Point Set Topology:

$B(a, r) = \{x \in \mathbb{R} \mid \|x-a\| < r\}$

A set $S \subseteq \mathbb{R}^n$ is open if $\forall x \in S,\ \exists B(x, r) \subset S$

① Interior point

H.W

② Show, A set $S \subseteq \mathbb{R}^n$ is open if all its pts are interior.

Open : $S = \text{int } S$

Show ① $\mathbb{R}$ is open
② $\mathbb{R}^n$ is open

Take $(a_1, b_1), (a_2, b_2), (a_3, b_3) \ldots (a_n, b_n)$

Show, the cartesian product $(a_1, b_1) \times (a_2 \times b_2) \times (a_3 \times b_3) \times \ldots (a_n \times b_n)$ is open.

Lemma:- ① Intersection of a finite collection of open sets is open
② Union of any collection of open sets is open.

Structure of open set in $\mathbb{R}$

Component Interval:

S open subset in $\mathbb{R}^1$. An open interval (finite/infinite) I is called component interval of S if $I \subseteq S$ & $\nexists$ any open interval, $J \neq I$ s.t $I \subseteq J \subseteq S$

Q. Show that every point of a non-empty open set S belongs to one & only one component interval of S.

Close Set:

Def^n ① $S \subseteq \mathbb{R}^n$ is closed if its complement $\mathbb{R}^n - S$ is open.

① $[a_1, b_1]^c = (-\infty, a_1) \cup (b_1, \infty)$

② $[a_1, b_1] \times [a_2 \times b_2] \times \dots \times [a_n, b_n]$

Lemma ③ i) Union of finite collection of closed set is closed.
ii) Arbitrary Intersection of close set is close.

Adherent point

$S \subseteq \mathbb{R}^n$ & $x \in \mathbb{R}^n$ (any point in $\mathbb{R}^n$)

Then x is adherent to S if every open ball $B(x, r)$ contains at least one point in S.

$$\text{Adh } S = \{x \in \mathbb{R}^n \,/\, B(x,r) \cap S \neq \emptyset \text{ for all } r > 0\}$$

Q. A open B closed.
Show, $A - B$ open, $B - A$ closed.

Q. If $S \subseteq \mathbb{R}$. S is bounded above, Then, $\sup S \in \text{Adh } S$.

Defⁿ (Accumulaⁿ pt) = (Limit pt.)

$S \subseteq \mathbb{R}^n$

$x \in \mathbb{R}^n$

x is called accumulaⁿ pt. of S if every $B(x, r)$ contains at least one point of S distinct from x.

Q. $\text{Acc } S = \text{Adh}(S - \{x\})$

$\boxed{x \in S \text{ but } x \notin \text{Acc}(S)}$ $\Rightarrow x$ is isolated.

Adherence pt. & Limit point

Q. ① $S = \{1/n\}_{n \in \mathbb{N}}$

Find limit pt. / accumulation pt.

② $\mathbb{Q}$ set of rationals, Find limit pt.

Thm :- x is acc. of $S \subseteq \mathbb{R}^n$. Then, every n-ball $B(x)$ contains an infinitely many pt. of S.

Closed Set :- A set $S \subseteq \mathbb{R}^n$ is closed iff it contains all its adherent pt.

Closure (defn): The set of all adherent pt. of a set S is called closure of S $(\bar{S})$.

$$\boxed{S \text{ is closed iff } S = \bar{S}}$$

$B(x, r) \subset S$

Derived Set:-
Set of all accumulation pt. of a set is called derived set of S (S')

$$\boxed{\bar{S} = S \cup S' \text{ for any set } S \subseteq \mathbb{R}^n}$$

$$\boxed{S \text{ is closed iff } S' \subseteq S}$$

Defn (Bounded set)
A set $S \subseteq \mathbb{R}^n$ is bounded if it lies entirely within a ball $B(\underset{\sim}{a}, r)$ for some $r > 0$ & for some $\underset{\sim}{a} \in \mathbb{R}^n$

Bolzano-Weierstrass:-
If a bounded set $S \subseteq \mathbb{R}^n$ contains infinitely many points, then there is at least one pt. in $\mathbb{R}^n$ which is an accumulation pt. of S.

Cantor Intersection Theorem:
Let $\{Q_1, Q_2, \dots\}$ be countable collection of non-empty sets in $\mathbb{R}^n$ such that
(i) $Q_{K+1} \subseteq Q_K$ $(K = 1, 2, 3, \dots)$
(ii) Each set Q_K is closed & Q_1 bounded.
Then the intersection $\bigcap_{K=1}^{\infty} Q_K$ is closed & non-empty.