Linearly Dept and Indepl. Vectors :-

$\{x_1 x_2 \dots x_n\} \xrightarrow{\text{indepl.}} \sum_{i=1}^{n} c_i x_i = 0 \Rightarrow c_i = 0 \ \forall i = 1(1)n$

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 \ ; \ c_3 \neq 0$$

$$x_3 = -\frac{c_1}{c_3} x_1 - \frac{c_2}{c_3} x_2 \leftarrow \text{dept.}$$

Null Vector $\longrightarrow$ dept vector

$c \times 0 = 0$, here 'c' can be non zero

Vector Space :- (addition & scalar multiplication)

$$x \in V \quad y \in V \Rightarrow x+y \in V$$

$$kx \in V$$

$\underline{x}'\underline{y} = 0 \longrightarrow$ Orthogonal Vector.

Orthogonality $\Rightarrow$ indept.

Graham-Smith Orthogo

Orthogonal + unit length = Orthonormal. $\sqrt{\sum y_i^2} = \|y_i\| = 1$

$x_1, x_2, \dots x_n$

$y_1, y_2 \dots y_n$

$x_1 = y_1$

$y_2 = x_2 + b_{21} x_1 \qquad y_2' y_1 = 0 \longrightarrow \quad x_2' + b_{21} x_1 = 0$

$x_2' x_1 + b_{21} x_1' x_1 = 0$

$\Rightarrow b_{21} = -\frac{x_2' x_1}{x_1' x_1}$

$\cancel{y_3 = x_3 + b_{32} x_2}$

$\Downarrow$

$\cancel{x_3 + b_{32} x_2 = 0}$

$\cancel{x_3' x_2 + b_{32} x_2' x_2 = 0. \Rightarrow b_{32} = -\frac{x_3' x_2}{x_2' x_2}}$

Check!

$$y_3 = x_3 + b_{31} x_1 + b_{32} x_2$$

$$= x_3' x_2 + b_{31} \cdot x_1' x_1 + b_{32} x_2' x_2$$

$$=$$

$y_3' y_1 = 0$

$y_3' y_2 = 0.$

$$x_3' y_2 + b_{31} x_1' y_2 + b_{32} x_2' y_2 = 0.$$

$$x_3' y_1 + b_{31} x_1' y_1 + b_{32} x_2' y_1 = 0$$

$$\boxed{\underline{b}_{ij} = -\frac{x_i' x_j}{x_j' x_j}}$$

$$Z_i = \frac{y_i'}{\|y_i\|}$$ $\left\{ \begin{array}{l} x \to \text{lin. indept} \\ \quad y \to \text{orthogonal} \\ \quad z \to \text{orthonormal} \end{array} \right.$

Graham-Smith Orthogonalization process

Rank:

$$A^{m \times n}$$

$$\text{Rank}(A) \le \min(m, n)$$

$$\text{Rank}(AB) \le r(A) \text{ or } r(B)$$

$$= r(A) \text{ if } B \text{ is n.s. } [\det \ne 0] \text{ [full Rank Matrix]}$$

$$= r(B) \text{ if } A \text{ is n.s.}$$

$$\begin{pmatrix} 8 & 3 \\ 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 8 & 3 \\ 0 & \frac{13}{4} \end{pmatrix} \quad R_2' = R_2 - \frac{1}{4} R_1$$

Premultiplied N.S. Matrix incase of Row operation. $\begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{pmatrix}$

Trace: $tr(AB) = tr(BA)$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Trace $\Rightarrow$ $(a_{11}b_{11} + a_{12}b_{21}$ $+ a_{21}b_{12} + a_{22}b_{22})$

Partition Matrix: (det, inverse).

Partition Matrix

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D - CA^{-1}B|$$
$$= |D|\,|A - BD^{-1}C|$$

$$A = \begin{pmatrix} 2 & 3 \\ 9 & 4 \end{pmatrix} = -19. = a_{22}a_{11} - a_{12}a_{21}$$

$$|A| = \sum_{j \in N(j)} (-1)^{N(j)} \prod_{i=1}^{n} a_{ij_i}$$

$N(j)$: no of inversion needed to get natural ordering

$$|A| = (-1)^0 a_{11}a_{22} + (-1)^1 a_{12}a_{21}$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

$j = (j_1, j_2)$

$\wedge \quad \wedge$

$1\ 2 \quad 1\ 2$

$$\frac{dF}{d\underline{x}} = \begin{pmatrix} \frac{dF}{dx_1} \\ \vdots \\ \frac{dF}{dx_n} \end{pmatrix}$$

$$\frac{d}{d\underline{x}} A = \left(\left(\frac{d}{dx} a_{ij}\right)\right)$$

- $\frac{d}{d\underline{x}}(\underline{a}'\underline{x}) = \frac{d}{dx}(\underline{x}'\underline{a}) = \underline{a}$
- $\frac{d}{d\underline{x}}(\underline{x}'A\underline{x}) = 2A\underline{x}$

* $|A - \lambda I_n| = 0$ → eigen values

* $(A - \lambda I_n)\underline{l_i} = \underline{0}$ → eigen vector

idempotent matrix $A^2 = A$

$\Rightarrow |A|\,|A - I| = 0$. So eigen values are 1 or 0.

→ det 0 or 1

→ Rank = no of nonzero lambda.

→ Rank = trace

$$A = \lambda_1 \underline{l_1}\underline{l_1}' + \lambda_2 \underline{l_2}\underline{l_2}' + \dots + \lambda_n \underline{l_n}\underline{l_n}'$$

Variance
Co-variance Matrix

$$V(X) = \Sigma$$
$$V(A\underline{X}) = A\Sigma A'$$

Linear Models

Normal Dist
↓
testing

$y = X\underline{\beta} + \underline{\epsilon}$

$V(\underline{\epsilon}) = \sigma^2 I$
$E(\underline{\epsilon}) = 0$ } indept — homoscatasticity

$\hat{\beta} = (X'X)^{-1}X'\underline{y}$

Problem of Multi Colinearity

$\hat{l'\beta}$ ⟶ linear parametric fn

~~Hom~~

Estimablity Criteria

Khidsagar → Ch-1, 2, 3

(1) Check estimable function. LD
(2) Gauss - Markov Stat & prov
(2) Cov (Blue, fune) = 0
(4) $\hat{l'\beta} = X'$. (5) error space example

Linear Models

$$\underset{n\times1}{\underline{y}} = \underset{n\times p}{X}\,\underset{p\times1}{\underline{\beta}} + \underset{n\times1}{\underline{\epsilon}}$$

$$\Rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\text{Known}} = \underbrace{\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & \cdots & \cdots & x_{2p} \\ \vdots & & & \vdots \\ x_{n1} & \cdots & \cdots & x_{np} \end{pmatrix}}_{\text{Known.}} \underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix}}_{\substack{\text{Unknown and} \\ \text{to be estimated. : objective}}} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

$\to$ random.

assumption: $E(\underline{\epsilon}) = \underline{0}$; $D(\underline{\epsilon}) = \sigma^2 I$

Col^n of $X = \underline{x}_1 \ldots \underline{x}_p$.

Rows of $X = \underline{X}'_{(1)}, \ldots \underline{X}'_{(n)}$

linear combination of rows of X : $a_1\underline{X}'_{(1)} + a_2\underline{X}'_{(2)} + \cdots + a_n\underline{X}'_{(n)} = \underline{b}' = \underline{a}'X$

linear combination of columns of X : $l_1\underline{x}_1 + l_2\underline{x}_2 + \cdots + l_p\underline{x}_p = X\underline{l} = \underline{m}$

$R(X) = r \leq \min(n, p)$

if $r = p$ $(\leq n)$: Full Rank Model.

$\hat{\beta}$ estimate of β, a function of x, y.

$\hat{\beta} = f(x, y)$ or $\underline{y} - X\hat{\underline{\beta}} = \underline{e}$ (Residuals) [Sampling quantity]

$Y - X\beta = \epsilon$ (Error) [Population "]

$$\underline{\epsilon}'\underline{\epsilon} = (\underline{y} - X\underline{\beta})'(\underline{y} - X\underline{\beta})$$

is equivalent to $\underline{e}'\underline{e} = (\underline{y} - X\hat{\underline{\beta}})'(\underline{y} - X\hat{\underline{\beta}})$

$$= \underline{y}'\underline{y} - 2\hat{\underline{\beta}}'X'\underline{y} + \hat{\underline{\beta}}'X'X\hat{\underline{\beta}}.$$

$$\Rightarrow \frac{d}{d\underline{\beta}}(\underline{e}'\underline{e}) = -2X'\underline{y} + 2(X'X)\hat{\underline{\beta}} = 0.$$

$$\Rightarrow \boxed{X'\underline{y} = (X'X)\hat{\underline{\beta}}} \leftarrow \text{Normal Equation.}$$

Notation $\to$ $\underline{q}^{p\times1} = S^{p\times p}$

Suppose we get 3 eq^n.

$$2x + 3y = 5$$
$$9x - 8y = 1$$
$$4x + 3y = 8.$$

$x=1,\ y=1.$

'p' equations for finding solution.
and, rest one for checking consistency.

Page - 13, 14

$$X'y = (X'X)\underline{\beta}$$

$$S = (X'X) : \text{symmetric}$$

$$\text{Rank}(S) = \text{Rank}(X)$$

$$\Rightarrow p - \dim N(S)$$
$$= p - \dim N(X).$$

$$\Rightarrow \dim N(S) = \dim N(X).$$

$X^{n \times p}$
$\Rightarrow (X'X)^{p \times p}$
$r(A^{m \times n}) = n - \dim(N(A))$

$X\alpha = 0 \Rightarrow \alpha \perp$ rows of X
$(X'X)\alpha = 0 \Rightarrow \alpha \perp$ " " $X'X$
inversely,
$X'X\alpha = 0$
$\Rightarrow \alpha' X' X\alpha = 0$
$\Rightarrow (X\alpha)'(X\alpha) = 0$
$\Rightarrow X\alpha = 0$.

Consistency: $AX = b$
here, $R(A : b) = R(A)$ then it is consistent

$$\boxed{R((X'X) : X'y) \geq R(X'X)} \quad \text{—— ①}$$

$R(AB) \leq R(A)$ / $R(B)$

$$R(X'X : X'\underline{y}) = R[X'(X : \underline{y})] \leq R(X')$$
$$= R(X'X) \quad \text{—— ②}$$

$$\because R((X'X) : X'y) = R(X'X)$$

$\therefore$ the system is Consistent.

($\hat{\beta}$) :-

$$(\underline{y} - X\underline{\beta_0})'(\underline{y} - X\underline{\beta_0})$$
$$= (\underline{y} - X\hat{\underline{\beta}} + X\hat{\underline{\beta}} - X\underline{\beta_0})'(\underline{y} - X\hat{\underline{\beta}} + X\hat{\underline{\beta}} - X\underline{\beta_0})$$
$$= (\underline{y} - X\hat{\underline{\beta}})'(\underline{y} - X\hat{\underline{\beta}}) + \underbrace{(\underline{y} - X\hat{\underline{\beta}})'X(\hat{\underline{\beta}} - \underline{\beta_0})}_{=0} + \underbrace{(\hat{\underline{\beta}} - \underline{\beta_0})'X'(\underline{y} - X\hat{\underline{\beta}})}_{=0}$$
$$+ (\hat{\beta} - \underline{\beta_0})'X'X(\hat{\underline{\beta}} - \underline{\beta_0})$$

$$= SSE + \underline{l}'\underline{l} \quad \text{where } l = X(\hat{\underline{\beta}} - \underline{\beta_0})$$
$$\geq SSE \quad \text{equality holds when } (\underline{\beta_0} = \hat{\underline{\beta}}).$$

Generalised Inverse Of Matrix

$$\underline{A\underline{x} = \underline{u}}$$

Consistent if $R(A) = R(A|\underline{u})$ —— (*)

Defn 1: An $n \times m$ matrix A^- is defined to be a gen. inversed of the $m \times n$ matrix A if for every vector u satisfying (*), $A^-\underline{u}$ is a solution of the equation $A\underline{x} = \underline{u}$.

One method of obtaining A^- is therefore to take an algebric vector u with elements $u_1, u_2 \ldots u_m$ assuming (*) holds and try to solve $A\underline{x} = \underline{u}$. Though $A\underline{x} = \underline{u}$ appears to be m equations in n unknowns actually they may have fewer equation; Suppose there are really only k equations then use any 'suitable', 'consistent' additional $(n-k)$ equations. Since Defn 1 needs only a solution of $A\underline{x} = \underline{u}$ it is immaterial what additional equations we take.

$$x_1 = a^{11}u_1 + \cdots + a^{1m}u_m$$
$$\vdots$$
$$x_n =$$

$$x = \underbrace{((a^{ij}))}_{g\cdot inv.}\underline{u}$$

Example:-

$$A = \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$\Rightarrow$ the equations are :

$$\underline{3x_1 + 5x_2 = u_1}$$
$$6x_1 + 10x_2 = u_2$$
$$9x_1 + 15x_2 = u_3$$

$$3x_1 + 5x_2 = u_1$$
$$\underline{x_2 = 0}$$
$$\therefore x_1 = -\tfrac{1}{3}u_1 + 0\cdot u_2 + 0\cdot u_3$$
$$x_2 = 0\cdot u_2 + 0\cdot u_2 + 0\cdot u_3$$

$$\therefore \underline{x} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\underline{u}$$

if $x_2 = u_2$ then,

$$3x_1 = u_1 - 5u_2$$

$$\therefore \quad x_1 = \frac{1}{3}u_1 - \frac{5}{3}u_2 + 0 \cdot u_3$$

$$x_2 = 0u_1 + 1 \cdot u_2 + 0 \cdot u_3$$

$$\underline{x} = \begin{pmatrix} 1/3 & -5/3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \underline{u}$$

$$A^- = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad AA^- = \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$AA^-A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} = A.$$

Now, $A^- = \begin{pmatrix} 1/3 & -5/3 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$AA^- = \begin{pmatrix} 3 & 5 \\ 6 & 10 \\ 9 & 15 \end{pmatrix} \begin{pmatrix} 1/3 & -5/3 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$\therefore \boxed{AA^-A = A}$$

Def 2

Date: 04.12.23

Def 2: Any $n \times m$ matrix A^- satisfying the relation $AA^-A = A$ is defined as a generalised inverse of $m \times n$ matrix A.

Show that both the Def 1 and Def 2 are equivalent—

Proof: Suppose def 2 holds then,

$$AA^-A\underline{x} = A\underline{x}$$

$$AA^-\underline{u} = \underline{u}$$

Showing that $A^-\underline{u}$ is a solution of $A\underline{x} = \underline{u}$ for every vector $\underline{u}$ for which $A\underline{x} = \underline{u}$ is consistent this shows that $\boxed{\text{defn. 1 holds.}}$

Now, suppose defn. 1 holds then,

let a_i be the ith column vector of a; $i = 1(1)n$

We know rank(A) = no of indept columns of a.

$$\cancel{\text{rank}(A|a_i)} = \text{rank}[A, \underline{a_i}]$$

So, the eq. $A\underline{x} = \underline{a_i}$ is obviously consistent and A^-a_i is a solution.

then, $AA^-\underline{a_i} = \underline{a_i} \;\; \forall i$

$$\Rightarrow \boxed{AA^-A = A}$$

So, both the ~~result~~ definitions are equivalent—

$$\boxed{A^-A = H^{n \times m}}$$

Property 1: $AH = A$

Property 2: $H^2 = A^-AA^-A = A^-A = H$ (idempotent matrix)

Property 3: $r(H) = r(A) = tr(H)$.

~~Property 4~~:

Now, $r(H) \leq$ ~~min~~ $r(A)$

again $r(A) \leq \text{rank}(H)$

combining these two

$r(H) = r(A)$.

The General Solution of the system of homogeneous equation can be expressed as $\tilde{x} = (I-H)z$ where z is any arbitrary vector.

Proof: $A(I-H)$

$$= A - AH$$
$$= A - A$$
$$= 0.$$

$\Rightarrow$ Columns of $(I-H)$, $(b_1, b_2, \ldots b_n)$ are orthogonal to the rows of A.

$$(I-H)^2 = (I-H)(I-H)$$
$$= I - H - H + H^2$$
$$= I - H \text{ , idempotent}$$

$$r(I-H) = tr(I-H).$$
$$= n - r \text{ assuming } rank(A) = r.$$

only $(n-r)$ of the column vectors $b_1, b_2 \ldots b_n$ are linearly indept.

WLG assume $(b_1, b_2 \ldots b_n)$ are linearly indept.

Since A is an $m \times n$ matrix of Rank r, its rows are n-vectors. and therefore we can find at most $n-r$ linearly indept vectors orthogonal to them.

let $(b_1, b_2 \ldots b_{n-r})$ is one such set

if there is any other vector orthogonal to the rows of A it must be a linear combination of $b_1, b_2, \ldots b_{n-r}$.

But this is also equivalent to say that x will be a linear combination of $b_1, b_2, \ldots b_n$. because $b_{n-r+1}, b_{n-r+2} \ldots b_n$ are linear combination of $b_1, b_2 \ldots b_{n-r}$.

Hence x must be of

$$x = z_1 b_1 + z_2 b_2 + \cdots + z_n b_n$$
$$= (b_1, b_2 \ldots b_n) z = (I-H) z$$

Conversely if these hold then $A\underline{x} = A(I-H)\underline{z} = 0$

that means its a general solution.

Date: 05-12-23

Gen Solution of Non-Homogeneous eq.

$$A\underline{x} = \underline{u} \quad \text{——} \ (1)$$

$A^{-}\underline{u}$ is a solution of (1)

$$AA^{-}\underline{u} = \underline{u} \quad \text{—— from (1)}$$

and, $A\underline{x} - AA^{-}\underline{u}$

$= \underline{u} - \underline{u}$

$= 0$

Let, $A\underbrace{(\underline{x} - A^{-}\underline{u})}_{\underline{y}} = 0$

$\therefore A\underline{y} = 0$

for non-homogeneous. $\tilde{\underline{x}} - A^{-}\underline{u} = (I-H)\underline{z}$

$$\Rightarrow \boxed{\tilde{\underline{x}} = A^{-}\underline{u} + (I-H)\underline{z}}$$

Solution of Normal Equation.

$$\underbrace{\underline{X}'\underline{y}}_{u} = \underbrace{(X'X)}_{A}\underbrace{\hat{\underline{\beta}}}_{x}$$

$$\therefore \hat{\underline{\beta}} = (X'X)^{-}\underline{X}'\underline{y} = S^{-}\underline{X}'\underline{y}$$

$$\therefore \hat{\underline{\beta}} = S^{-}\underline{X}'\underline{y} \quad \Rightarrow \boxed{\hat{\beta} = S^{-}q}$$

$$\Rightarrow \boxed{\cancel{\hat{\underline{\beta}} = S^{-}g}}$$

$$\tilde{\beta} = S^{-}g + (I-H)\underline{z}$$

here $H = S^{-}S$

Result 1: if S^- is a gen inv. of $X^-X = S$, it's transpose $(S^-)'$ is also a gen. inv.

$\Rightarrow$ from the def $SS^-S = S$

$\Rightarrow S'(S^-)'S' = S'$

$\Rightarrow S(S^-)'S = S.$ as $S = X'X$

$\Rightarrow S' = XX' = S$

Proved

Result 2: $X = XH$

$\Rightarrow$

$SH = SS^-S = S.$ [def of gen. inv] $H = S^-S$, $S = X'X$

now, $S - SH = 0$

$\Rightarrow (I-H)'(S-SH) = 0.$

ie $(I-H)'S(I-H) = 0$

$\Rightarrow (I-H)'X'X(I-H) = 0$

$\Rightarrow (X(I-H))'(X(I-H)) = 0$

$\Rightarrow X - XH = 0$

$\Rightarrow X = XH$ Proved

Rlt 3: if S_a^- and S_b^- are two G. inverses of $(X'X)$; then $XS_a^-X' = XS_b^-X'$

$\Rightarrow$ by def $H_a = S_a^- S_a$ and $S_b^- S_b = H_b$

$X = XH_a = XS_a^-S_a = XS_a^-X'X$

Similarly $X = XH_b = XS_b^-S_b = XS_b^-X'X$

$\therefore XS_a^-X'X = XS_b^-X'X$

$\Rightarrow XS_a^-X'X - XS_b^-X'X = 0$

$\Rightarrow (XS_a^- X'X - XS_b^- X'X)(XS_a^- - XS_b^-)' = 0$

$\Rightarrow (XS_a^- X' - XS_b^- X')(XS_a^- X' - XS_b^- X')' = 0$

$\Rightarrow XS_a^- X' - XS_b^- X' = 0$

$\Rightarrow \boxed{XS_a^- X' = XS_b^- X'}$ Proved

Result 4 : A Solution of the Normal Eqn is unique if and only if

$\text{rank}(X) = \text{rank}(X'X) = p$

from gen sol. of non homog. eq.

$I - H = 0$ for unique solution.

$\Rightarrow I = H$

$\Rightarrow S^- S = I$

$\therefore S^-$ will be the true inv. of S.

that means we will get unique solution.

Theorem 1 : A necessary and sufficient condition for the expression $\underline{\lambda}'\hat{\underline{\beta}}$; where $\hat{\underline{\beta}}$ is any solution of the Normal Equations $X'y = (X'X)\hat{\beta}$ to have a unique value is $\lambda' = \lambda' H$ where $\hat{\underline{\beta}} = S^- q$ and $H = S^- S$ and $S^- S$ is a Gen. inv. of S.

here, $\underline{\lambda}'\hat{\underline{\beta}} = \lambda_1 \hat{\beta}_1 + \lambda_2 \hat{\beta}_2 + \cdots + \lambda_p \hat{\beta}_p$

Proof

For a non-full rank model there will be an infinite number of solution of the equation $\underline{X}'\underline{y} = \underline{X}'\underline{X}\hat{\beta}$ for $\hat{\beta}$

however, if we don't focus on all the eliments of $\hat{\beta}$, but only a linear function of them say $\underline{\lambda}'\hat{\beta} = \lambda_1\hat{\beta}_1 + \dots + \lambda_p\hat{\beta}_p$

then for different solutions $\hat{\beta}_{(1)}, \hat{\beta}_{(2)}, \dots$ of the normal equation, the expressions $\lambda'\hat{\beta}_1, \lambda'\hat{\beta}_2, \dots$ will be different.

$$\underline{\lambda}'\hat{\underline{\beta}} = \lambda'\hat{\beta}_{(i)} + \lambda(I-H)z_i \qquad i=1,2,\dots \quad \text{---} (*)$$

this shows that if and only if $\lambda'(I-H)=0$ (*) will not involve any arbitary z_i and $\underline{\lambda}'\hat{\underline{\beta}}_{(i)}$ will all have the same value.

necessary sufficient condition

$$\lambda'(I-H)=0 \Rightarrow \boxed{\lambda' = \lambda' H}$$

Defn. Estimability of a Linear Parametric Function :-

If $\hat{\beta}$ is a solution of the normal equation $X'y = X'X\hat{\beta}$, there are two difficulties that arrives in using $\hat{\beta}$ for estimating β;

the first is that $\hat{\beta}$ is not unique

2nd is that $E(\hat{\beta}) = E(S^{-}X'y)$

$$= S^{-}X'E(y)$$
$$= S^{-}X'X\underline{\beta}$$
$$= S^{-}S\underline{\beta}$$
$$= H\underline{\beta}$$

Thus $\hat{\beta}$ is not biased in general.

We therefore abandon the idea of estimating all the elements of β and see whether we can estimate at least some ~~minimum~~ linear ~~function~~ combination of them.

for that we introduce the defn of estimability.

(*) A linear parametric fn $\lambda'\beta$ where $\lambda' = (\lambda_1, \lambda_2, \ldots \lambda_p)$ is said to be estimable if ~~and only if~~ there exists at least one linear function of observation $u'y$ where $u' = (u_1, u_2, \ldots u_n)$ such that $\boxed{E(u'y) = \lambda'\beta}$

using $u' \longrightarrow u'X\beta = \lambda'\beta$

$$\Rightarrow u'X = \lambda'$$

~~for β~~ this means λ' is a linear combination of rows of X.

Conversely if $u'X = \lambda'$ then, $E(u'y)$

$\Rightarrow$ ~~$E(u'X)$~~ $= u'X\beta$

$= \lambda'\beta$

Theorem 2 :- We thus have the following theorem —

(A necessary and sufficient condition for a linear parametric function $\lambda'\beta$ to be estimable is that λ' is a linear combination of the row vectors of the matrix X.)

Theorem 3 :- A necessary and sufficient condition for estimability of a parametric fn $\lambda\beta$ is $\lambda' = \lambda'H$.

(Proof) $\lambda'H = u'XH$

$\lambda'H = u'XH$ (using theorem 2)

$= u'X$ [show $XH = X$]

$= \lambda'$

Conversely let $\lambda' = \lambda' H$ then,

$$\lambda' = \lambda' S^- S$$
$$= \underbrace{\lambda' S^-}\, X'X$$
$$= U'X$$
$$\Rightarrow \boxed{\lambda' = U'X} \text{ where } U' = \lambda' S^- X'.$$

The definition of estimability gurantees only the existance of at least one unbiased estimate of an estimable parametric function.

It does not explicitly give a method of obtaining it, Nor does it say it is the best estimate.

→ mvue.

Problem 1 Consider the model $y_1 = \beta_1 + \beta_2 + \epsilon_1$

$$y_2 = \beta_2 + \beta_3 + \epsilon_2 \quad \text{(Check)}$$
$$y_3 = \beta_1 + \beta_3 + \epsilon_3$$

Show that $\lambda_1\beta_1 + \lambda_2\beta_2 + \lambda_3\beta_3$ is estimable if and only if $\lambda_1 = \lambda_2 + \lambda_3$.

$$E(a_1 y_1 + a_2 y_2 + a_3 y_3)$$
$$= E\big(a_1(\beta_1+\beta_2) + a_2(\beta_2+\beta_3) + a_3(\beta_1+\beta_3)\big)$$
$$= a_1 + a_2 + a_3$$

Proof using G-inverse

$$y_1 = \beta_1 + \beta_2 + \epsilon_1$$
$$y_2 = \beta_1 + \beta_3 + \epsilon_2$$
$$y_3 = \beta_1 + \beta_2 + \epsilon_3$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} \approx \boxed{\underline{y} = X\underline{\beta} + \underline{\epsilon}}$$

Normal equation: $X'\underline{y} = (X'X)\beta = q$

$\therefore \hat{\beta} = (X'X)^{-}q$

[coefficient matrix of q is the solution]

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\therefore y_1 + y_2 + y_3 = 3\beta_1 + 2\beta_2 + \beta_3 = q_1$$
$$y_1 + y_3 = 2\beta_1 + 2\beta_2 = q_2$$
$$y_2 = \beta_1 + \beta_3 = q_3$$

let, $\hat{\beta}_2 = 0$

$$\hat{\beta}_1 = \frac{q_2}{2} \quad ; \quad \hat{\beta}_3 = q_1 - \frac{3}{2}q_2$$

$$\cancel{\hat{\beta}_3 = q_3 - \frac{q_2}{2}}$$

$$(X'X)^{-} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 1 & -3/2 & 0 \end{pmatrix}$$

Estimability Check $\rightarrow$

$$H = S^{-}S$$

$$= \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 1 & -3/2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

now, we have to check $\lambda' = \lambda' H$ or not

$$(\lambda_1 \ \lambda_2 \ \lambda_3) \cancel{\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}} = (\lambda_1 \ \lambda_2 \ \lambda_3) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} \lambda_1 \\ \lambda_1 - \lambda_3 \\ \lambda_3 \end{pmatrix}} \ (\lambda_1 \quad \lambda_1 - \lambda_3 \quad \lambda_3)$$

now, $\lambda_2 = \lambda_1 - \lambda_3$

$$\Rightarrow \boxed{\lambda_1 = \lambda_2 + \lambda_3}$$

Problem 2:

$$y_1 = \mu + \alpha_1 + \beta_1 + \epsilon_1$$
$$y_2 = \mu + \alpha_1 + \beta_2 + \epsilon_2$$
$$y_3 = \mu + \alpha_2 + \beta_1 + \epsilon_3$$
$$y_4 = \mu + \alpha_2 + \beta_2 + \epsilon_4$$
$$y_5 = \mu + \alpha_3 + \beta_1 + \epsilon_5$$
$$y_6 = \mu + \alpha_3 + \beta_2 + \epsilon_6$$

(i) When is $\lambda_0 \mu + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3 + \lambda_4 \beta_1 + \lambda_5 \beta_2$ estimable

(ii) is $(\alpha_1 + \alpha_2)$ estimable ?

(iii) is $(\beta_1 - \beta_2)$ estimable ?

(iv) is $(\mu + \alpha_1)$ " ?

(v) is $(6\mu + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\beta_1 + 3\beta_2)$ estimable ?

(vi) is $(\alpha_1 - 2\alpha_2 + \alpha_3)$ estimable ?

Solution : Normal Eqn $X'\underline{y} = (X'X)\underline{\beta}$

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$X' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\therefore X'X = \begin{pmatrix} 6 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 0 & 0 & 1 & 1 \\ 2 & 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 & 3 & 0 \\ 3 & 1 & 1 & 1 & 0 & 3 \end{pmatrix}$$

Now,
$$q_1 = \sum_{i=1}^{6} y_i = 6\mu + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3(\beta_1 + \beta_2)$$
$$q_2 = y_1 + y_2 = 2\mu + 2\alpha_1 + \beta_1 + \beta_2$$
$$q_3 = y_3 + y_4 = 2\mu + 2\alpha_2 + \beta_1 + \beta_2$$
$$q_4 = y_5 + y_6 = 2\mu + 2\alpha_3 + \beta_1 + \beta_2$$
$$q_5 = y_1 + y_3 + y_5 = 3\mu + \alpha_1 + \alpha_2 + \alpha_3 + 3\underline{\beta_1}$$
$$q_6 = y_2 + y_4 + y_6 = 3\mu + \alpha_1 + \alpha_2 + \alpha_3 + 3\underline{\beta_2}$$

$$\beta_1 = \left\{ q_5 - 3\mu - (\alpha_1 + \alpha_2 + \alpha_3) \right\} \tfrac{1}{3}$$
$$\beta_2 = \left\{ q_6 - 3\mu - (\alpha_1 + \alpha_2 + \alpha_3) \right\} \tfrac{1}{3}$$

now put β_1 and β_2 in the rest of the eqn.

additional equation $\Sigma \alpha_i = 0$.

let, $\hat{\beta}_1+\hat{\beta}_2=0$

and $\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3=0$

now,

$$\hat{\mu}=\frac{1}{6}q_1$$

$$\hat{\alpha}_1=\frac{q_2}{2}-\frac{q_1}{6}$$

$$\hat{\alpha}_2=\frac{q_3}{2}-\frac{q_1}{6}$$

$$\hat{\alpha}_3=\frac{q_4}{2}-\frac{q_1}{6}$$

$$\hat{\beta}_1=\frac{q_5}{3}-\frac{q_1}{6}$$

$$\hat{\beta}_2=\frac{q_6}{3}-\frac{q_1}{6}$$

$$X'X=\begin{pmatrix}6&2&2&2&3&3\\2&2&0&0&1&1\\2&0&2&0&1&1\\2&0&0&2&1&1\\3&1&1&1&3&0\\3&1&1&1&0&3\end{pmatrix}$$

$$(X'X)^-=\begin{pmatrix}\frac{1}{6}&0&0&0&0&0\\-\frac{1}{6}&\frac{1}{2}&0&0&0&0\\-\frac{1}{6}&0&\frac{1}{2}&0&0&0\\-\frac{1}{6}&0&0&\frac{1}{2}&0&0\\-\frac{1}{6}&0&0&0&\frac{1}{3}&0\\-\frac{1}{6}&0&0&0&0&\frac{1}{3}\end{pmatrix}$$

$-1+1$

$-\frac{1}{3}+1$

$-\frac{1}{2}$

$$H=(X'X)^-(X'X)=\begin{pmatrix}1&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{2}&\frac{1}{2}\\0&\frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}&0&0\\0&-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}&0&0\\0&-\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}&0&0\\0&0&0&0&\frac{1}{2}&-\frac{1}{2}\\0&0&0&0&-\frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

now, $(\lambda_0\ \lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5)=(\lambda_0\ \lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5)\cdot H$

$$=\Big(\lambda_0\quad \frac{\lambda_0}{3}+\frac{2\lambda_1}{3}-\frac{1}{3}(\lambda_2+\lambda_3)\quad \frac{\lambda_0}{3}-\frac{\lambda_1}{3}+\frac{2\lambda_2}{3}-\frac{\lambda_3}{3}$$

$$\frac{\lambda_0}{3}-\frac{\lambda_1}{3}-\frac{\lambda_2}{3}+\frac{2\lambda_3}{3}$$

$$\frac{\lambda_0}{2}+\frac{\lambda_4}{2}-\frac{\lambda_5}{2}\qquad \frac{\lambda_0}{2}-\frac{\lambda_4}{2}+\frac{\lambda_5}{2}\Big)$$

$$\therefore \quad \lambda_1 = \frac{\lambda_0}{3} + \frac{2\lambda_1}{3} - \frac{\lambda_2+\lambda_3}{3}$$

$$\lambda_2 = \frac{\lambda_0}{3} + \frac{2\lambda_2}{3} - \frac{\lambda_1+\lambda_3}{3} \Rightarrow \boxed{\lambda_2+\lambda_1+\lambda_3 = \lambda_0}$$

$$\lambda_3 = \frac{\lambda_0}{3} + \frac{2\lambda_3}{3} - \frac{\lambda_1+\lambda_2}{3}$$

$$\lambda_4 = \frac{1}{2}(\lambda_0 + \lambda_4 - \lambda_5) \Rightarrow \lambda_4 + \lambda_5 = \lambda_0$$

$$\lambda_5 = \frac{1}{2}(\lambda_0 - \lambda_4 + \lambda_5) \Rightarrow \lambda_5 = \lambda_0 - \lambda_4$$

$$\Rightarrow \boxed{\lambda_4 = \lambda_5}$$

Theorem 4 : Gauss- Markov Theorem.

For the model $\underline{y} = X\underline{\beta} + \underline{\epsilon}$; $E(\epsilon) = 0$ · $V(\epsilon) = \sigma^2 I$

Where $\underline{y}$ is observed; X is known; β, σ^2 unknown ;

~~for the model~~ the best linear unbiased estimate (BLUE) of an estimable linear parametric function $\underline{\lambda}'\underline{\beta}$ (where $\underline{\lambda}$ is known) is $\underline{\lambda}'\hat{\underline{\beta}}$; $\hat{\underline{\beta}}$ being any solution of the Normal Equation

$$\underline{X}'\underline{y} = (X'X)\hat{\beta}$$ which is obtained

by minimising w.r.t $\underline{\beta}$.

* BLUE unique ⇒ full Rank Matrix

Proof: first observe that $\underline{\lambda}'\hat{\underline{\beta}}$ is unbiased for $\underline{\lambda}'\underline{\beta}$. and is thus eligable for being BLUE.

$$E(\underline{\lambda}'\hat{\underline{\beta}})$$
$$= E(\underline{\lambda}' S^- X'\underline{y})$$
$$= \underline{\lambda}' S^- X'(E(\underline{y}))$$
$$= \underline{\lambda}' S^- S\underline{\beta}$$
$$= \underline{\lambda}' H\underline{\beta}$$
$$= \underline{\lambda}'\underline{\beta}$$

it remains to prove now that the variance of $\underline{\lambda}'\hat{\underline{\beta}}$ is not larger than that of any other unbiased estimator of $\underline{\lambda}'\underline{\beta}$.

let $\underline{U}'\underline{y}$ be any other unbiased estimator of $\underline{\lambda}'\underline{\beta}$.

$$\Rightarrow E(\underline{U}'\underline{y}) = \underline{\lambda}'\underline{\beta}$$
$$\therefore \underline{U}'X\underline{\beta} = \underline{\lambda}'\underline{\beta}$$
$$\Rightarrow \underline{U}'X = \underline{\lambda}'$$

Again, $\underline{U}'\underline{y} = \underline{U}'\underline{y} - \underline{\lambda}'\hat{\underline{\beta}} + \underline{\lambda}'\hat{\underline{\beta}}$

$$\therefore V(\underline{U}'\underline{y}) = V(\underline{U}'\underline{y} - \underline{\lambda}'\hat{\underline{\beta}}) + V(\underline{\lambda}'\hat{\underline{\beta}}) + 2\,Cov(\underline{U}'\underline{y} - \underline{\lambda}'\hat{\underline{\beta}}, \underline{\lambda}'\hat{\underline{\beta}})$$

⟶

$$\mathrm{Cov}(\underline{U}'\underline{Y} - \underline{\lambda}'\hat{\beta},\ \underline{\lambda}'\hat{\beta})$$

$$= \mathrm{Cov}(\underline{U}'\underline{Y} - \underline{\lambda}'S^{-}X'\underline{Y},\ \underline{\lambda}'S^{-}X'\underline{Y})$$

$$= \mathrm{Cov}((\underline{U}' - \lambda'S^{-}X')\underline{Y},\ \underline{\lambda}'S^{-}X'\underline{Y})$$

$$= (\underline{U}' - \lambda'S^{-}X')\,V(\underline{Y})\,(\lambda'S^{-}X'\underline{Y})'$$

$$= \sigma^2(\underline{U}' - \lambda'S^{-}X')(X(S^{-})'\lambda)$$

$$= \sigma^2(\underline{U}'X - \lambda'S^{-}X'X)((S^{-})'\lambda)$$

$$= \sigma^2\underbrace{(\lambda' - \underline{\lambda}'H)}_{0}((S^{-})'\lambda)$$

$$= 0$$

So, $\boxed{V(\lambda'\hat{\beta}) \leq V(\underline{U}'\underline{Y})}$

equality holds if ~~when $\to U'Y = \lambda'\hat{\beta}$~~,

$$V(\underline{U}'Y - \underline{\lambda}'\hat{\beta}) = 0$$

$\Rightarrow E[(U'\underline{Y} - \lambda'\hat{\beta})^2] = 0$ as $V(\ ) = 0$ and $E(\ ) = 0$.

that means. $\boxed{U'\underline{Y} = \underline{\lambda}'\hat{\underline{\beta}}}$

Proved

Equality holds if $U'\underline{Y} = \lambda'\hat{\beta}$ with prob 1; in other words

$\lambda'\beta$ is estimable; $\lambda'\hat{\beta}$ is its BLUE & if any other unbiased estimate of $\lambda'\beta$ has the same variance as $\lambda'\hat{\beta}$ it can't be diff. from $\lambda'\hat{\beta}$.

We therefore conclude that the blue of an estimable function is Unique.

The Gauss-Markov Theorem thus provide a very convenient method of obtaining the BLUE of an estimable parametric function $\underline{\lambda}'\underline{\beta}$. Obtain any solution $\hat{\beta}$ of the Normal Equation and Substitute $\hat{\underline{\beta}}$ for $\underline{\beta}$ in the Linear Parametric function to get its BLUE.

Suppose $\hat{\beta}_1$ and $\hat{\beta}_2$ are two diff solution of the normal equation. if their substituted in an estimable parametric function $\lambda'\beta$, apperently it looks as if we have two diff. BLUEs namely $\lambda'\hat{\beta}_1$ and $\lambda'\hat{\beta}_2$ but it is not so they are the same.

However if $\lambda'\beta$ is not estimable substituting two diff solutions may result in diff expression.

Variances and Co-Variances of BLUEs

$$\hat{\beta} = S^- X'\underline{y}$$

$$V(\hat{\beta}) = V(S^- X'\underline{y})$$

$$= S^- X' V(\underline{y}) X (S^-)'$$

$$= \sigma^2 S^-(X'X)(S^-)'$$

$$= \sigma^2 S^- S (S^-)'$$

In general $V(\hat{\beta}) \neq \sigma^2 S^-$

$$\therefore V(\lambda'\hat{\beta}) = \underline{\lambda}' V(\hat{\beta}) \underline{\lambda}$$

$$= \sigma^2 \underline{\lambda}' \underbrace{S^- S}_{H} (S^-)' \underline{\lambda}$$

$$= \sigma^2 \underline{\lambda}' S^- \underbrace{H'\lambda}_{(\lambda' H)'} \qquad = \sigma^2 \lambda' H (S^-)'\lambda$$

$$= \sigma^2 \underline{\lambda}' S^- \lambda \qquad = \sigma^2 \underline{\lambda}' (S^-)' \underline{\lambda}$$

$$\Rightarrow \boxed{\underline{\lambda}' S^- \underline{\lambda} = \underline{\lambda}'(S^-)'\underline{\lambda}}$$

two BLUEs $\lambda'_{(1)}\hat{\beta}$, $\lambda'_{(2)}\hat{\beta}$

$$Cov(\lambda'_{(1)}\hat{\beta}, \lambda'_{(2)}\hat{\beta})$$

$$= \underline{\lambda}'_{(1)} V(\hat{\beta}) \lambda_{(2)}$$

$$= \sigma^2 \underline{\lambda}'_{(1)} S^- S (S^-)' \underline{\lambda}_{(2)}$$

$$= \sigma^2 \underline{\lambda}'_{(1)} H (S^-)' \underline{\lambda}_{(2)} = \boxed{\sigma^2 \underline{\lambda}'_{(1)} (S^-)' \underline{\lambda}_{(2)}}$$

Let there be m estimable parametric function $\lambda_{(i)}'\underline{\beta}$, $i=1(1)m$.

Denote by $\Lambda = \begin{pmatrix} \lambda_{(1)}' \\ \vdots \\ \lambda_{(m)}' \end{pmatrix}$ then all Linear parametric functions may be expressed as $\Lambda\underline{\beta}$. and $\Lambda H = \Lambda$ (due to estimability)

$$V(\Lambda\hat{\beta}) = \Lambda V(\hat{\beta})\Lambda'$$

$$= \Lambda S^- S (S^-)'\Lambda'\sigma^2$$

$$= \Lambda H (S^-)'\Lambda'\sigma^2$$

$$= \Lambda (S^-)'\Lambda'\sigma^2 \quad = \Lambda S^- \Lambda'\sigma^2$$

☐ if the m parametric function $\Lambda\underline{\beta}$ are linearly independent — that is if $r(\Lambda) = m$. then Prove ~~show~~ that the Variance Covariance Matrix $\Lambda(S^-)'\Lambda'\sigma^2$ or $\Lambda S^- \Lambda'\sigma^2$ is non-singular.

$\Rightarrow$ Estimibility criteria $\Lambda = \Lambda H = \Lambda S^- S = \Lambda S^- X'X$

$$m = r(\Lambda) = r(\Lambda S^- X'X)$$

$$\leq r(\Lambda S^- X') \leq r(\Lambda) = m.$$

$\therefore (\Lambda S^- X') = m.$ as. $m \leq r(\Lambda S^- X')$ and $r(\Lambda S^- X') \leq m$.

$$m = r(\Lambda S^- X')$$

$$= r(\Lambda S^- X' X (S^-)'\Lambda')$$

$$= r(\Lambda S^- S (S^-)'\Lambda')$$

$$= r(\Lambda H (S^-)'\Lambda')$$

$$= r(\Lambda (S^-)'\Lambda')$$

$\Rightarrow \Lambda(S^-)'\Lambda'$ non-singular matrix

$\Lambda \rightarrow m \times p$

$(S^-)' \rightarrow p \times p \quad \Lambda' \rightarrow p \times m.$

det| $\quad$ | $\neq 0$

Estimation Space

$$\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}} = \underset{\sim}{\lambda}' S^{-} X'\underset{\sim}{y}$$
$$= \underset{\sim}{l}'\underset{\sim}{q} \quad ; \quad q = X'\underset{\sim}{y}$$
$$l = (\underset{\sim}{S}^{-})'\lambda$$

The BLUE $\underset{\sim}{\lambda}'\hat{\underset{\sim}{\beta}}$ is thus a linear combination of the left hand sides $q_1, q_2 \ldots q_p$ of the normal equation: $\underline{X'\underset{\sim}{y} = (X'X)\underset{\sim}{\beta}}$.

Conversely if we consider a linear combination $l'q = \sum_{i=1}^{p} l_i q_i$ of the left hand sides of the Normal Equations it is the BLUE its expected value; because $E(\underset{\sim}{l}'\underset{\sim}{q})$

$$= l'\underbrace{X'X\underset{\sim}{\beta}}$$

By Gauss-Markov, BLUE $(l'X'X\underset{\sim}{\beta})$ is $\underset{\sim}{l}'\underbrace{X'X\hat{\underset{\sim}{\beta}}}$

$$= \underset{\sim}{l}'X'\underset{\sim}{y}$$
$$= \underset{\sim}{l}'\underset{\sim}{q}$$

So we have the following (*) theorem 5: For the Model $Y = X\beta + \epsilon$ a BLUE of every estimable parametric function is a linear combination of the left Hand Side $X'y = q$ of the Normal Equations and conversely any linear combination of the LHS q of the Normal Eqn is the BLUE of its expected value.

Corollary: A necessary and sufficient condition for a linear parametric function $\lambda'\beta$ to be estimable is that λ is a linear combination of rows of $X'X$.

$\Longrightarrow$

$r(X) = r(X'X)$ $\quad$ $|\ r(A) = q - \dim N(A)$

rows of X & rows of $X'X$

$\boxed{\text{span}}$ same vector space

Theorem 6: [The BLUE of any linear combinations of estimable parametric function is the same linear combination of their BLUEs.]

In other words if $\lambda_i'\beta$; $i = 1(1)m$ are all estimable the BLUE of $\lambda'\beta$ equals $(K_1\lambda_1'\beta + K_2\lambda_2'\beta + \dots + K_m\lambda_m'\beta)$ is $\lambda'\hat{\beta} = K_1\lambda_1'\hat{\beta} + \dots + K_m\lambda_m'\hat{\beta}$

The proof follows from the fact that $\lambda' = \lambda'H$ and each λ_i' satisfies. $\lambda_i' = \lambda_i'H$ and by the Gauss-Markov theorem. $\lambda'\hat{\beta}$ is the BLUE of $\lambda'\beta$.

Theorem 7: If every BLUE is expressed in terms of the observations y as $\underline{a}'\underline{y}$, the coefficient vector $\underline{a}$ is a linear combination of the columns of X and conversely every linear function $\underline{a}'\underline{y}$ of the observation such that the coefficient vector $\underline{a}$ is a linear combination of the columns of X, is the BLUE of its expected value.

Proof: if $\lambda'\beta$ is estimable, its BLUE is $\lambda'\hat{\underline{\beta}} = \lambda' S^{-} X' y$

$$= \underline{a}'y$$

$$\Rightarrow \underline{a}' = \lambda' S^{-} X'$$

$$\Rightarrow \underline{a} = X(S^{-})'\lambda = X\underline{l}$$

$\therefore$ $\underline{a}$ is a linear combination of the columns of X.

Conversely $\underline{a} = X\underline{l}$

$$\Rightarrow E(\underline{a}'\underline{y}) = E(\underline{l}'X'\underline{y})$$

$$= \underline{l}'X'X\underline{\beta}$$

BLUE of $\underline{l}'X'X\underline{\beta}$ is $\underline{l}'\underbrace{X'X\hat{\underline{\beta}}}$

$$= \underline{l}'X'\underline{y}$$

$$= \underline{a}'\underline{y}$$

$\therefore$ $\underline{a}'\underline{y}$ is the BLUE of expected value of $\underline{a}'\underline{y}$

Note: We must check the estimability of $\underline{l}'X'X\underline{\beta}$.
as $\underline{a}'\underline{y}$ is a linear function such that its expected value is $\underline{l}'X'X\underline{\beta}$,
By definition it is estimable.

Error Space

A Linear Function of the observations is said to belong to the error space if and only if its expected value is identically equal to 0, irrespective of the value of $\underline{\beta}$.

Thus if $\underline{b}'\underline{y}$ belongs to the error space $E(\underline{b}'\underline{y}) = 0$.

That is, $\underline{b}'X\underline{\beta} = 0$

$$\Rightarrow \underline{b}'X \text{ or } X'\underline{b} = 0$$

$$\Rightarrow \underline{b} \text{ is } \perp \text{ to the columns of } X.$$

$\longrightarrow$

$$X'\underline{b} = 0$$

$$\Rightarrow \underline{b}'X = 0$$

$$\Rightarrow \underline{b}'X\underline{\beta} = 0$$

$$\Rightarrow E(\underline{b}'\underline{y}) = 0$$

Theorem 8 : A linear function of observation belongs to the error space if and only if the coefficient vector is orthogonal to the columns of X.

$$(i)\ E(\underline{\lambda}'\underline{y}) = \underline{\lambda}'\underline{\beta}$$

$$\therefore E(\underline{\lambda}'\underline{y} - \underline{\lambda}'\hat{\underline{\beta}}) \quad \text{by the defn of BLUE}$$

$$= \lambda'\beta - \lambda'\beta$$

$$= 0.$$

$$(ii)\ E(\underline{y} - X\hat{\underline{\beta}})$$

$$= X\underline{\beta} - E(XS^{-}X'\underline{y})$$

$$= X\underline{\beta} - XS^{-}X'X\underline{\beta}$$

$$= X\underline{\beta} - XS^{-}S\underline{\beta}$$

$$= X\beta - XH\beta$$

$$= X\beta - X\beta$$

$$= 0.$$

☑ if $X'\underline{b}_{(i)} = 0$ then $X'\underbrace{(c_1 b_{(1)} + c_2 b_{(2)} + \dots + c_K b_{(K)})}_{\in \text{ Error Space}} = 0$

☑ **Theorem 9 :** The coefficient vector of any blue, when expressed in terms of the observation, is orthogonal to the coefficient vector of any linear function of the obs. belonging to the Error Space.

$\longrightarrow$ The proof of the theorem is obvious from the fact that if $\underline{b}'\underline{y}$ belongs to $\in$ Error space $(=0)$; $\mathbf{b}$ is orthogonal to the columns of X and by Theorem 7 — the coefficient vector of any blue is a linear combination of columns of X.

Thus any vector in the estimation space is orthogonal to any vector in the Error Space., and so we say that Error Space is orthogonal to the estimation Space.

Since, the Estimation Space generated by columns of x has rank 'r'. and since we can find at most $(n-r)$ indept vectors orthogonal to columns of x, the Rank of the Error Space is $(n-r)$

Ex. $f(\underline{y})\cdot \underline{u}'\underline{y} - \underbrace{\underline{\lambda}'\hat{\underline{\beta}}}_{\to \text{BLUE of } \underline{\lambda}'\underline{\beta}}$.

$$E(\underline{u}'\underline{y} - \underline{\lambda}'\hat{\underline{\beta}})$$

$$= \underline{u}'X\underline{\beta} - \underline{\lambda}'\underline{\beta}$$

$$= \underline{\lambda}'\underline{\beta} - \underline{\lambda}'\underline{\beta}$$

$$= 0.$$

Check for, $\underline{y} - X\hat{\underline{\beta}}$.

Theorem 10 The covariance between any Linear functions $\in$ Error space and any BLUE is zero.

$$\mathrm{Cov}(\underline{b}'\underline{y}, \underline{\lambda}'\hat{\underline{\beta}}) = \mathrm{Cov}(\underline{b}'\underline{y}, \underline{\lambda}'S^{-}X'\underline{y})$$

$$= \underline{b}'\underline{y}\ V(\underline{y})\ X(S^{-})'\lambda$$

$$= b'X(S^{-})'\lambda\sigma^2$$

$$= 0$$

Role of Error Space (i) estimate σ^2;

if $\underline{b}'\underline{y} \in$ Error Space and $b'b = 1$

then $E(\underline{b}'\underline{y}) = 0 \Rightarrow E(\underline{b}'\underline{y})^2 = V(\underline{b}'\underline{y})$

$$= \underline{b}'b\,\sigma^2$$

$$= \sigma^2.$$

σ^2 is the UE of $(\underline{b}'\underline{y})^2$.

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$$B_{(1)} = \begin{pmatrix} b_{(1)} \\ \vdots \\ b_{(n-r)} \end{pmatrix} \qquad \text{Now, } b_{(i)}' b_{(j)} = 0$$

$$b_{(i)}' X = 0$$

$$b_{(i)}' b_{(i)} = 1 \text{ (orthonormality)}$$

Now, $B_1 X = 0.$

$$B_1 B_1' = I_{(n-r)} \text{ (orthogonality)}$$

$$(b_1' \underline{Y})^2 + (b_2' \underline{Y})^2 + \cdots + (b_{n-r}' \underline{Y})^2$$

$= \underline{Y}' B_1' B_1 \underline{Y} \longrightarrow$ This is the sum of squares of a complete set of $(n-r)$ unit, mutually orthogonal linear functions belonging to the error space. This is why we called it as SSE.

$$E(\underline{Y}' B_1' B_1 \underline{Y}) = (n-r)\sigma^2$$

Thus by pulling together all the linearly indept functions belonging to the error space, we can obtain the estimate $\left[\frac{SSE}{n-r} = \underline{Y}' B_1' B_1 \underline{Y} / n-r\right]$ of σ^2

We know $SSE = (Y - X\hat{\beta})'(Y - X\hat{\beta})$ where $\hat{\beta}$ is any solution of normal equation.

To establish equivalency of the defn let us consider r mutually orthogonal rows such that $B = \begin{pmatrix} B_{1(n-r)} \\ B_{2(r)} \end{pmatrix}$ becomes a $(n \times n)$ orthogonal matrix.

By definition $[B_1 B_2' = 0]$; due to orthogonality

again $[B_1 X = 0] \Rightarrow$ rows of B_1 are orthogonal to the columns of X. Also, rows of B_2 are orthogonal to the rows of B_1.

But there can't be more than $(n-r)$ linearly indept. vectors orthogonal to the rows of B_1 and so, rows of B_2 must be a linear combination of columns of X.

$\therefore B_2 = CX'$ $[C_{(n-r) \times n}$ Matrix$]$

and $B_2 X \hat{\underline{\beta}} = CX'X\hat{\underline{\beta}}$

$= CX'\underline{Y}$

$= B_2\underline{Y}$

$$\Rightarrow \boxed{B_2 X\hat{\beta} = B_2 \underline{Y}}$$

$I = B'B$

$= (B_1' \vdots B_2')\begin{pmatrix} B_1 \\ \cdots \\ B_2 \end{pmatrix}$

$= B_1'B_1 + B_2'B_2$

$$SSE = (Y - X\hat{\beta})'(Y - X\hat{\beta})$$

$$= (\underline{Y} - X\hat{\underline{\beta}})'(B_1'B_1 + B_2'B_2)(\underline{Y} - X\hat{\underline{\beta}})$$

$$= (Y - X\hat{\beta})'B_1'B_1(Y - X\hat{\beta}) + (Y - X\hat{\beta})'B_2'B_2(Y - X\hat{\beta})$$

$$= (B_1Y - \underbrace{B_1X\hat{\beta}})'(B_1Y - \underbrace{B_1X\hat{\beta}}) + (B_2Y - B_2X\hat{\beta})'(B_2Y - B_2X\hat{\beta})$$

$$= (B_1Y)'(B_1Y)$$

$$= Y'B_1'B_1Y$$

Again, $SSE = (Y - X\hat{\beta})'(Y - X\hat{\beta})$

$$= Y'Y - 2\hat{\underline{\beta}}'X'X\hat{\underline{\beta}} + (X\hat{\beta})'(X\hat{\beta})$$

$$= \sum_{i=1}^{n} y_i^2 - \hat{\beta}'X'X\hat{\beta}$$

$$= \sum y_i^2 - \hat{\beta}'X'Y$$

$$= \sum y_i^2 - \underbrace{(\hat{\beta}_1 q_1 + \cdots + \hat{\beta}_p q_p)}_{SSR}$$

$$\boxed{\therefore E(SSR) = E(\sum y_i^2) - E(SSE)}$$

$$= \sum[V(y_i) + (E(y_i))^2] - (n-r)\sigma^2$$

$$= r\sigma^2 + n(X\beta)'(X\beta)$$

$$= r\sigma^2 + n\beta'X'X\beta$$

Source	df	SS	E(MS)
Regression	r	$\hat{\underline{\beta}}'\underline{q}$	$\sigma^2 + \frac{1}{r}\underline{\beta}'X'X\underline{\beta}$
Error	$n-r$	$\underline{y}'\underline{y} - \hat{\underline{\beta}}'\underline{q}$	σ^2
Total	n	$\underline{y}'\underline{y}$	

$\Rightarrow \boxed{E(MSR) \geq E(MSE)}$

'=' holds when $\boxed{X\beta = 0.}$ otherwise $F > 1$.

(*) $\boxed{\text{Examples from Book}}$ Page-45 onwards 64.