

$$\boxed{\frac{SSE}{\sigma^2} \sim \chi^2_{n-r}}$$

Theorem 6:

$$\boxed{\frac{(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta})'(\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta})/m}{SSE/n-r} \sim F_{m,n-r}}$$

sol, $\lambda'^{*}\beta = \lambda'\beta$

$$= \frac{(\lambda'\hat{\beta} - \lambda'\beta)'(\lambda' S^{-} \lambda)^{-1}(\lambda'\hat{\beta} - \lambda'\beta)}{SSE/n-r}$$

$$= \left[\frac{(\lambda'\hat{\beta} - \lambda'\beta)(\lambda' S^{-} \lambda)^{-1/2}}{\sqrt{SSE/n-r}}\right]^2$$

$\rightarrow$ It follows t_{n-r}

Q. Find the dist^n of W which is $(\Lambda\hat{\beta} - d)'(\Lambda S^{-} \Lambda' \sigma^2)^{-1}(\Lambda\hat{\beta} - d)$

$\Rightarrow$

$$y = (\Lambda S^{-} \Lambda')^{-1/2}(\Lambda\hat{\beta} - d)$$

$$\boxed{W = y'y/\sigma^2}$$

$$E(y) = (\Lambda S^{-} \Lambda')^{-1/2}(\Lambda\beta - d) = \mu$$

$$V(y) = (\Lambda S^{-} \Lambda')^{-1/2}(\Lambda S^{-} \Lambda' \sigma^2)(\Lambda S^{-} \Lambda')^{-1/2}$$

$$= \sigma^2 I$$

Thus, $(y_1, y_2, \ldots, y_m)$ are normal independent variables with mean $(\mu_1, \mu_2, \ldots, \mu_m)$ and variance σ^2, hence $y'y/\sigma^2$ will follow a non-central chi-square with m d.f and non-centrality parameter $\sum_{i=1}^{m} \mu_i^2/\sigma^2$

Conditional SSE:

$$SSE = (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}),$$

We shall now minimize $(\underset{\sim}{y} - X\tilde{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\tilde{\underset{\sim}{\beta}})$ subject to the consistent cond^{n}s $\Lambda\tilde{\underset{\sim}{\beta}} = \underset{\sim}{d}$. To find this we use Lagrangian Multipliers $2k_1, 2k_2, \ldots, 2k_m$

$\boxed{\underset{\sim}{\lambda}'_{(i)}\tilde{\underset{\sim}{\beta}} = d_{(i)}}$

$$\phi = (\underset{\sim}{y} - X\tilde{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\tilde{\underset{\sim}{\beta}}) + 2k_1(\underset{\sim}{\lambda}'_{(1)}\tilde{\underset{\sim}{\beta}} - d_{(1)}) + \ldots + 2k_m(\underset{\sim}{\lambda}'_{(m)}\tilde{\underset{\sim}{\beta}} - d_{(m)})$$

$$\Rightarrow \frac{d\phi}{d\tilde{\underset{\sim}{\beta}}} = -2X'\underset{\sim}{y} + 2X'X\tilde{\underset{\sim}{\beta}} + 2k_1(\underset{\sim}{\lambda}_{(1)}) + \ldots + 2k_m\underset{\sim}{\lambda}_{(m)} = 0$$

$$\Rightarrow X'\underset{\sim}{y} = X'X\tilde{\underset{\sim}{\beta}} + \Lambda'\underset{\sim}{k} \quad \text{--(i)}$$

From, normal eq^n we get,

$$\langle X'\underset{\sim}{y} = X'X\hat{\underset{\sim}{\beta}} \rangle \quad \text{--(ii)}$$

(ii) - (i) we will get,

$$\boxed{X'X(\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}}) = \Lambda'\underset{\sim}{k}} \quad \text{--(iii)}$$

$$\Rightarrow S(\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}}) = \Lambda'\underset{\sim}{k}$$

$$\Rightarrow \underbrace{\Lambda \underbrace{S^{-}S}_{H}}_{\Lambda}(\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}}) = \Lambda S^{-}\Lambda'\underset{\sim}{k}$$

$$\Rightarrow \Lambda(\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}}) = (\Lambda S^{-}\Lambda')\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{k} = (\Lambda S^{-}\Lambda')^{-1}\Lambda(\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}}) \quad \text{--(iv)}$$

putting (iv) in (iii) we will get,

$$X'X\underbrace{(\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}})}_{\beta^*} = \Lambda'(\Lambda S^{-}\Lambda')^{-1}\Lambda(\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}})$$

$$= \Lambda'(\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\underset{\sim}{\beta}} - \underset{\sim}{d})$$

General solⁿ

$$\hat{\beta}^* = S^{-}\Lambda'(\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\beta} - d) + (I-H)z$$

$$\boxed{\hat{\beta} - \tilde{\beta} = S^{-}\Lambda'(\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\beta} - d)}$$

→ particular solⁿ

It doesn't matter what solⁿ of eqⁿ $X'y = X'X\tilde{\beta} + \Lambda' k$, we take, as we get the same value for $(y - X\beta)'(y - X\beta)$, when it is minimize w.r.t β subject to $\Lambda\beta = d$.

Proof ⟹ Let, $\Lambda\beta_0 = d$, $\Lambda\tilde{\beta} = d$

$$(y - X\beta_0)'(y - X\beta_0)$$

$$= (\underbrace{y - X\tilde{\beta}}_{A} + \underbrace{X\tilde{\beta} - X\beta_0}_{X(\tilde{\beta}-\beta_0)})'(A + B)$$

$$= (y - X\tilde{\beta})'(y - X\tilde{\beta}) + 2(\tilde{\beta} - \beta_0)'X'(y - X\tilde{\beta}) + (\tilde{\beta} - \beta_0)'X'X(\tilde{\beta} - \beta_0)$$

$$= (y - X\tilde{\beta})'(y - X\tilde{\beta}) + 2(\tilde{\beta} - \beta_0)'\Lambda' k + (\tilde{\beta} - \beta_0)'X'X(\tilde{\beta} - \beta_0)$$

$$[\because \Lambda\tilde{\beta} - \Lambda\beta_0 = 0] \quad [\because X'y - X'X\tilde{\beta} = \Lambda' k]$$

$$= (\underbrace{y - X\tilde{\beta}}_{a})'(y - X\tilde{\beta}) + (\tilde{\beta} - \beta_0)'X'\underbrace{X(\tilde{\beta} - \beta_0)}_{b}$$

$$\geq (y - X\tilde{\beta})'(y - X\tilde{\beta})$$

Conditional SSE

$$\min_{\underset{\sim}{\beta}} (\underset{\sim}{y} - X\underset{\sim}{\beta})'(\underset{\sim}{y} - X\underset{\sim}{\beta}) , \boxed{\begin{matrix} \Lambda\underset{\sim}{\beta} = \underset{\sim}{d} \\ \Lambda H = \Lambda \end{matrix}}$$

$$= (\underset{\sim}{y} - X\tilde{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\tilde{\underset{\sim}{\beta}})$$

$$= (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}} + X\hat{\underset{\sim}{\beta}} - X\tilde{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}} + X\hat{\underset{\sim}{\beta}} - X\tilde{\underset{\sim}{\beta}})$$

$$= SSE + (\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}})' \underbrace{X'X(\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}})}_{\Lambda'\underset{\sim}{k}}$$

$$= SSE + (\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}})'\Lambda'\underset{\sim}{k}$$

$\left[\because (\hat{\underset{\sim}{\beta}} - \tilde{\underset{\sim}{\beta}})X'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}) = 0 \left[\text{as } X'\underset{\sim}{y} - X'X\hat{\underset{\sim}{\beta}} = 0 \text{ from normal eq}^{n}\right]\right]$

$$= SSE + (\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\tilde{\underset{\sim}{\beta}})'\underset{\sim}{k}$$

$$= SSE + (\Lambda\hat{\underset{\sim}{\beta}} - \underset{\sim}{d})'(\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\underset{\sim}{\beta}} - \underset{\sim}{d})$$

[putting the value of $\underset{\sim}{k}$]

Theorem 7:

The cond^{n}al minimum of the sum of squares of the residuals $(\underset{\sim}{y} - X\underset{\sim}{\beta})'(\underset{\sim}{y} - X\underset{\sim}{\beta})$ in the model $\left[\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}, \langle E(\underset{\sim}{\varepsilon}) = 0, V(\underset{\sim}{\varepsilon}) = \sigma^2 I\rangle\right.$, subject to m cond^{n}s $\boxed{\Lambda\underset{\sim}{\beta} = \underset{\sim}{d}}$ where $\Lambda\underset{\sim}{\beta}$ are estimable and $\left.\langle \text{rank}(\Lambda) = m\rangle\right]$ exceeds the unconditional minimum or SSE by a quantity which is a quadratic form in the BLUEs of the parametric fu^{n} $\Lambda\underset{\sim}{\beta}$ measure from $\underset{\sim}{d}$; The matrix of this quadratic form is the inverse of the var-cov matrix of the BLUEs excluding the factor σ^2

Q) prove that the diff. b/w the cond^{n}al and uncond^{n}al SSEs can be expressed as $\underset{\underset{\sim}{a}^{m+1}}{\text{Max}} \dfrac{\{\underset{\sim}{a}'(X\hat{\underset{\sim}{\beta}}-\underset{\sim}{d})\}^2}{\underset{\sim}{a}'(\Lambda S^{-}\Lambda')\underset{\sim}{a}}$,

proof

Let, $(\Lambda S^{-}\Lambda')^{1/2}\underset{\sim}{a} = \underset{\sim}{u}$

$(\Lambda S^{-}\Lambda')^{-1/2}(\Lambda\hat{\underset{\sim}{\beta}}-\underset{\sim}{d}) = \underset{\sim}{v}$

$\underset{\sim}{u}'\underset{\sim}{v} = \underset{\sim}{a}'(\Lambda S^{-}\Lambda')^{1/2}(\Lambda S^{-}\Lambda')^{-1/2}(\Lambda\hat{\underset{\sim}{\beta}}-\underset{\sim}{d})$

$= \underset{\sim}{a}'(\Lambda\hat{\underset{\sim}{\beta}}-\underset{\sim}{d})$ [$\because (\Lambda S^{-}\Lambda')$ is symmetric as it is coming from var-cov matrix]

$\underset{\sim}{u}'\underset{\sim}{u}$

$= \underset{\sim}{a}'(\Lambda S^{-}\Lambda')\underset{\sim}{a}$

$$\therefore \frac{\{\underset{\sim}{a}'(X\hat{\underset{\sim}{\beta}}-\underset{\sim}{d})\}^2}{\underset{\sim}{a}'(\Lambda S^{-}\Lambda')\underset{\sim}{a}} = \frac{(\underset{\sim}{u}'\underset{\sim}{v})^2}{\underset{\sim}{u}'\underset{\sim}{u}} \le (\underset{\sim}{v}'\underset{\sim}{v})$$

[From C-S inequality]

$$\boxed{\underset{\sim}{v}'\underset{\sim}{v} = (\Lambda\hat{\underset{\sim}{\beta}}-\underset{\sim}{d})'(\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\underset{\sim}{\beta}}-\underset{\sim}{d})}$$

'=' holds when,

$\underset{\sim}{u} \propto \underset{\sim}{v}$

$(\Lambda S^{-}\Lambda')^{1/2}\underset{\sim}{a} \propto (\Lambda S^{-}\Lambda')^{-1/2}(\Lambda\hat{\underset{\sim}{\beta}}-\underset{\sim}{d})$

$$\Rightarrow \underbrace{\underset{\sim}{a} \propto (\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\underset{\sim}{\beta}}-\underset{\sim}{d})}_{\underset{\sim}{a}\text{ is maximum}}$$

Testing:

$H_0: \underset{\sim}{\lambda}_{(1)}'\underset{\sim}{\beta} = d_1$

$\underset{\sim}{\lambda}_{(2)}\underset{\sim}{\beta} = d_2$

$\vdots$

$\underset{\sim}{\lambda}_{(m)}\underset{\sim}{\beta} = d_m$

$H_0: \Lambda\underset{\sim}{\beta} = \underset{\sim}{d}$

⟨The hypothesis H_0 is called "testable" if $\Lambda \underset{\sim}{\beta}$ is estimable i.e $\Lambda H = \Lambda$⟩

■ By thm. 6, $$\frac{\frac{(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta})'(\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\underset{\sim}{\beta}} - \Lambda\underset{\sim}{\beta})}{m}}{SSE/n-r}$$ has an F-distⁿ

with $(m, n-r)$ d.f, and if H_0 is true then, $\Lambda\underset{\sim}{\beta} = \underset{\sim}{d}$, Hence to test H_0, we use the statistic, $\boxed{\frac{SSE_{H_0}/m}{SSE/(n-r)}}$

where, $\boxed{SSE_{H_0} = (\Lambda\hat{\underset{\sim}{\beta}} - \underset{\sim}{d})'(\Lambda S^{-}\Lambda')^{-1}(\Lambda\hat{\underset{\sim}{\beta}} - \underset{\sim}{d})}$, the above statistic will also follow F-distⁿ with $(m, n-r)$ d.f.

Distribution Theory

$X \sim N(\mu, \sigma^2)$

$Y = aX + b$

$Y \sim N(a\mu + b, a^2\sigma^2)$

$$f_y = \frac{1}{\sqrt{2\pi}\,|a|\sigma} e^{-\frac{1}{2}\frac{(y-a\mu-b)^2}{a^2\sigma^2}}$$

$X \sim N(\mu, \sigma^2)$

$Y = X^2 \Rightarrow X = \begin{cases} \sqrt{Y} & \text{if } x \in (0,\infty) \\ -\sqrt{Y} & \text{if } x \in (-\infty, 0) \end{cases}$

$\boxed{\cancel{F_y(Y \le y)}}$

$\boxed{\cancel{X = R}}$ $= \boxed{\cancel{R(X}}$

$$G_Y(y) = P(Y \le y)$$

$$= P(X^2 \le y)$$

$$= P(X \le \sqrt{y}) + P(X \ge -\sqrt{y})$$

$$= P(X \le \sqrt{y}) + 1 - P(X \le -\sqrt{y})$$

$$= F(\sqrt{y}) + 1 - F(-\sqrt{y})$$

$$\boxed{= \int_{-\infty}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx}$$

$$\frac{d}{dy} G_Y(y) = g_Y(y) = f(\sqrt{y})\frac{1}{2\sqrt{y}} + f(-\sqrt{y})\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\sqrt{y}-\mu}{\sigma}\right)^2} + \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{-\sqrt{y}-\mu}{\sigma}\right)^2}\right]$$

$$= \frac{1}{2\sqrt{2\pi y}\,\sigma} e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}} \left[e^{-\frac{1}{2}\left(\frac{y-2\sqrt{y}\mu}{\sigma^2}\right)} + e^{-\frac{1}{2}\left(\frac{y+2\sqrt{y}\mu}{\sigma^2}\right)}\right]$$

$$= \frac{e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}}}{\sigma\sqrt{2\pi}} e^{-\frac{y}{2\sigma^2}} y^{-1/2} \times \underbrace{\frac{e^{-\mu\sqrt{y}/\sigma^2} + e^{\mu\sqrt{y}/\sigma^2}}{2}}_{\cosh\left(\frac{\mu\sqrt{y}}{\sigma^2}\right)}$$

$; 0 < y < \infty$

$\frac{Y}{\sigma^2} = \frac{X^2}{\sigma^2}$ is called a non-central χ^2 with one d.f. and non-centrality parameter $\psi^2 = \mu^2/\sigma^2$

Non-central χ^2

Let, $x_1, x_2, \ldots, x_n$ be n independent Normally distributed r.v with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ then the statistic $\sum_{i=1}^{n}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2$ would be a central chi-square with n d.f. But if we take $\sum_{i=1}^{n}\frac{x_i^2}{\sigma_i^2}$ then we have a non-central chi-square with n d.f and non-centrality parameter $\psi^2 = \sum_{i=1}^{n} \mu_i^2/\sigma_i^2$

property:-

If Y_1^2 and Y_2^2 are independently distributed as non-central chi-square r.vs with ~~μ_1 and μ_2 d.f~~ n_1 and n_2 d.f with non-centrality parameters ψ_1^2 and ψ_2^2 respectively then $Y_1^2 + Y_2^2$ has also non-central chi-square distn with d.f n_1+n_2 and non-centrality parameters $\psi_1^2 + \psi_2^2$

$$\boxed{f(\chi^2) = e^{-\psi^2/2} e^{-\chi^2/2} \sum_{j=0}^{\infty} \frac{(\psi^2/2)^j}{j!} \times \frac{(\chi^2)^{j+(n-2)/2}}{2^{n/2}\,\Gamma\left(\frac{n}{2}+j\right)}}$$

Non-central t

Let x be distributed as $N(\mu, \sigma^2)$ and let Y^2, independent of x have the central χ^2 distn with n d.f then $\frac{x-\mu}{\sigma}\Big/\sqrt{Y^2/n}$ has a central t-distn with d.f n, now $t = \frac{x/\sigma}{\sqrt{Y^2/n}}$ will have a different distn unless $\mu = 0$. This is called non-central t with n d.f and with non-centrality parameter μ/σ.

$\Rightarrow \quad t = \dfrac{X/\sigma}{\sqrt{Y^2/n}}$

Let, $Z = \dfrac{X}{\sigma} \sim N\left(\dfrac{\mu}{\sigma}, 1\right)$

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(Z - \frac{\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-Z^2/2 - \mu^2/2\sigma^2} \cdot e^{Z\mu/\sigma}$$

$$e^{Z\mu/\sigma} = \sum_{r=0}^{\infty} \left(\frac{Z\mu}{\sigma}\right)^r / r! = \sum_{r=0}^{\infty} \frac{\left(\frac{\mu}{\sigma}\right)^r Z^r}{r!}$$

$$f(y) = \frac{1}{2^{n/2 - 1/2}\Gamma n/2} e^{-y^2/2} y^{n-1}$$

$$\therefore t = \frac{\sqrt{n}\, Z}{Y} \Rightarrow Z = \frac{tu}{\sqrt{n}}$$

$$u = Y$$

$$\frac{1}{J} = \begin{vmatrix} \frac{\partial t}{\partial z} & \frac{\partial t}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial u}{\partial y} \end{vmatrix}$$

$$\Rightarrow \frac{1}{J} = \begin{vmatrix} \frac{\sqrt{n}}{Y} & -\frac{\sqrt{n}\,Z}{Y^2} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{\sqrt{n}}{Y} = \frac{\sqrt{n}}{u}$$

$$\therefore J = \frac{u}{\sqrt{n}}$$

$$f(t,u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2u^2}{2n} - \frac{\mu^2}{2\sigma^2}} \sum_{r=0}^{\infty} \frac{\left(\frac{\mu}{\sigma}\right)^r}{r!} \frac{(tu)^r}{n^{r/2}}$$

$$\times \frac{1}{2^{n/2 - 1/2}\Gamma n/2} e^{-u^2/2} u^{n-1} \cdot \frac{u}{\sqrt{n}}$$

$$= \frac{1}{2^{n/2}\Gamma n/2 \, \Gamma 1/2} e^{-\mu^2/2\sigma^2} \times \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\mu}{\sigma}\right)^r e^{-\frac{u^2t^2}{2n} - u^2/2} \left(\frac{tu}{\sqrt{n}}\right)^r \frac{u^n}{\sqrt{n}}$$

Non-central F

Let χ_1^2 be a non-central chi-square with df n_1 and non-centrality parameter ψ^2 and let χ_2^2 be a central chi-square with df n_2 then the ratio $\dfrac{\chi_1^2/n_1}{\chi_2^2/n_2}$ is defined to be a non-central F statistic (n_1, n_2) d.f and with non-centrality parameter ψ^2.

Let $\underset{\sim}{x} = (x_1, x_2, \ldots, x_p)$ be distributed in the multivariate normal form as $N_p(\underset{\sim}{\mu}, \Sigma)$. so the joint dist^n will be, $\dfrac{1}{(2\pi)^{p/2}\Sigma^{-1/2}} e^{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})}$, since Σ^{-1} is p.d we can have a full rank matrix V such that, $\Sigma^{-1} = VV'$, then, $\underset{\sim}{y} = V'(\underset{\sim}{x}-\underset{\sim}{\mu}) \Rightarrow \underset{\sim}{x} = (V')^{-1}\underset{\sim}{y} + \underset{\sim}{\mu}$

$$f(\underset{\sim}{y}) = \frac{1}{(2\pi)^{p/2}} e^{-\underset{\sim}{y}'\underset{\sim}{y}/2}$$

$$\underset{\sim}{Z}^{k\times 1} = \underset{\sim}{a}^{k\times 1} + B^{k\times p}\underset{\sim}{X}^{p\times 1}$$

$$= \underset{\sim}{a} + B\mu + B(V')^{-1}\underset{\sim}{y}$$

$$\therefore E(\underset{\sim}{Z}) = a + B\underset{\sim}{\mu}$$

$$D(\underset{\sim}{Z}) = B(V')^{-1} D(\underset{\sim}{y}) V^{-1} B'$$

$$= B(VV')^{-1}B' \quad [\because D(\underset{\sim}{y}) = I]$$

$$= B\Sigma B'$$

Define, $Q(\underset{\sim}{x}) = (\underset{\sim}{x}-\underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})$

$$= \underset{\sim}{y}'\underset{\sim}{y} \sim \chi^2_p$$

Let, $Q(\underset{\sim}{x}) = \underset{\sim}{x}'\Sigma^{-1}\underset{\sim}{x}$

$Z = V'x \Rightarrow E(Z) = V'\underset{\sim}{\mu}$

$Z'Z = x'VV'x$
$= x'\Sigma^{-1}x$
$= Q(\underset{\sim}{x})$

$Z \sim \chi^2_{p, \psi^2}$

$D(Z)$
$= V'\Sigma V$
$= V'(VV')^{-1}V$
$= I$

$\psi^2 = (V'\underset{\sim}{\mu})'(V'\underset{\sim}{\mu})$
$= \underset{\sim}{\mu}'VV'\underset{\sim}{\mu}$
$= \underset{\sim}{\mu}'\Sigma^{-1}\underset{\sim}{\mu}$

<u>Consider the following lin. model</u>

$8 = \mu + t_1 + e_{11}$
$6 = \mu + t_1 + e_{12}$
$5 = \mu + t_2 + e_{21}$
$3 = \mu + t_2 + e_{23}$
$12 = \mu + t_3 + e_{31}$
$14 = \mu + t_3 + e_{32}$

Find BLUE's

i) $t_1 - t_2$
ii) $t_2 - t_3$
iii) μ
iv) t_1
v) $t_1 + t_2$
vi) $2t_1 + 7t_2 - 2t_3$
vii) $2\mu + t_1 + t_2$
viii) $\mu + \left(\frac{t_1 + t_2 + t_3}{3}\right)$
ix) $\mu + t_1$
x) $\frac{t_1 + t_2}{2} - t_3$

$\Rightarrow$

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$X'X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}_{4\times 6} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}_{6\times 4} = \begin{pmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

$$6\mu + 2t_1 + 2t_2 + 2t_3 = 48 = q_1$$
$$2\mu + 2t_1 + 0.t_2 + 0.t_3 = 14 = q_2$$
$$2\mu + 2t_2 + 0.t_1 + 0.t_3 = 8 = q_3$$
$$2\mu + 2t_3 + 0.t_1 + 0.t_2 = 26 = q_4$$

$$q_1 = q_2 + q_3 + q_4 \qquad X'\underset{\sim}{y} = \begin{pmatrix} 48 \\ 14 \\ 8 \\ 26 \end{pmatrix}$$

Let, $\hat{\mu} = 0$

$$\therefore 2t_1 + 2t_2 + 2t_3 = q_1$$
$$2t_1 = q_2$$
$$2t_2 = q_3$$
$$2t_3 = q_4$$

$$\therefore \hat{\mu} = 0.q_1 + 0.q_2 + 0.q_3 + 0.q_4$$
$$\hat{t}_1 = 0.q_1 + \frac{1}{2}.q_2 + 0.q_3 + 0.q_4$$
$$\hat{t}_2 = 0.q_1 + 0.q_2 + \frac{1}{2}.q_3 + 0.q_4$$
$$\hat{t}_3 = 0.q_1 + 0.q_2 + 0.q_3 + \frac{1}{2}.q_4$$

$$\therefore (X'X)^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$H = (X'X)^-(X'X) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{\lambda}'H = (\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$(\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4) = (\lambda_2+\lambda_3+\lambda_4\quad \lambda_2\quad \lambda_3\quad \lambda_4)$$

Estimable if, $\boxed{\lambda_1 = \lambda_2+\lambda_3+\lambda_4}$

i) Estimable vi) Not estimable
ii) " vii) "
iii) Not " viii) "
iv) " " ix) "
v) " " x) "

$$(X'X)^{-}(X'\underset{\sim}{y}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}_{4\times 4}\begin{pmatrix} 48 \\ 14 \\ 8 \\ 26 \end{pmatrix}_{4\times 1} = \begin{pmatrix} 0 \\ 7 \\ 4 \\ 13 \end{pmatrix}$$

BLUE

i) $t_1 - t_2$ $\quad (0\ 1\ -1\ 0)\begin{pmatrix} 0 \\ 7 \\ 4 \\ 13 \end{pmatrix} = 3$

i) $t_2 - t_3$ $\quad (0\ 0\ 1\ -1)\begin{pmatrix} 0 \\ 7 \\ 4 \\ 13 \end{pmatrix} = -9$

vii) $2\mu + t_1 + t_2$ $\quad (2\ 1\ 1\ 0)\begin{pmatrix} 0 \\ 7 \\ 4 \\ 13 \end{pmatrix} = 11$

viii) $\mu + \left(\frac{t_1+t_2+t_3}{3}\right)$ $\quad (1\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{3})\begin{pmatrix} 0 \\ 7 \\ 4 \\ 13 \end{pmatrix} = 8$

ix) $\mu + t_1$ $\quad (1\ 1\ 0\ 0)\begin{pmatrix} 0 \\ 7 \\ 4 \\ 13 \end{pmatrix} = 7$

x) $\frac{t_1+t_2}{2} - t_3$ $\quad (0\ \frac{1}{2}\ \frac{1}{2}\ -1)\begin{pmatrix} 0 \\ 7 \\ 4 \\ 13 \end{pmatrix} = -\frac{15}{2}$

Quadratic Forms

Thm 1:- If $\underset{\sim}{y} \sim N(\underset{\sim}{\mu}_y, \Sigma_y)$ then,

$$Z = A\underset{\sim}{y} \sim N(\mu_Z = A\underset{\sim}{\mu}_y, \Sigma_Z = A\Sigma_y A')$$

Thm 2:-

$\underset{\sim}{y}^{n\times 1} \sim N(0,1)$, then $\underset{\sim}{y}'\underset{\sim}{y} \sim \chi^2_n$

Thm 3:-

$\underset{\sim}{y} \sim N(0, \sigma^2 I)$ and M is a symmetric idempotent matrix of rank m,

then, $$\frac{\underset{\sim}{y}' M \underset{\sim}{y}}{\sigma^2} \sim \chi^2_{(tr(M))}$$

proof:

$$Q' M Q' = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

Q be an orthogonal matrix

Let, $\underset{\sim}{v} = Q'\underset{\sim}{y}$ $\quad V(\underset{\sim}{v}) = Q'\sigma^2 I Q'$

$\Rightarrow E(\underset{\sim}{v}) = 0$ $\quad = \sigma^2 Q' Q'$

$= \sigma^2 I \quad [\because Q'Q' = I]$

$$\underset{\sim}{y} = (Q')^{-1}\underset{\sim}{v}$$
$$= (Q^{-1})'\underset{\sim}{v}$$
$$= Q'\underset{\sim}{v} \quad \left[\begin{array}{l}\because Q \text{ orthogonal} \\ \therefore Q^{-1} = Q'\end{array}\right]$$

$$\therefore \frac{\underset{\sim}{y}' M \underset{\sim}{y}}{\sigma^2} = v' Q' M Q' v$$
$$= \frac{1}{\sigma^2} \underset{\sim}{v}' \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \underset{\sim}{v}$$
$$= \frac{1}{\sigma^2} \sum_{i=1}^{tr\, m} v_i^2$$

~~This is~~

Thm 4:

$\underset{\sim}{y} \sim N(0, \sigma^2 I)$

M is a symmetric idempotent matrix of order n.

$L^{k\times n}$ then $L\underset{\sim}{y}$ and $\underset{\sim}{y}'M\underset{\sim}{y}$ they are independently distributed if $LM = 0$

Proof:

$$Q M Q' = \begin{bmatrix} I^{r\times r} & 0 \\ 0 & 0 \end{bmatrix}$$

Let, $\boxed{tr\, M = r}$

$$\underset{\sim}{v} = Q\underset{\sim}{y} \Rightarrow \underset{\sim}{y} = (Q)^{-1}\underset{\sim}{v} = Q'\underset{\sim}{v}$$

$$\underset{\sim}{v} = \begin{pmatrix} \underset{\sim}{v}_1 \\ \underset{\sim}{v}_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \\ v_{r+1} \\ \vdots \\ v_n \end{pmatrix}$$

$\therefore \underset{\sim}{y}'M\underset{\sim}{y}$

$= \underset{\sim}{v}'QMQ'\underset{\sim}{v}$

$= \underset{\sim}{v}'\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\underset{\sim}{v}$

$= \underset{\sim}{v}_1'\underset{\sim}{v}_1$

Let, $C = L^{k\times n} Q'^{\,n\times n}$

$$C = \left(C_1^{k\times r} \;\; C_2^{k\times(n-r)}\right) = (C_1 \;\; C_2)\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$C(QMQ')$ $\Rightarrow C_1 = 0$

$= LQ'QMQ'$

$= LMQ' = 0 \quad [\because LM = 0]$

$\therefore L\underset{\sim}{y} = LQ'Q\underset{\sim}{y}$

$= C\underset{\sim}{v}$

$= (0 \;\; C_2)\begin{pmatrix} \underset{\sim}{v}_1 \\ \underset{\sim}{v}_2 \end{pmatrix}$

$= C_2\underset{\sim}{v}_2$

Thm 5:

$\underset{\sim}{y} \sim N(0, I)$

A & B is a symmetric idempotent matrix of ~~order~~ rank p and s.

$$BA = 0$$

then,

$\underset{\sim}{y}' A \underset{\sim}{y}$ & $\underset{\sim}{y}' B \underset{\sim}{y}$ independent.

Proof:

$$QAQ' = \Lambda = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \qquad \underset{\sim}{v} = Q\underset{\sim}{y}$$

$$\underset{\sim}{y}' A \underset{\sim}{y} = \underset{\sim}{v}_1' \underset{\sim}{v}_1$$

Define,

$$G = QBQ'$$

$$\Rightarrow G\Lambda = QBQ'QAQ'$$

$$= QBAQ' \quad [\because Q'Q = I]$$

$$= 0 \quad [\because BA = 0]$$

$$GQAQ'$$

$$= \begin{pmatrix} G_1 & G_2 \\ G_2' & G_3 \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \overset{p \times p}{G_1} & 0 \\ \overset{n-p \times p}{G_2'} & 0 \end{pmatrix}$$

$$= 0$$

$$\therefore G_1 = 0, \; G_2 = 0$$

$$\therefore G = \begin{pmatrix} 0 & 0 \\ 0 & G_3 \end{pmatrix}$$

$\underset{\sim}{y}' B \underset{\sim}{y}$

$= \underset{\sim}{y}' Q'Q B Q'Q \underset{\sim}{y}$

$= \underset{\sim}{v}' G \underset{\sim}{v}$

$= \begin{pmatrix} \underset{\sim}{v}_1' & \underset{\sim}{v}_2' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix} \begin{pmatrix} \underset{\sim}{v}_1 \\ \underset{\sim}{v}_2 \end{pmatrix}$

$= \underset{\sim}{v}_2' G_3 \underset{\sim}{v}_2$

Th Wishart dist^n

$$M^{p\times p} = X'X^{m\times p} \qquad X \sim N_p(0, \Sigma)$$

$$M \sim \text{Wishart}_p(\Sigma, m)$$

If, $\quad W_p(\Sigma, m)$

$p = 1, \quad X'X = \sum x_i^2 \sim \sigma^2 \chi^2_m$

proo

$$X'X = \sum_{i=1}^{m} x_i x_i'$$

$$\therefore E(M) = \sum_{i=1}^{m} E(x_i x_i')$$

$$= \sum_{i=1}^{m} \Sigma = m\Sigma$$

$$M \sim W_p(\Sigma, m)$$

$$B'MB^{p\times q} \sim W_q(B'\Sigma B, m)$$

$$B'MB = B'X'XB = (XB)'(XB)$$

$$= Y'Y$$

$$Y \sim N(0, B'\Sigma B)$$

If, $B = \Sigma^{-1/2}$

$$\Sigma^{-1/2} M \Sigma^{1/2} \sim W_q(I, m)$$

$\circledast$ $M \sim W_p(\Sigma, m)$, $\underset{\sim}{a} \in \mathbb{R}^p$, $\underset{\sim}{a}'\Sigma \underset{\sim}{a} \neq 0$

then $\dfrac{\underset{\sim}{a}'M\underset{\sim}{a}}{\underset{\sim}{a}'\Sigma\underset{\sim}{a}}$ is χ^2_m

proof $\Rightarrow$ $\underset{\sim}{a}'^{1\times p} M \underset{\sim}{a}^{p\times 1} \sim W_1(\underset{\sim}{a}'\Sigma \underset{\sim}{a}, m)$

$$\therefore \frac{\underset{\sim}{a}'M\underset{\sim}{a}}{\underset{\sim}{a}'\Sigma\underset{\sim}{a}} \sim W_1(I, m)$$

$$\sim \chi^2_m$$

Hotelling T^2

$$T^2 = N\bar{x}'S^{-1}\bar{x}$$

$$x \sim N_m(\underset{\sim}{\mu}, \Sigma) \quad \begin{matrix} N = n+1, \\ N_m(\underset{\sim}{\mu}, \Sigma) \end{matrix}$$

$$\frac{T^2}{n} \cdot \frac{n-m+1}{m} \sim F_{m, n-m+1}(\delta)$$

$$\delta = N\underset{\sim}{\mu}'\Sigma^{-1}\underset{\sim}{\mu}$$

$$\therefore \bar{x} \sim N_m\left(\mu, \frac{1}{n}\Sigma\right)$$

$$S \sim W_m\left(\frac{1}{n}\Sigma, n\right)$$

$$\therefore \frac{T^2}{n} = \frac{N\bar{x}'S^{-1}\bar{x}}{n\ \bar{x}'\Sigma^{-1}\bar{x}} \cdot \bar{x}'\Sigma^{-1}\bar{x}$$

$$= \frac{N \cdot x'\Sigma^{-1}\bar{x}}{n \cdot \dfrac{\bar{x}'\Sigma^{-1}\bar{x}}{\bar{x}'S^{-1}\bar{x}}}$$

$$N\bar{x}'\Sigma^{-1}\bar{x} \sim \chi^2_m(\delta), \ \delta = N\underset{\sim}{\mu}'\Sigma^{-1}\underset{\sim}{\mu}$$

$$\boxed{n \cdot \frac{\bar{x}'S^{-1}\bar{x}}{\bar{x}'\Sigma^{-1}\bar{x}} \sim \chi^2_{n-m+1}}$$

$$d^2 = (\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)' \Sigma^{-1} (\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)'$$

$$D^2 = (\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2)' S^{-1} (\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2)$$

$$\underset{\sim}{x}_i \overset{iid}{\sim} N_p(\mu_i, \Sigma_i) \quad i = 1(1)2$$

when, $\underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2$, $\Sigma_1 = \Sigma_2$

$$\frac{n_1 n_2}{n} D^2 \sim \text{Hotelling } T^2(p, n-2)$$

$$\bar{x}_i \sim N_p\left(\mu_i, \frac{1}{n_i}\Sigma_i\right)$$

$$\therefore \mathcal{D}^* = \bar{x}_1 - \bar{x}_2 \sim N_p\left(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2, \frac{1}{n_1}\Sigma_1 + \frac{1}{n_2}\Sigma_2\right)$$

when, $\underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2$, $\Sigma_1 = \Sigma_2$

$$\therefore \mathcal{D}^* \sim N_p(0, c\Sigma)$$

$$m_i = n_i S_i \sim W_p(\Sigma_i, n_i - 1)$$

$$m = (n-2) S_u = m_1 + m_2 \sim W_p(\Sigma, n-2)$$

$$cm \sim W_p(c\Sigma, n-2)$$

$$(n-2)\mathcal{D}^{*\prime}(cm)^{-1}\mathcal{D}^* \sim T^2(p, n-2)$$

$$\Rightarrow \frac{n_1 n_2}{n} \mathcal{D}^2 \sim T^2(p, n-2)$$

Wilks lambda

$$\Lambda = \frac{|A|}{|A+B|} \qquad A \sim W_p(I, m) \qquad B \sim W_p(I, n)$$

$3K, 3K+1, 3K+2$ $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$

a = # of elements

$\boxed{a \ngtr 7}$ $\boxed{a \le 6}$

$3K \to 3, 6, 9, 12, 15$

$3K+1 \to 1, 4, 7, 10, 13$

$3K+2 \to 2, 5, 8, 11, 14$

$\boxed{a=6} \to \left[\begin{array}{c|c} 3K \to 1 & 3K \to 1 \\ 3K+1 \to 5 & 3K+2 \to 5 \end{array}\right]$

$\boxed{5+5=10}$

$\boxed{a=5} \to \left[\begin{array}{c|c} 3K+1 \to 4 & 3K+2 \to 4 \\ 3K \to 1 & 3K \to 1 \end{array}\right]$ $\left[\begin{array}{c} 3K+2 \to 5 \to 1 \\ 3K+1 \to 5 \to 1 \end{array}\right]$

$2 \times {}^5C_4 \times {}^5C_1$ $\boxed{2}$

$= \boxed{25 \times 2 = 50}$

$a=4 \to \left[\begin{array}{c|c} 3K+1 \to 3 & 3K+2 \to 3 \\ 3K \to 1 & 3K+2 \to 1 \end{array}\right]$ $\left[\begin{array}{c} 3K+2 \to 4 \to 5 \\ 3K+1 \to 4 \to 5 \end{array}\right]$

$2 \times {}^5C_3 \times {}^5C_1$ $\boxed{10}$

$= 100$

$a=3 \to \left[\begin{array}{c|c} 3K+2 \to 2 & 3K+1 \to 2 \\ 3K \to 1 & 3K \to 1 \end{array}\right]$ $\left[\begin{array}{c} 3K+2 \to 3 \to 10 \\ 3K+1 \to 3 \to 10 \end{array}\right]$

$2 \times {}^5C_2 \times {}^5C_1$ $\boxed{20}$

$= \boxed{100}$

$a=2$ $\left[\begin{array}{c|c} 3K+2 \to 1 & 3K+1 \to 1 \\ 3K \to 1 & 3K \to 1 \end{array}\right]$

$2 \times {}^5C_1 \times {}^5C_1$

$= \boxed{50}$

$3K+1 \to 2$

$3K+2 \to 2$

${}^5C_2 + {}^5C_2$

$= \boxed{20}$ $10 + 10$

$a=1, \to \boxed{15}$

$\boxed{\phi \to \boxed{1}}$