Measure Theory

Q1 Let $\mathcal{A}$ be a collection of subsets of X such that $X \in \mathcal{A}$ & $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$. Show $\mathcal{A}$ is an algebra.

Q2

(A_n): Seq. of sets

Let, $\lim\limits_{n\to\infty} \inf A_n = \bigcup\limits_{n\in\mathbb{N}} \bigcap\limits_{k\geq n} A_k$

$\lim\limits_{n\to\infty} \sup A_n = \bigcap\limits_{n\in\mathbb{N}} \bigcup\limits_{k\geq n} A_k$

Show $\lim\limits_{n\to\infty} \inf A_n \subseteq \lim\limits_{n\to\infty} \sup A_n$

Extended Real Number

$$\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

Convention: $\infty + x = \infty \quad \forall x \in \overline{\mathbb{R}}$ except $x = -\infty$

[$\infty - \infty$ is undefined]

Let, $A_1, A_2, A_3, \ldots \subseteq \overline{\mathbb{R}}$

Def^n:

$\overline{\lim\limits_{n\to\infty}} A_n = \lim\limits_{n\to\infty} \sup A_n = \bigcap\limits_{n\geq 1} \bigcup\limits_{k\geq n} A_k$

$\underline{\lim\limits_{n\to\infty}} A_n \quad \lim\limits_{n\to\infty} \inf A_n = \bigcup\limits_{n\geq 1} \bigcap\limits_{k\geq n} A_k$

$$\overline{\lim_{n\to\infty}} A_n \supseteq \underline{\lim_{n\to\infty}} A_n$$

$\overline{\lim_{n\to\infty}} A_n$ = "independently many often sets"

$\underline{\lim_{n\to\infty}} A_n$ = "eventually sets"

④ **Probability Space**

A triplet $(\Omega, \mathcal{F}, P)$

Ω = Set of outcomes

$\mathcal{F}$ = Set of events

$P : \mathcal{F} \to [0,1]$ is a fn that assigns probabilities to event.

$\mathcal{F}$ = a σ-field (σ-algebra)

① $\mathcal{F}$ is called σ-algebra (σ-field) which is a (non-empty) collecn of subsets of Ω satisfying

i) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

ii) If A_i is a countable seq of sets in $\mathcal{F}$ then $\bigcup_i A_i \in \mathcal{F}$.

② **Measure space:**

$(\Omega, \mathcal{F})$ is called a measure space (space where we can put a measure)

③ **Measure:**

A measure is a non-neg countably additive set fn. i.e, a fn $\mu : \mathcal{F} \to \mathbb{R}$

with i) $\mu(A) \geq \mu(\phi) = 0 \quad \forall A \in \mathcal{F}$

ii) If $A_i \in \mathcal{F}$ is a countable seq of disjoint sets, then

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$$

[If $\mu(\Omega) = 1$, then μ = prob. measure]

Theorem:-

Let μ be a measure on $(\Omega, \mathcal{F})$

(i) monotonicity :

$$A \subset B \Rightarrow \mu(A) \leq \mu(B)$$

(ii) (Sub - additivity)

If $A \subset \bigcup_{m=1}^{\infty} A_m$

then $\mu(A) \leq \sum_{m=1}^{\infty} \mu(A_m)$

(iii) (Continuity from below)

If $A_i \uparrow A$ $[A_1 \subset A_2 \subset \dots]$

& $\bigcap_i A_i = A$ then $\mu(A_i) \uparrow \mu(A)$

(iv) (Continuity from above)

If $A_i \downarrow A$ $(A_1 \supset A_2 \supset \dots)$

& $\bigcap_i A_i = A$ & $\mu(A_1) < \infty$

then $\mu(A_i) \downarrow \mu(A)$

ii) $\bigcup_{m=1}^{\infty} A_m = \bigcup_{i=1}^{\infty} \left(A_i \left(\bigcap_{j=1}^{i-1} A_j^c\right)\right)$

$$\Rightarrow \mu\left(\bigcup_{m=1}^{\infty} A_m\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i \bigcap_{j<i} A_j^c\right)$$

$$= \sum \mu\left(A_i \bigcap_{j<i} A_j^c\right)$$

$$\leq \sum \mu(A_i)$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup (A_2 - A_1) \cup \dots \cup (A_n - A_{n-1})$$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{n=1}^{\infty} (A_n - A_{n-1}) \quad [A_0 = \phi]$$

$$\overline{\lim_{n\to\infty}} A_n := \inf_{n\geq 1} \sup_{k\geq n} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

[$\omega \in \overline{\lim_{n\to\infty}} A_n$ iff for all n, $\omega \in A_k$ for some $k \geq n$]

<u>Corollary:</u> $\omega \in \overline{\lim_{n\to\infty}} A_n$ iff $\omega \in A_n$ for infinitely many n.

$$\underline{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \sup_n \inf_{k\geq n} A_n$$

[$\omega \in \underline{\lim} A_n$ iff for some n, $\omega \in A_k \ \forall k \geq n$]

<u>Corollary:</u> $\omega \in \underline{\lim} A_n$ iff $\omega \in A_n$ eventually
[all but finitely many]

$$(\overline{\lim} A_n)^c = \underline{\lim} A_n^c, \quad (\underline{\lim} A_n)^c = \overline{\lim} A_n^c$$

$$\underline{\lim} A_n \subseteq \overline{\lim} A_n$$

<u>Defn:</u> (Limit of a seq. of sets)
$A_1, A_2, A_3, \dots$

$$\lim_{n\to\infty} A_n := \overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n$$

[If $\overline{\lim} = \underline{\lim}$ then lim exists & defined as the equal values]

<u>Ex:</u> Let $A = (a,b)$, $B = (c,d)$

$A \cap B = \phi$

$a, b, c, d \in \mathbb{R}$

Find $\overline{\lim} C_n$ & $\underline{\lim} C_n$ where $C_n = \begin{cases} A & \text{if } n \text{ odd} \\ B & \text{" } n \text{ even} \end{cases}$

$\Rightarrow \overline{\lim}\, C_n = (a,b) \cup (c,d)$

$\underline{\lim}\, C_n = \phi$

Examples ①

Largest σ-field : Collecⁿ of all subsets of Ω

Smallest σ-field : $\{\Omega, \phi\}$

[Let $A \subset \Omega$, $\mathcal{F} = \{\phi, \Omega, A, A^c\}$ = smallest σ-field containing A]

//Count Measure//:

Ω = any set

$\mathcal{F}$ = all subsets of Ω

Define: $\mu(A)$ = # points of A

If A has n members, $n = 0, 1, 2 \ldots$

Then $\mu(A) = n$

Show. μ is a measure (count measure)

$\Rightarrow$ We have to show that μ satisfies the property of measures.

② Let $\Omega = \{x_1, x_2, \ldots\}$ finite or countably many

Let $p_1, p_2, \ldots$ (non-neg numbers)

$\mathcal{F}$ = all subsets of Ω

Define: $\mu(A) := \sum_{x_i \in A} p_i$

[If $A = \{x_{i_1}, x_{i_2}, \ldots\}$ then $\mu(A) = p_{i_1} + p_{i_2} + \ldots$

Show μ is a measure on $\mathcal{F}$.

If $\sum p_i = 1$ then μ is a (discrete) prob. measure

If all $p_i = 1$ then μ is a count measure.

Defn (Length Measure)

Let $A \subset \mathbb{R}$, A = interval (open, close, semi-close) end pts. a, b

Define $\mu(A) = b - a$

If A is complicated set, (then ...)

μ is determined on collection of "Borel sets" of $\mathbb{R}$.

Define: $\mathcal{B}(\mathbb{R})$ = smallest σ-field of $\mathbb{R}$ ~~con~~ containing all intervals $(a, b]$

Measures on Real Line:

Distribution

Defn:- (Random variable)

A real valued fn $X : \Omega \to \mathbb{R}$ is called random variable if $\forall B \in \mathcal{B}(\mathbb{R})$, $X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$ where $(\Omega, \mathcal{F}, P)$ is a prob. space.

X : (Indicator f^n of a set $A \in \mathcal{F}$ be defined by $I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$

$(\Omega, \mathcal{F}, P)$

$\boxed{X^{-1}(A)}$

$$P(\{\omega \in \Omega \mid I_A(\omega) = 1\}) = P(A)$$

$$A_n = \{\omega \in \Omega \mid X(\omega) \le n\}$$

$$P(\{\omega \in \Omega / X(\omega) \le n\}) = P(A_n)$$

H.W Durret page : 10 - Dist^n f^n & its properties w.r.t measure space

Let, $f : \mathbb{R} \to \mathbb{R}$ measurable f^n,
Prove that, $\{x : f(x) = \alpha\}$ is ~~measu~~ mble $\forall \alpha \in \overline{\mathbb{R}}$

$\Rightarrow$ $\{x : f(x) = \alpha\}$

$= \{x : \underbrace{f(x) \le \alpha / x : f(x) < \alpha}\}$

We have to show that both belongs to the $\mathcal{B}$ (Borel set) from the def^n of $\mathcal{B}$ (Borel set)

$(\Omega, \mathcal{A}, P)$: prob. space

$\Omega \xrightarrow{f} \mathbb{R}$

Defn: ~~A real valued~~

Let $(\Omega, \mathcal{A})$ and (E, ξ) be two measurable spaces.

A fun $f: \Omega \to E$ is called measurable (with respect to the σ-field $\mathcal{A}$ and ξ) if for every $B \in \xi$, the set $f^{-1}(B) \in \mathcal{A}$

Defn: A real valued random variable is a real-valued measurable fun on a prob. space.

(2) A d-dimensional random variable (random vector) is a $\mathbb{R}^d$-valued mble fun on a prob space.

Lemma: If $\mathcal{C}$ be any collec of subsets of E such that, $\xi = \sigma(\mathcal{C})$ Then; $f: (\Omega, \mathcal{A}) \to (E, \xi)$ is m'ble iff $f^{-1}(C) \in \mathcal{A} \; \forall C \in \mathcal{C}$

Defn: If $\Omega = \mathbb{R}^m$, $E = \mathbb{R}^d$, $f: \mathbb{R}^m \to \mathbb{R}^d$ is m'ble if f is m'ble w.r.t $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$

Lemma: Suppose $f : (\Omega, \mathcal{A}) \to \mathbb{R}$

TFAE (The following are Equivalent)

(1) f is m'ble

(2) $f^{-1}((a,b]) \in \mathcal{A} \quad \forall a,b \in \mathbb{R}$

(3) $f^{-1}((-\infty, x]) \in \mathcal{A} \quad \forall x \in \mathbb{R}$

(4) $f^{-1}((-\infty, x)) \in \mathcal{A} \quad \forall x \in \mathbb{R}$

$$\sigma((-\infty, x) : x \in \mathbb{R}) \equiv \sigma((a,b] : a,b \in \mathbb{R})$$

Proposition: Let $(\Omega, \mathcal{a})$ m'ble space, & $C \in \mathbb{R}$

(1) $f : \Omega \to \mathbb{R}$ m'ble
$\Rightarrow (C \cdot f)$ is also m'ble

(2) $\left.\begin{array}{l} f : \Omega \to \mathbb{R} \\ g : \Omega \to \mathbb{R} \end{array}\right\}$ m'ble

$\Rightarrow (f + g)$ m'ble

$(f \cdot g)$ m'ble

(3) $f : \Omega \to \mathbb{R}$ m'ble

$f \neq 0$

$\Rightarrow \frac{1}{f}$ m'ble

(4) $f = C$ is m'ble (constant fn)

4) $f^{-1}((-\infty, x]) = \begin{cases} \phi & \text{if } x < C \\ \Omega & \text{if } x \geq C \end{cases}$

We know, $\phi, \Omega \in \mathcal{A}$ (Hence proved)

1) If $c=0$, nothing to prove,

If $c>0$, $(c\cdot f)^{-1}((-\infty,x])$

$$= f^{-1}\left(\left(-\infty, \frac{x}{c}\right]\right)$$

If $c<0$, $(c\cdot f)^{-1}(-\infty,x]$

$$= f^{-1}\left(\left[\frac{x}{c}, \infty\right)\right) \in \mathcal{A}$$

$x_n \to x$

$\lim\limits_{n\to\infty} x_n = x$

$\lim\limits_{n\to\infty} X_n = X$

Weak   Strong   almost sure

1) If f m'ble,

$g = f^2$ is also m'ble.

$f:\Omega\to\mathbb{R}$ $(\Omega,\mathcal{A})\to(\mathbb{R},\mathcal{B})$

$\Rightarrow$ Given, $f^{-1}(B)\in\mathcal{A}$ $(B\in\mathcal{B})$

To show, $g^{-1}(B)\in\mathcal{A}$

$f^{-1}(-\infty, a] \in \mathcal{A}$

$(-\infty,0]\cup(0,a]$

Characteristic Function

PGF: $g(s) = E(s^X)$

MGF: $M(t) = E[e^{tX}] = \int e^{tx} dF$

Laplace transform: $L(t) = E[e^{-tX}]$

Fourier transform: $E[e^{-itX}] = \int e^{-itx} dF$

Characteristic fn: $\varphi_X(t) = E[e^{itX}] = \int e^{itx} dF$

Def^n

$$\varphi_X(t) = E(e^{itX}) = E(\cos tX) + i\,E(\sin tX)$$

Prop:-

1) φ exists $\forall$ dist^n

2) $\varphi(0) = 1$

3) $|\varphi(t)| \leq 1 \;\forall t$

4) $\varphi(a+bX) = e^{iat}\varphi(bt)$

5) $\varphi_{-X}(t) = \overline{\varphi_X(t)}$

$E(|e^{itX}|) \geq |E(e^{itX})|$

$\Rightarrow 1 \geq |E(e^{itX})|$

$\Rightarrow 1 \geq |\varphi(t)|$

HW

① Suppose $X \sim \text{Uniform}(0, \theta)$

find $\phi_X(t)$

② Let, $X_n \sim \text{Unif}[-n, n]$

Show, $\phi_n(t) = \dfrac{\sin tn}{tn}$

③ Let $X_1, X_2, \ldots$ iid Cauchy with pdf

$$f(x) = \frac{1}{\pi(1+x^2)}$$

Then, $\dfrac{\sum X_i}{n} = \bar{X}$ show $\bar{X}$ have same dist^n

ch. fn. φ is

Q.: (Uniformly Continuous) for all $\varepsilon>0$, there exists $\delta>0$.

S.t $|\varphi(t)-\varphi(s)|\le\varepsilon$ whenever $|t-s|<\delta$

$\varphi(t)$ is cts. at pt. t

$\forall\ \varepsilon>0, \exists\ \delta>0 \ni \forall\ |\varphi(t+\delta)-\varphi(t)|<\varepsilon$

proof:

$h=s-t$

Assume wlog that $s>t$

$\Rightarrow |\varphi(t)-\varphi(s)| = |E(e^{itx}(e^{ihx}-1))|$

$\le E[|e^{itx}(e^{ihx}-1)|]$

$\le E[|e^{itx}||e^{ihx}-1|]$

$\le E[|e^{ihx}-1|]$

$\le\varepsilon$

Observe that,

as $h\to 0$

$e^{ihx(\omega)}-1 \to 0$

for each $\omega\in\Omega$

and $|e^{ihx}-1|\le|e^{ihx}|+|-1|\le 2$

$\Rightarrow |e^{ihx}-1|\le 2$ (bounded)

$\Rightarrow E[|e^{ihx}-1|]\to 0$ as $h\to 0$

$\Rightarrow \forall\ \varepsilon>0$,
we can choose $h>0$ suff. small

s.t $|\varphi(t)-\varphi(s)|\le\varepsilon$

Inversion formula:-

Let X has ch. fn. $\varphi_X(t)$. Then for any interval (a,b)

$$P[a<X<b]+\frac{P[X=a]+P[X=b]}{2}=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\varphi(t)\,dt$$

Corollary:- If the ch. fun of two r.v X & Y are same, then X & Y have same distn

$$X \overset{d}{=} Y$$

Q

1) $X \sim N(0,1)$

find $\varphi_X(t)$

Soln

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-it)^2 - t^2/2}\,dx$$

$$= \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{2}\,du$$

$\frac{x-it}{\sqrt{2}} = u$

$\Rightarrow \frac{dx}{\sqrt{2}} = du$

$$= \frac{\cancel{2}e^{-t^2/2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-v} \frac{dv}{\cancel{2}\sqrt{v}}$$

$u^2 = v$

$\Rightarrow 2u\,du = dv$

$\Rightarrow du = \frac{dv}{2\sqrt{v}}$

$$= e^{-t^2/2}$$