

## Point Set Topology:

$$B(a, r) = \{x \in \mathbb{R}^n \mid \|x - a\| < r\}$$

A set  $S \subseteq \mathbb{R}^n$  is open if  
 $\forall x \in S, \exists B(x, r) \subset S$

### ① Interior point

HW

② Show, A set  $S \subseteq \mathbb{R}^n$  is open if all its pts are interior.

Open:  $S = \text{int } S$

Show

- ①  $\mathbb{R}$  is open
- ②  $\mathbb{R}^n$  is open

Take  $(a_1, b_1), (a_2, b_2), (a_3, b_3) \dots (a_n, b_n)$

Show, the cartesian product  $(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \times \dots \times (a_n, b_n)$  is open.

Lemma:-

- ① Intersection of a finite collection of open sets is open
- ② Union of any collection of open sets is open.

## Structure of open set in $\mathbb{R}$

### Component Interval:

$S$  open subset in  $\mathbb{R}^1$ . An open interval (finite/infinite)  $I$  is called component interval of  $S$  if  $I \subseteq S$  &  $\nexists$  any open interval,  $J \neq I$  s.t.  $I \subseteq J \subseteq S$

Q. Show that every point of a non-empty open set  $S$  belongs to one & only one component interval of  $S$ .

### Close Set:

Def<sup>n</sup> ①  $S \subseteq \mathbb{R}^n$  is closed if its complement  $\mathbb{R}^n - S$  is open.

$$\textcircled{1} [a_1, b_1]^c = \underbrace{(-\infty, a_1)} \cup \underbrace{(b_1, \infty)}$$

$$\textcircled{2} [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$$

Lemma ③ Union of finite collection of closed set is closed.  
ii) Arbitrary Intersection of closed set is closed.

### Adherent point

$S \subseteq \mathbb{R}^n$  &  $x \in \mathbb{R}^n$  (any point in  $\mathbb{R}^n$ )

Then  $x$  is adherent to  $S$  if every open ball  $B(x, r)$  contains at least one point in  $S$ .

$$\text{Adh } S = \left\{ x \in \mathbb{R}^n \mid B(x, r) \cap S \neq \emptyset \text{ for all } r > 0 \right\}$$

Q.  $A$  open  $B$  closed.  
Show,  $A - B$  open,  $B - A$  closed.

Q If  $S \subseteq \mathbb{R}$ .  $S$  is bounded above,  
then,  $\text{Sup } S \in \text{Adh } S$ .

Def<sup>n</sup> (accumula<sup>n</sup> pt) = (limit pt.)

$$S \subseteq \mathbb{R}^n$$

$$x \in \mathbb{R}^n$$

$x$  is called accumula<sup>n</sup> pt. of  $S$  if  
every  $B(x, \epsilon)$  contains at least one point  
of  $S$  distinct from  $x$ .

Q  $\text{Acc } S = \text{Adh}(S - \{x\})$

$x \in S$  but  $x \notin \text{Acc}(S)$

$\Rightarrow x$  is isolated.

Adherence pt. & Limit point

Q ①  $S = \{1/n\}_{n \in \mathbb{N}}$   
Find limit pt./accumulation pt.

②  $\mathbb{Q}$  set of rationals,  
Find limit pt.

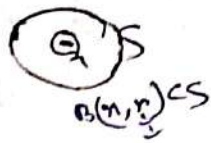
~~Thm~~ Thm  $\Rightarrow$   $x$  is acc. of  $S \subseteq \mathbb{R}^n$ . Then, every  $n$ -ball  
 $B(x)$  contains an infinitely many pt. of  $S$ .

Closed Set :-

A set  $S \subseteq \mathbb{R}^n$  is closed iff it contains  
all its adherent pt.

Closure (Def<sup>n</sup>): The set of all adherent pt. of a set  $S$  is called a closure of  $S$  ( $\bar{S}$ ).

$$\boxed{S \text{ is closed iff } S = \bar{S}}$$



Derived Set :-

Set of all accumulation pt. of a set is called derived set of  $S$  ( $S'$ )

$$\boxed{* \bar{S} = S \cup S' \text{ for any set } S \subseteq \mathbb{R}^n}$$

$$\boxed{S \text{ is closed iff } S' \subseteq S}$$

Def<sup>n</sup> (Bounded set)

A set  $S \subseteq \mathbb{R}^n$  is bounded if it lies entirely within a ball  $B(a, r)$  for some  $r > 0$  & for some  $a \in \mathbb{R}^n$ .

Bolzano-Weierstrass :-

If a bounded set  $S \subseteq \mathbb{R}^n$  contains infinitely many points, then there is at least one pt. in  $\mathbb{R}^n$  which is an accumulation pt. of  $S$ .

Cantor Intersection Theorem :-

Let  $\{Q_1, Q_2, \dots\}$  be countable collection of non-empty sets in  $\mathbb{R}^n$  such that

- (i)  $Q_{k+1} \subseteq Q_k$  ( $k=1, 2, 3, \dots$ )
  - (ii) Each set  $Q_k$  is closed &  $Q_1$  bounded.
- Then the intersection  $\bigcap_{k=1}^{\infty} Q_k$  is closed & non-empty.