CHAPTER 3

ELEMENTS OF POINT SET TOPOLOGY

3.1 INTRODUCTION

A large part of the previous chapter dealt with "abstract" sets, that is, sets of arbitrary objects. In this chapter we specialize our sets to be sets of real numbers, sets of complex numbers, and more generally, sets in higher-dimensional spaces.

In this area of study it is convenient and helpful to use geometric terminology. Thus, we speak about sets of points on the real line, sets of points in the plane, or sets of points in some higher-dimensional space. Later in this book we will study functions defined on point sets, and it is desirable to become acquainted with certain fundamental types of point sets, such as *open* sets, *closed* sets, and *compact* sets, before beginning the study of functions. The study of these sets is called *point set topology*.

3.2 EUCLIDEAN SPACE $\mathbf{R}^n$

A point in two-dimensional space is an ordered pair of real numbers (x_1, x_2). Similarly, a point in three-dimensional space is an ordered triple of real numbers (x_1, x_2, x_3). It is just as easy to consider an ordered n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ and to refer to this as a point in n-dimensional space.

Definition 3.1. *Let $n > 0$ be an integer. An ordered set of n real numbers $(x_1, x_2, \ldots, x_n)$ is called an n-dimensional point or a vector with n components. Points or vectors will usually be denoted by single bold-face letters; for example,*

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \qquad \textit{or} \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n).$$

The number x_k is called the kth *coordinate of the point* $\mathbf{x}$ *or the* kth *component of the vector* $\mathbf{x}$. *The set of all n-dimensional points is called n-dimensional Euclidean space or simply n-space, and is denoted by* $\mathbf{R}^n$.

The reader may wonder whether there is any advantage in discussing spaces of dimension greater than three. Actually, the language of n-space makes many complicated situations much easier to comprehend. The reader is probably familiar enough with three-dimensional vector analysis to realize the advantage of writing the equations of motion of a system having three degrees of freedom as a single vector equation rather than as three scalar equations. There is a similar advantage if the system has n degrees of freedom.

Another advantage in studying n-space for a general n is that we are able to deal in one stroke with many properties common to 1-space, 2-space, 3-space, etc., that is, properties independent of the dimensionality of the space.

Higher-dimensional spaces arise quite naturally in such fields as relativity, and statistical and quantum mechanics. In fact, even infinite-dimensional spaces are quite common in quantum mechanics.

Algebraic operations on n-dimensional points are defined as follows:

Definition 3.2. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *be in* $\mathbf{R}^n$. *We define:*

a) *Equality:*

$$\mathbf{x} = \mathbf{y} \text{ if, and only if, } x_1 = y_1, \ldots, x_n = y_n.$$

b) *Sum:*

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n).$$

c) *Multiplication by real numbers (scalars):*

$$a\mathbf{x} = (ax_1, \ldots, ax_n) \qquad (a \text{ real}).$$

d) *Difference:*

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-1)\mathbf{y}.$$

e) *Zero vector or origin:*

$$\mathbf{0} = (0, \ldots, 0).$$

f) *Inner product or dot product:*

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{n} x_k y_k.$$

g) *Norm or length:*

$$\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{k=1}^{n} x_k^2\right)^{1/2}.$$

The norm $\|\mathbf{x} - \mathbf{y}\|$ is called the *distance* between $\mathbf{x}$ and $\mathbf{y}$.

NOTE. In the terminology of linear algebra, $\mathbf{R}^n$ is an example of a *linear space.*

Theorem 3.3. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *denote points in* $\mathbf{R}^n$. *Then we have:*

a) $\|\mathbf{x}\| \geq 0$, *and* $\|\mathbf{x}\| = 0$ *if, and only if,* $\mathbf{x} = \mathbf{0}$.

b) $\|a\mathbf{x}\| = |a| \, \|\mathbf{x}\|$ *for every real a.*

c) $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$.

d) $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$ *(Cauchy–Schwarz inequality).*

e) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *(triangle inequality).*

Proof. Statements (a), (b) and (c) are immediate from the definition, and the Cauchy–Schwarz inequality was proved in Theorem 1.23. Statement (e) follows from (d) because

$$\|\mathbf{x} + \mathbf{y}\|^2 = \sum_{k=1}^{n} (x_k + y_k)^2 = \sum_{k=1}^{n} (x_k^2 + 2x_k y_k + y_k^2)$$
$$= \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 \le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

NOTE. Sometimes the triangle inequality is written in the form

$$\|\mathbf{x} - \mathbf{z}\| \le \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\|.$$

This follows from (e) by replacing $\mathbf{x}$ by $\mathbf{x} - \mathbf{y}$ and $\mathbf{y}$ by $\mathbf{y} - \mathbf{z}$. We also have

$$\big|\|\mathbf{x}\| - \|\mathbf{y}\|\big| \le \|\mathbf{x} - \mathbf{y}\|.$$

Definition 3.4. *The unit coordinate vector* $\mathbf{u}_k$ *in* $\mathbf{R}^n$ *is the vector whose* kth *component is* 1 *and whose remaining components are zero. Thus,*

$$\mathbf{u}_1 = (1, 0, \ldots, 0), \qquad \mathbf{u}_2 = (0, 1, 0, \ldots, 0), \ldots, \mathbf{u}_n = (0, 0, \ldots, 0, 1).$$

If $\mathbf{x} = (x_1, \ldots, x_n)$ then $\mathbf{x} = x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n$ and $x_1 = \mathbf{x}\cdot\mathbf{u}_1$, $x_2 = \mathbf{x}\cdot\mathbf{u}_2, \ldots, x_n = \mathbf{x}\cdot\mathbf{u}_n$. The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are also called *basis vectors.*

3.3 OPEN BALLS AND OPEN SETS IN $\mathbf{R}^n$

Let $\mathbf{a}$ be a given point in $\mathbf{R}^n$ and let r be a given positive number. The set of all points $\mathbf{x}$ in $\mathbf{R}^n$ such that

$$\|\mathbf{x} - \mathbf{a}\| < r,$$

is called an open *n-ball* of radius r and center $\mathbf{a}$. We denote this set by $B(\mathbf{a})$ or by $B(\mathbf{a}; r)$.

The ball $B(\mathbf{a}; r)$ consists of all points whose distance from $\mathbf{a}$ is less than r. In $\mathbf{R}^1$ this is simply an open interval with center at $\mathbf{a}$. In $\mathbf{R}^2$ it is a circular disk, and in $\mathbf{R}^3$ it is a spherical solid with center at $\mathbf{a}$ and radius r.

3.5 Definition of an interior point. *Let S be a subset of* $\mathbf{R}^n$, *and assume that* $\mathbf{a} \in S$. *Then* $\mathbf{a}$ *is called an interior point of S if there is an open n-ball with center at* $\mathbf{a}$, *all of whose points belong to S.*

In other words, every interior point $\mathbf{a}$ of S can be surrounded by an n-ball $B(\mathbf{a}) \subseteq S$. The set of all interior points of S is called the *interior* of S and is denoted by int S. Any set containing a ball with center $\mathbf{a}$ is sometimes called a *neighborhood* of $\mathbf{a}$.

3.6 Definition of an open set. *A set S in* $\mathbf{R}^n$ *is called open if all its points are interior points.*

NOTE. A set S is open if and only if $S = \operatorname{int} S$. (See Exercise 3.9.)

Examples. In $\mathbf{R}^1$ the simplest type of nonempty open set is an open interval. The union of two or more open intervals is also open. A closed interval $[a, b]$ is not an open set because the endpoints a and b are not interior points of the interval.

Examples of open sets in the plane are: the interior of a disk; the cartesian product of two one-dimensional open intervals. The reader should be cautioned that an open interval in $\mathbf{R}^1$ is no longer an open set when it is considered as a subset of the plane. In fact, *no* subset of $\mathbf{R}^1$ (except the empty set) can be open in $\mathbf{R}^2$, because such a set cannot contain a 2-ball.

In $\mathbf{R}^n$ the empty set is open (Why?) as is the whole space $\mathbf{R}^n$. Every open n-ball is an open set in $\mathbf{R}^n$. The cartesian product

$$(a_1, b_1) \times \cdots \times (a_n, b_n)$$

of n one-dimensional open intervals $(a_1, b_1), \ldots, (a_n, b_n)$ is an open set in $\mathbf{R}^n$ called an *n-dimensional open interval.* We denote it by $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$.

The next two theorems show how additional open sets in $\mathbf{R}^n$ can be constructed from given open sets.

Theorem 3.7. *The union of any collection of open sets is an open set.*

Proof. Let F be a collection of open sets and let S denote their union, $S = \bigcup_{A \in F} A$. Assume $\mathbf{x} \in S$. Then $\mathbf{x}$ must belong to at least one of the sets in F, say $\mathbf{x} \in A$. Since A is open, there exists an open n-ball $B(\mathbf{x}) \subseteq A$. But $A \subseteq S$, so $B(\mathbf{x}) \subseteq S$ and hence $\mathbf{x}$ is an interior point of S. Since every point of S is an interior point, S is open.

Theorem 3.8. *The intersection of a finite collection of open sets is open.*

Proof. Let $S = \bigcap_{k=1}^{m} A_k$ where each A_k is open. Assume $\mathbf{x} \in S$. (If S is empty, there is nothing to prove.) Then $\mathbf{x} \in A_k$ for every $k = 1, 2, \ldots, m$, and hence there is an open n-ball $B(\mathbf{x}; r_k) \subseteq A_k$. Let r be the smallest of the positive numbers $r_1, r_2, \ldots, r_m$. Then $\mathbf{x} \in B(\mathbf{x}; r) \subseteq S$. That is, $\mathbf{x}$ is an interior point, so S is open.

Thus we see that from given open sets, new open sets can be formed by taking arbitrary unions or finite intersections. Arbitrary intersections, on the other hand, will not always lead to open sets. For example, the intersection of all open intervals of the form $(-1/n, 1/n)$, where $n = 1, 2, 3, \ldots$, is the set consisting of 0 alone.

3.4 THE STRUCTURE OF OPEN SETS IN $\mathbf{R}^1$

In $\mathbf{R}^1$ the union of a countable collection of disjoint open intervals is an open set and, remarkably enough, every nonempty open set in $\mathbf{R}^1$ can be obtained in this way. This section is devoted to a proof of this statement.

First we introduce the concept of a component interval.

3.9 Definition of component interval. *Let S be an open subset of $\mathbf{R}^1$. An open interval I (which may be finite or infinite) is called a component interval of S if $I \subseteq S$ and if there is no open interval $J \neq I$ such that $I \subseteq J \subseteq S$.*

In other words, a component interval of S is not a proper subset of any other open interval contained in S.

Theorem 3.10. *Every point of a nonempty open set S belongs to one and only one component interval of S.*

Proof. Assume $x \in S$. Then x is contained in some open interval I with $I \subseteq S$. There are many such intervals but the "largest" of these will be the desired component interval. We leave it to the reader to verify that this largest interval is $I_x = (a(x), b(x))$, where

$$a(x) = \inf \{a : (a, x) \subseteq S\}, \qquad b(x) = \sup \{b : (x, b) \subseteq S\}.$$

Here $a(x)$ might be $-\infty$ and $b(x)$ might be $+\infty$. Clearly, there is no open interval J such that $I_x \subseteq J \subseteq S$, so I_x is a component interval of S containing x. If J_x is another component interval of S containing x, then the union $I_x \cup J_x$ is an open interval contained in S and containing both I_x and J_x. Hence, by the definition of component interval, it follows that $I_x \cup J_x = I_x$ and $I_x \cup J_x = J_x$, so $I_x = J_x$.

Theorem 3.11 (Representation theorem for open sets on the real line). *Every nonempty open set S in $\mathbf{R}^1$ is the union of a countable collection of disjoint component intervals of S.*

Proof. If $x \in S$, let I_x denote the component interval of S containing x. The union of all such intervals I_x is clearly S. If two of them, I_x and I_y, have a point in common, then their union $I_x \cup I_y$ is an open interval contained in S and containing both I_x and I_y. Hence $I_x \cup I_y = I_x$ and $I_x \cup I_y = I_y$ so $I_x = I_y$. Therefore the intervals I_x form a disjoint collection.

It remains to show that they form a countable collection. For this purpose, let $\{x_1, x_2, x_3, \dots\}$ denote the countable set of rational numbers. In each component interval I_x there will be infinitely many x_n, but among these there will be exactly one with *smallest index n*. We then define a function F by means of the equation $F(I_x) = n$, if x_n is the rational number in I_x with smallest index n. This function F is one-to-one since $F(I_x) = F(I_y) = n$ implies that I_x and I_y have x_n in common and this implies $I_x = I_y$. Therefore F establishes a one-to-one correspondence between the intervals I_x and a subset of the positive integers. This completes the proof.

NOTE. This representation of S is unique. In fact, if S is a union of disjoint open intervals, then these intervals must be the component intervals of S. This is an immediate consequence of Theorem 3.10.

If S is an open interval, then the representation contains only one component interval, namely S itself. Therefore an open interval in $\mathbf{R}^1$ cannot be expressed as the union of two nonempty disjoint open sets. This property is also described by saying that an open interval is *connected.* The concept of connectedness for sets in $\mathbf{R}^n$ will be discussed further in Section 4.16.

3.5 CLOSED SETS

3.12 Definition of a closed set. *A set S in $\mathbf{R}^n$ is called closed if its complement $\mathbf{R}^n - S$ is open.*

Examples. A closed interval $[a, b]$ in $\mathbf{R}^1$ is a closed set. The cartesian product

$$[a_1, b_1] \times \cdots \times [a_n, b_n]$$

of n one-dimensional closed intervals is a closed set in $\mathbf{R}^n$ called an *n-dimensional closed interval* $[\mathbf{a}, \mathbf{b}]$.

The next theorem, a consequence of Theorems 3.7 and 3.8, shows how to construct further closed sets from given ones.

Theorem 3.13. *The union of a finite collection of closed sets is closed, and the intersection of an arbitrary collection of closed sets is closed.*

A further relation between open and closed sets is described by the following theorem.

Theorem 3.14. *If A is open and B is closed, then $A - B$ is open and $B - A$ is closed.*

Proof. We simply note that $A - B = A \cap (\mathbf{R}^n - B)$, the intersection of two open sets, and that $B - A = B \cap (\mathbf{R}^n - A)$, the intersection of two closed sets.

3.6 ADHERENT POINTS. ACCUMULATION POINTS

Closed sets can also be described in terms of adherent points and accumulation points.

3.15 Definition of an adherent point. *Let S be a subset of $\mathbf{R}^n$, and $\mathbf{x}$ a point in $\mathbf{R}^n$, $\mathbf{x}$ not necessarily in S. Then $\mathbf{x}$ is said to be adherent to S if every n-ball $B(\mathbf{x})$ contains at least one point of S.*

Examples

1. If $\mathbf{x} \in S$, then $\mathbf{x}$ adheres to S for the trivial reason that every n-ball $B(\mathbf{x})$ contains $\mathbf{x}$.
2. If S is a subset of $\mathbf{R}$ which is bounded above, then sup S is adherent to S.

Some points adhere to S because every ball $B(\mathbf{x})$ contains points of S *distinct* from $\mathbf{x}$. These are called accumulation points.

3.16 Definition of an accumulation point. *If $S \subseteq \mathbf{R}^n$ and $\mathbf{x} \in \mathbf{R}^n$, then $\mathbf{x}$ is called an accumulation point of S if every n-ball $B(\mathbf{x})$ contains at least one point of S distinct from $\mathbf{x}$.*

In other words, $\mathbf{x}$ is an accumulation point of S if, and only if, $\mathbf{x}$ adheres to $S - \{\mathbf{x}\}$. If $\mathbf{x} \in S$ but $\mathbf{x}$ is not an accumulation point of S, then $\mathbf{x}$ is called an *isolated point* of S.

Examples

1. The set of numbers of the form $1/n$, $n = 1, 2, 3, \ldots$, has 0 as an accumulation point.
2. The set of rational numbers has every real number as an accumulation point.
3. Every point of the closed interval $[a, b]$ is an accumulation point of the set of numbers in the open interval (a, b).

Theorem 3.17. *If* $\mathbf{x}$ *is an accumulation point of* S*, then every n-ball* $B(\mathbf{x})$ *contains infinitely many points of* S.

Proof. Assume the contrary; that is, suppose an n-ball $B(\mathbf{x})$ exists which contains only a finite number of points of S distinct from $\mathbf{x}$, say $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. If r denotes the smallest of the positive numbers

$$\|\mathbf{x} - \mathbf{a}_1\|, \qquad \|\mathbf{x} - \mathbf{a}_2\|, \qquad \ldots, \qquad \|\mathbf{x} - \mathbf{a}_m\|,$$

then $B(\mathbf{x}; r/2)$ will be an n-ball about $\mathbf{x}$ which contains no points of S distinct from $\mathbf{x}$. This is a contradiction.

This theorem implies, in particular, that a set cannot have an accumulation point unless it contains infinitely many points to begin with. The converse, however, is not true in general. For example, the set of integers $\{1, 2, 3, \ldots\}$ is an infinite set with no accumulation points. In a later section we will show that infinite sets contained in some n-ball always have an accumulation point. This is an important result known as the Bolzano–Weierstrass theorem.

3.7 CLOSED SETS AND ADHERENT POINTS

A closed set was defined to be the complement of an open set. The next theorem describes closed sets in another way.

Theorem 3.18. *A set* S *in* $\mathbf{R}^n$ *is closed if, and only if, it contains all its adherent points.*

Proof. Assume S is closed and let $\mathbf{x}$ be adherent to S. We wish to prove that $\mathbf{x} \in S$. We assume $\mathbf{x} \notin S$ and obtain a contradiction. If $\mathbf{x} \notin S$ then $\mathbf{x} \in \mathbf{R}^n - S$ and, since $\mathbf{R}^n - S$ is open, some n-ball $B(\mathbf{x})$ lies in $\mathbf{R}^n - S$. Thus $B(\mathbf{x})$ contains no points of S, contradicting the fact that $\mathbf{x}$ adheres to S.

To prove the converse, we assume S contains all its adherent points and show that S is closed. Assume $\mathbf{x} \in \mathbf{R}^n - S$. Then $\mathbf{x} \notin S$, so $\mathbf{x}$ does not adhere to S. Hence some ball $B(\mathbf{x})$ does not intersect S, so $B(\mathbf{x}) \subseteq \mathbf{R}^n - S$. Therefore $\mathbf{R}^n - S$ is open, and hence S is closed.

3.19 Definition of closure. *The set of all adherent points of a set* S *is called the closure of* S *and is denoted by* $\bar{S}$.

For any set we have $S \subseteq \bar{S}$ since every point of S adheres to S. Theorem 3.18 shows that the opposite inclusion $\bar{S} \subseteq S$ holds if and only if S is closed. Therefore we have:

Theorem 3.20. *A set S is closed if and only if $S = \bar{S}$.*

3.21 Definition of derived set. *The set of all accumulation points of a set S is called the derived set of S and is denoted by S'.*

Clearly, we have $\bar{S} = S \cup S'$ for any set S. Hence Theorem 3.20 implies that S is closed if and only if $S' \subseteq S$. In other words, we have:

Theorem 3.22. *A set S in $\mathbf{R}^n$ is closed if, and only if, it contains all its accumulation points.*

3.8 THE BOLZANO–WEIERSTRASS THEOREM

3.23 Definition of a bounded set. *A set S in $\mathbf{R}^n$ is said to be bounded if it lies entirely within an n-ball $B(\mathbf{a}; r)$ for some $r > 0$ and some $\mathbf{a}$ in $\mathbf{R}^n$.*

Theorem 3.24 (Bolzano–Weierstrass). *If a bounded set S in $\mathbf{R}^n$ contains infinitely many points, then there is at least one point in $\mathbf{R}^n$ which is an accumulation point of S.*

Proof. To help fix the ideas we give the proof first for $\mathbf{R}^1$. Since S is bounded, it lies in some interval $[-a, a]$. At least one of the subintervals $[-a, 0]$ or $[0, a]$ contains an infinite subset of S. Call one such subinterval $[a_1, b_1]$. Bisect $[a_1, b_1]$ and obtain a subinterval $[a_2, b_2]$ containing an infinite subset of S, and continue this process. In this way a countable collection of intervals is obtained, the nth interval $[a_n, b_n]$ being of length $b_n - a_n = a/2^{n-1}$. Clearly, the sup of the left endpoints a_n and the inf of the right endpoints b_n must be equal, say to x. [Why are they equal?] The point x will be an accumulation point of S because, if r is any positive number, the interval $[a_n, b_n]$ will be contained in $B(x; r)$ as soon as n is large enough so that $b_n - a_n < r/2$. The interval $B(x; r)$ contains a point of S distinct from x and hence x is an accumulation point of S. This proves the theorem for $\mathbf{R}^1$. (Observe that the accumulation point x may or may not belong to S.)

Next we give a proof for $\mathbf{R}^n$, $n > 1$, by an extension of the ideas used in treating $\mathbf{R}^1$. (The reader may find it helpful to visualize the proof in $\mathbf{R}^2$ by referring to Fig. 3.1.)

Since S is bounded, S lies in some n-ball $B(\mathbf{0}; a)$, $a > 0$, and therefore within the n-dimensional interval J_1 defined by the inequalities

$$-a \le x_k \le a \qquad (k = 1, 2, \ldots, n).$$

Here J_1 denotes the cartesian product

$$J_1 = I_1^{(1)} \times I_2^{(1)} \times \cdots \times I_n^{(1)};$$

that is, the set of points $(x_1, \ldots, x_n)$, where $x_k \in I_k^{(1)}$ and where each $I_k^{(1)}$ is a one-dimensional interval $-a \le x_k \le a$. Each interval $I_k^{(1)}$ can be bisected to

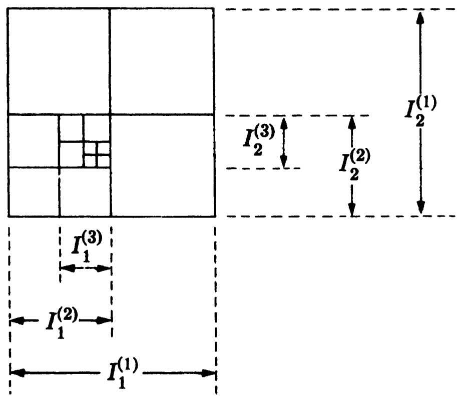

Figure 3.1

form two subintervals $I_{k,1}^{(1)}$ and $I_{k,2}^{(1)}$, defined by the inequalities

$$I_{k,1}^{(1)}: -a \leq x_k \leq 0; \qquad I_{k,2}^{(1)}: 0 \leq x_k \leq a.$$

Next, we consider all possible cartesian products of the form

$$I_{1,k_1}^{(1)} \times I_{2,k_2}^{(1)} \times \cdots \times I_{n,k_n}^{(1)}, \tag{a}$$

where each $k_i = 1$ or 2. There are exactly 2^n such products and, of course, each such product is an n-dimensional interval. The union of these 2^n intervals is the original interval J_1, which contains S; and hence at least one of the 2^n intervals in (a) must contain infinitely many points of S. One of these we denote by J_2, which can then be expressed as

$$J_2 = I_1^{(2)} \times I_2^{(2)} \times \cdots \times I_n^{(2)},$$

where each $I_k^{(2)}$ is one of the subintervals of $I_k^{(1)}$ of length a. We now proceed with J_2 as we did with J_1, bisecting each interval $I_k^{(2)}$ and arriving at an n-dimensional interval J_3 containing an infinite subset of S. If we continue the process, we obtain a countable collection of n-dimensional intervals $J_1, J_2, J_3, \ldots$, where the mth interval J_m has the property that it contains an infinite subset of S and can be expressed in the form

$$J_m = I_1^{(m)} \times I_2^{(m)} \times \cdots \times I_n^{(m)}, \qquad \text{where } I_k^{(m)} \subseteq I_k^{(1)}.$$

Writing

$$I_k^{(m)} = [a_k^{(m)}, b_k^{(m)}],$$

we have

$$b_k^{(m)} - a_k^{(m)} = \frac{a}{2^{m-2}} \qquad (k = 1, 2, \cdots, n).$$

For each fixed k, the sup of all left endpoints $a_k^{(m)}$, $(m = 1, 2, \ldots)$, must therefore be equal to the inf of all right endpoints $b_k^{(m)}$, $(m = 1, 2, \ldots)$, and their common value we denote by t_k. We now assert that the point $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ is an accumulation point of S. To see this, take any n-ball $B(\mathbf{t}; r)$. The point $\mathbf{t}$, of course, belongs to each of the intervals $J_1, J_2, \ldots$ constructed above, and when m is such that $a/2^{m-2} < r/2$, this neighborhood will include J_m. But since J_m contains infinitely many points of S, so will $B(\mathbf{t}; r)$, which proves that $\mathbf{t}$ is indeed an accumulation point of S.

3.9 THE CANTOR INTERSECTION THEOREM

As an application of the Bolzano–Weierstrass theorem we prove the Cantor intersection theorem.

Theorem 3.25. *Let $\{Q_1, Q_2, \ldots\}$ be a countable collection of nonempty sets in $\mathbf{R}^n$ such that:*

i) $Q_{k+1} \subseteq Q_k \qquad (k = 1, 2, 3, \ldots)$.

ii) *Each set Q_k is closed and Q_1 is bounded.*

Then the intersection $\bigcap_{k=1}^{\infty} Q_k$ is closed and nonempty.

Proof. Let $S = \bigcap_{k=1}^{\infty} Q_k$. Then S is closed because of Theorem 3.13. To show that S is nonempty, we exhibit a point $\mathbf{x}$ in S. We can assume that each Q_k contains infinitely many points; otherwise the proof is trivial. Now form a collection of distinct points $A = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$, where $\mathbf{x}_k \in Q_k$. Since A is an infinite set contained in the bounded set Q_1, it has an accumulation point, say $\mathbf{x}$. We shall show that $\mathbf{x} \in S$ by verifying that $\mathbf{x} \in Q_k$ for each k. It will suffice to show that $\mathbf{x}$ is an accumulation point of each Q_k, since they are all closed sets. But every neighborhood of $\mathbf{x}$ contains infinitely many points of A, and since all except (possibly) a finite number of the points of A belong to Q_k, this neighborhood also contains infinitely many points of Q_k. Therefore $\mathbf{x}$ is an accumulation point of Q_k and the theorem is proved.

3.10 THE LINDELÖF COVERING THEOREM

In this section we introduce the concept of a *covering* of a set and prove the *Lindelöf covering theorem.* The usefulness of this concept will become apparent in some of the later work.

3.26 Definition of a covering. *A collection F of sets is said to be a covering of a given set S if $S \subseteq \bigcup_{A \in F} A$. The collection F is also said to cover S. If F is a collection of open sets, then F is called an open covering of S.*

Examples

1. The collection of all intervals of the form $1/n < x < 2/n$, $(n = 2, 3, 4, \ldots)$, is an open covering of the interval $0 < x < 1$. This is an example of a countable covering.
2. The real line $\mathbf{R}^1$ is covered by the collection of all open intervals (a, b). This covering is not countable. However, it contains a countable covering of $\mathbf{R}^1$, namely, all intervals of the form $(n, n + 2)$, where n runs through the integers.

3. Let $S = \{(x, y) : x > 0, y > 0\}$. The collection F of all circular disks with centers at (x, x) and with radius x, where $x > 0$, is a covering of S. This covering is not countable. However, it contains a countable covering of S, namely, all those disks in which x is rational. (See Exercise 3.18.)

The Lindelöf covering theorem states that every open covering of a set S in $\mathbf{R}^n$ contains a countable subcollection which also covers S. The proof makes use of the following preliminary result:

Theorem 3.27 *Let $G = \{A_1, A_2, \dots\}$ denote the countable collection of all n-balls having rational radii and centers at points with rational coordinates. Assume $\mathbf{x} \in \mathbf{R}^n$ and let S be an open set in $\mathbf{R}^n$ which contains $\mathbf{x}$. Then at least one of the n-balls in G contains $\mathbf{x}$ and is contained in S. That is, we have*

$$\mathbf{x} \in A_k \subseteq S \qquad \text{for some } A_k \text{ in } G.$$

Proof. The collection G is countable because of Theorem 2.27. If $\mathbf{x} \in \mathbf{R}^n$ and if S is an open set containing $\mathbf{x}$, then there is an n-ball $B(\mathbf{x}; r) \subseteq S$. We shall find a point $\mathbf{y}$ in S with rational coordinates that is "near" $\mathbf{x}$ and, using this point as center, will then find a neighborhood in G which lies within $B(\mathbf{x}; r)$ and which contains $\mathbf{x}$. Write

$$\mathbf{x} = (x_1, x_2, \dots, x_n),$$

and let y_k be a rational number such that $|y_k - x_k| < r/(4n)$ for each $k = 1, 2, \dots, n$. Then

$$\|\mathbf{y} - \mathbf{x}\| \le |y_1 - x_1| + \cdots + |y_n - x_n| < \frac{r}{4}.$$

Next, let q be a rational number such that $r/4 < q < r/2$. Then $\mathbf{x} \in B(\mathbf{y}; q)$ and $B(\mathbf{y}; q) \subseteq B(\mathbf{x}; r) \subseteq S$. But $B(\mathbf{y}; q) \in G$ and hence the theorem is proved. (See Fig. 3.2 for the situation in $\mathbf{R}^2$.)

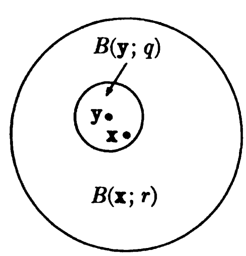

Figure 3.2

Theorem 3.28 (Lindelöf covering theorem). *Assume $A \subseteq \mathbf{R}^n$ and let F be an open covering of A. Then there is a countable subcollection of F which also covers A.*

Proof. Let $G = \{A_1, A_2, \dots\}$ denote the countable collection of all n-balls having rational centers and rational radii. This set G will be used to help us extract a countable subcollection of F which covers A.

Assume $\mathbf{x} \in A$. Then there is an open set S in F such that $\mathbf{x} \in S$. By Theorem 3.27 there is an n-ball A_k in G such that $\mathbf{x} \in A_k \subseteq S$. There are, of course, infinitely many such A_k corresponding to each S, but we choose only one of these, for example, the one of smallest index, say $m = m(\mathbf{x})$. Then we have $\mathbf{x} \in A_{m(\mathbf{x})} \subseteq S$. The set of all n-balls $A_{m(\mathbf{x})}$ obtained as $\mathbf{x}$ varies over all elements of A is a countable collection of open sets which covers A. To get a countable subcollection of F which covers A, we simply correlate to each set $A_{k(\mathbf{x})}$ one of the sets S of F which contained $A_{k(\mathbf{x})}$. This completes the proof.

3.11 THE HEINE–BOREL COVERING THEOREM

The Lindelöf covering theorem states that from any open covering of an arbitrary set A in $\mathbf{R}^n$ we can extract a countable covering. The Heine–Borel theorem tells us that if, in addition, we know that A is closed and bounded, we can reduce the covering to a finite covering. The proof makes use of the Cantor intersection theorem.

Theorem 3.29 (Heine–Borel). *Let F be an open covering of a closed and bounded set A in $\mathbf{R}^n$. Then a finite subcollection of F also covers A.*

Proof. A countable subcollection of F, say $\{I_1, I_2, \dots\}$, covers A, by Theorem 3.28. Consider, for $m \geq 1$, the finite union

$$S_m = \bigcup_{k=1}^{m} I_k.$$

This is open, since it is the union of open sets. We shall show that for some value of m the union S_m covers A.

For this purpose we consider the complement $\mathbf{R}^n - S_m$, which is closed. Define a countable collection of sets $\{Q_1, Q_2, \dots\}$ as follows: $Q_1 = A$, and for $m > 1$,

$$Q_m = A \cap (\mathbf{R}^n - S_m).$$

That is, Q_m consists of those points of A which lie outside of S_m. If we can show that for some value of m the set Q_m is empty, then we will have shown that for this m no point of A lies outside S_m; in other words, we will have shown that some S_m covers A.

Observe the following properties of the sets Q_m: Each set Q_m is closed, since it is the intersection of the closed set A and the closed set $\mathbf{R}^n - S_m$. The sets Q_m are decreasing, since the S_m are increasing; that is, $Q_{m+1} \subseteq Q_m$. The sets Q_m, being subsets of A, are all bounded. Therefore, if no set Q_m is empty, we can apply the Cantor intersection theorem to conclude that the intersection $\bigcap_{k=1}^{\infty} Q_k$ is also not empty. This means that there is some point in A which is in all the sets Q_m, or, what is the same thing, outside all the sets S_m. But this is impossible, since $A \subseteq \bigcup_{k=1}^{\infty} S_k$. Therefore some Q_m must be empty, and this completes the proof.