$\left(\frac{1}{n} - \frac{1}{N}\right)$

Double Sampling For Ratio Estimation:

First sample of size m is ~~__~~ used to obtain ~~$\bar{x}_m$ as an estimate of $\bar{X}$~~ the sample mean $\bar{x}_m$ of x as an estimate of $\bar{X}$. The second sample of size n is enumerated for both X and Y to give $\bar{x}_n$ & $\bar{y}_n$ as the sample means.

Estimator of the population mean

$$\hat{\bar{Y}}_{Rd} = \frac{\bar{y}_n}{\bar{x}_n} \cdot \bar{x}_m$$

We know, that $E(\bar{x}_m) = E(\bar{x}_n) = \bar{X}$

$$E(\bar{y}_n) = \bar{Y}$$

Define,
$$e = \frac{\bar{y}_n - \bar{Y}}{\bar{Y}}$$

$$e_1 = \frac{\bar{x}_n - \bar{X}}{\bar{X}}$$

& $$e_2 = \frac{\bar{x}_m - \bar{X}}{\bar{X}}$$ where, $E(e) = E(e_1) = E(e_2) = 0$

Thus,
$$\bar{y}_n = \bar{Y}(1+e)$$
$$\bar{x}_n = \bar{X}(1+e_1)$$
$$\bar{x}_m = \bar{X}(1+e_2)$$

$$\hat{\bar{Y}}_{Rd} = \frac{\bar{Y}(1+e)(1+e_2)}{(1+e_1)} = \bar{Y}(1+e)(1+e_2)(1+e_1)^{-1}$$

$$\simeq \bar{Y}(1+e)(1+e_2)(1-e_1+e_1^2)$$ [Keeping terms upto 2^{nd} degree where. $(1+e_1)^{-1} = 1 - e_1 + e_1^2 - e_1^3 + \ldots$ provided, $|e_1| < 1$]. The expansion is valid since the difference $|\bar{x}_n - \bar{X}|$ is small much smaller than $\bar{X}$, hence, $|e_1| < 1$]

$$= \bar{Y}(1+e+e_2+ee_2)(1-e_1+e_1^2)$$

$$\simeq \bar{Y}(1+e+e_2-e_1-ee_1-e_1e_2+ee_2+e_1^2)$$

Thus, $E(\hat{\bar{Y}}_{Rd}) \simeq \bar{Y}\left[1+E\left(ee_2 - ee_1 - e_1e_2 + e_1^2\right)\right]$

Now,

$$E(e_1^2) = Var(e_1) = Var\left(\frac{\bar{x}_n - \bar{X}}{\bar{X}}\right) = \frac{1}{\bar{X}^2} Var(\bar{x}_n) = \frac{S_X^2}{\bar{X}^2}\left(\frac{1}{n}-\frac{1}{N}\right)$$

$[\because E(e_1)=0]$

$$E(ee_1) = Cov(e, e_1) = Cov\left(\frac{\bar{y}_n - \bar{Y}}{\bar{Y}}, \frac{\bar{x}_n - \bar{X}}{\bar{X}}\right)$$

$$= \frac{1}{\bar{X}\bar{Y}} Cov(\bar{y}_n, \bar{x}_n) \qquad [\because E(e) = E(e_1) = 0]$$

$$= \frac{1}{\bar{X}\bar{Y}} S_{XY}\left(\frac{1}{n}-\frac{1}{N}\right)$$

~~$E(e_1e_2) = E_1E_2$~~

$$E(ee_2) = Cov(e, e_2) = Cov\left(\frac{\bar{y}_n - \bar{Y}}{\bar{Y}}, \frac{\bar{x}_m - \bar{X}}{\bar{X}}\right)$$

$$= \frac{1}{\bar{X}\bar{Y}} Cov(\bar{y}_n, \bar{x}_m)$$

$$= \frac{1}{\bar{X}\bar{Y}}\left[E\, Cov(\bar{y}_n, \bar{x}_m \mid \bar{x}_m) + Cov(\bar{x}_m, E(\bar{y}_n \mid \bar{x}_m))\right]$$

$[\because Cov(g(X), h(Y))$
$= E\, Cov(h(Y), g(X) \mid X) + Cov(g(X), E(h(Y) \mid X))]$

$$= \frac{1}{\bar{X}\bar{Y}}\left[0 + Cov(\bar{x}_m, \bar{y}_m)\right]$$

[Given, $\bar{x}_m$, $\bar{x}_m$ can be treated as constant and covariance between a variable & a constant is 0, Also, $E(\bar{y}_n \mid \bar{x}_m) = \bar{y}_m$]

$$= \frac{1}{\bar{X}\bar{Y}} S_{XY}\left(\frac{1}{m}-\frac{1}{N}\right)$$

$$E(e_1 e_2) = Cov(e_1, e_2) = \cancel{Cov} \frac{1}{\bar{X}^2} Cov(\bar{x}_n, \bar{x}_m)$$

$$= \frac{1}{\bar{X}^2}\left[E\, Cov(\bar{x}_n, \bar{x}_m \mid \bar{x}_m) + Cov(\bar{x}_m, E(\bar{x}_n \mid \bar{x}_m))\right]$$

$$= \frac{1}{\bar{X}^2}\left[0 + Cov(\bar{x}_m, \bar{x}_m)\right]$$

$$= \frac{Var(\bar{x}_m)}{\bar{X}^2} = \frac{1}{\bar{X}^2} S_x^2 \left(\frac{1}{m} - \frac{1}{N}\right)$$

$$\therefore E(\hat{\bar{Y}}_{Rd}) \simeq \bar{Y}\left[1 + \frac{S_x^2}{\bar{X}^2}\left(\frac{1}{n} - \frac{1}{N}\right) - \frac{1}{\bar{X}\bar{Y}} S_{xy}\left(\frac{1}{n} - \frac{1}{N}\right) + \frac{1}{\bar{X}\bar{Y}} S_{xy}\left(\frac{1}{m} - \frac{1}{N}\right) - \frac{1}{\bar{X}^2} S_x^2\left(\frac{1}{m} - \frac{1}{N}\right)\right]$$

$$= \bar{Y}\left[1 + \frac{S_x^2}{\bar{X}^2}\left(\frac{1}{n} - \frac{1}{m}\right) + \frac{1}{\bar{X}\bar{Y}} S_{xy}\left(\frac{1}{m} - \frac{1}{n}\right)\right]$$

$$= \bar{Y}\left[1 + \frac{S_x^2}{\bar{X}^2}\left(\frac{1}{n} - \frac{1}{m}\right) - \frac{\rho S_x S_y}{\bar{X}\bar{Y}}\left(\frac{1}{n} - \frac{1}{m}\right)\right]$$

$$\cancel{\therefore Bias(\hat{\bar{Y}}_{Rd}) =}$$

$$\therefore Bias(\hat{\bar{Y}}_{Rd}) = E(\hat{\bar{Y}}_{Rd}) - \bar{Y}$$

$$\simeq \bar{Y}\left(\frac{1}{n} - \frac{1}{m}\right)\left(C_x^2 - \rho C_x C_y\right)$$, where, $C_x = \frac{S_x}{\bar{X}}$, $C_y = \frac{S_y}{\bar{Y}}$, $\rho = \frac{S_{xy}}{S_x S_y}$

Remark:

The bias is neighligible if n as well as m is large.

$$MSE(\hat{\bar{Y}}_{Rd}) = E(\hat{\bar{Y}}_{Rd} - \bar{Y})^2$$

$$\simeq E[\bar{Y}(1+e+e_2-e_1) - \bar{Y}]^2$$ [Keeping terms upto ~~2nd order~~ 1st degree]

$$= \bar{Y}^2 E[(e+e_2-e_1)^2]$$

$$= \bar{Y}^2 E[e^2+e_2^2+e_1^2+2ee_2-2e_1e_2-2ee_1]$$

Here, $E(e^2) = Var(e) = Var\left(\frac{\bar{y}_n-\bar{Y}}{\bar{Y}}\right) = \frac{1}{\bar{Y}^2} Var(\bar{y}_n) = \frac{S_Y^2}{\bar{Y}^2}\left(\frac{1}{n}-\frac{1}{N}\right)$

$E(e_2^2) = Var(e_2) = Var\left(\frac{\bar{x}_m-\bar{X}}{\bar{X}}\right) = \frac{1}{\bar{X}^2} Var(\bar{x}_m)$

$$= \frac{1}{\bar{X}^2} S_X^2\left(\frac{1}{m}-\frac{1}{N}\right)$$

$$\therefore MSE(\hat{\bar{Y}}_{Rd}) = \bar{Y}^2\Big[\frac{S_Y^2}{\bar{Y}^2}\left(\frac{1}{n}-\frac{1}{N}\right) + \frac{S_X^2}{\bar{X}^2}\left(\frac{1}{m}-\frac{1}{N}\right) + \frac{S_X^2}{\bar{X}^2}\left(\frac{1}{n}-\frac{1}{N}\right)$$

$$+2\Big\{\frac{S_{XY}}{\bar{X}\bar{Y}}\left(\frac{1}{m}-\frac{1}{N}\right) - \frac{S_X^2}{\bar{X}^2}\left(\frac{1}{m}-\frac{1}{N}\right) - \frac{S_{XY}}{\bar{X}\bar{Y}}\left(\frac{1}{n}-\frac{1}{N}\right)\Big]$$

$$= S_Y^2\left(\frac{1}{n}-\frac{1}{N}\right) - R^2S_X^2\left(\frac{1}{m}-\frac{1}{N}\right) + R^2S_X^2\left(\frac{1}{n}-\frac{1}{N}\right)$$

$$+ 2RS_{XY}\left(\frac{1}{m}-\frac{1}{n}\right)$$

$$\therefore MSE(\hat{\bar{Y}}_{Rd}) \simeq \left(\frac{1}{n}-\frac{1}{N}\right)S_Y^2 + R^2S_X^2\left(\frac{1}{n}-\frac{1}{m}\right)$$

$$-2RS_{XY}\left(\frac{1}{n}-\frac{1}{m}\right)$$

$$\therefore MSE(\hat{\bar{Y}}_{Rd}) \simeq \left(\frac{1}{m}-\frac{1}{N}\right)S_Y^2 + \left(\frac{1}{n}-\frac{1}{m}\right)\left(S_Y^2+R^2S_X^2-2R\rho S_XS_Y\right)$$

where, $R = \frac{\bar{Y}}{\bar{X}}$, $\rho = \frac{S_{XY}}{S_XS_Y}$

For the ratio estimator under the single sampling.

(1) $MSE\left(\hat{\bar{Y}}_R\right) \approx \frac{1-f}{m}\left(S_Y^2 - 2\rho R S_x S_Y + R^2 S_x^2\right)$, $f = \frac{m}{N}$

$= \left(\frac{1}{m} - \frac{1}{N}\right) S_Y^2 + \left(\frac{1}{m} - \frac{1}{N}\right)\left(R^2 S_x^2 - 2\rho R S_x S_Y\right)$

Under double sampling

(2) $MSE\left(\hat{\bar{Y}}_{Rd}\right) \approx \left(\frac{1}{m} - \frac{1}{N}\right) S_Y^2 + \left(\frac{1}{n} - \frac{1}{m}\right)\left(S_Y^2 - 2\rho R S_x S_Y + R^2 S_x^2\right)$

$= \left(\frac{1}{n} - \frac{1}{N}\right) S_Y^2 + \left(\frac{1}{n} - \frac{1}{m}\right)\left(R^2 S_x^2 - 2\rho R S_x S_Y\right)$

[$MSE\left(\hat{\bar{Y}}_{Rd}\right) < MSE\left(\hat{\bar{Y}}_R\right)$

$\Rightarrow \left(\frac{1}{n} - \frac{1}{N}\right) S_Y^2 + \left(\frac{1}{n} - \frac{1}{m}\right)\left(R^2 S_x^2 - 2\rho R S_x S_Y\right)$

$< \left(\frac{1}{m} - \frac{1}{N}\right)\left(S_Y^2 - 2\rho R S_x S_Y + R^2 S_x^2\right)$

$\Rightarrow \left(\frac{2}{m} - \frac{1}{n} - \frac{1}{N}\right)\left(R^2 S_x^2 - 2\rho R S_x S_Y\right) \cancel{\not> 0} + \left(\frac{1}{n} - \frac{1}{m}\right) S_Y^2 > 0$

$\Rightarrow R^2 S_x^2 > 2\rho R S_x S_Y$

$\Rightarrow \rho < \frac{1}{2} R \frac{S_x}{S_Y}$

$\Rightarrow \rho < \frac{1}{2} \frac{\bar{Y}}{\bar{X}} \frac{S_x}{S_Y}$,

$\Rightarrow \rho < \frac{1}{2} \frac{C_x}{C_Y}$, where $C_x = \frac{S_x}{\bar{X}}$, $C_Y = \frac{S_Y}{\bar{Y}}$]

Coeff. of S_Y^2 in (1) $\left(\frac{1}{m} - \frac{1}{N}\right)$ $\Rightarrow$ (2) $\frac{1}{n} - \frac{1}{N} >$ (1) $\frac{1}{m} - \frac{1}{N}$

Coeff. of S_Y^2 in (2) $\left(\frac{1}{n} - \frac{1}{N}\right)$ [$\because \frac{1}{n} > \frac{1}{m}$], usually $m > n$

Coeff. of the other term in both case (1) $\left(\frac{1}{m} - \frac{1}{N}\right)$ & (2) $\left(\frac{1}{n} - \frac{1}{m}\right)$.

Usually, (1) $\frac{1}{m} - \frac{1}{N} <$ (2) $\frac{1}{n} - \frac{1}{m}$

Thus usually, $Var\left(\hat{\bar{Y}}_{Rd}\right) > Var\left(\hat{\bar{Y}}_R\right)$

DOUBLE SAMPLING FOR REGRESSION ESTIMATOR

Estimator of the population mean

$$\hat{\bar{Y}}_{rd} = \bar{y}_n - b_n(\bar{x}_n - \bar{x}_m) \qquad [\bar{y} - b(\bar{x} - \bar{X})]$$

$$E(\hat{\bar{Y}}_{rd}) = E_1 E_2 \left[\bar{y}_n - b_n(\bar{x}_n - \bar{x}_m) \,\middle|\, \bar{x}_m \right]$$

$$= \bar{Y} - E_1 E_2 \left[b_n(\bar{x}_n - \bar{x}_m) \,\middle|\, \bar{x}_m \right]$$

$$\therefore \text{Bias}(\hat{\bar{Y}}_{rd}) = E(\hat{\bar{Y}}_{rd}) - \bar{Y}$$

$$= -E_1 E_2 \left[b_n(\bar{x}_n - \bar{x}_m) \,\middle|\, \bar{x}_m \right]$$

We write,

$$\bar{x}_n = \bar{X}(1+e)$$

$$\bar{x}_m = \bar{X}(1+e') \qquad \text{where,}$$

$$s_{xy} = S_{XY}(1+e_1) \qquad E(e) = E(e') = E(e_1) = E(e_2)$$

$$s_x^2 = S_x^2(1+e_2) \qquad = 0$$

$$\bar{x}_n - \bar{x}_m = \bar{X}(e - e')$$

$$b_n = \frac{s_{xy}}{s_x^2} = \left(\frac{S_{XY}}{S_x^2}\right)\left(\frac{1+e_1}{1+e_2}\right) = \frac{S_{XY}}{S_x^2}(1+e_1)(1+e_2)^{-1}$$

$$= \beta(1+e_1)(1+e_2)^{-1} \qquad \left[\because \beta = \frac{S_{XY}}{S_x^2}\right]$$

$$\simeq \beta(1+e_1)(1-e_2) \qquad [\text{Keeping terms up to 1st order}]$$

$$\simeq \beta(1+e_1-e_2)$$

$$\therefore b_n(\bar{x}_n - \bar{x}_m) \simeq \beta\bar{X}(e-e')(1+e_1-e_2)$$

$$= \beta\bar{X}(e - e' + ee_1 - ee_2 - e'e_1 + e'e_2)$$

Thus,

$$\text{Bias}\left(\hat{\bar{Y}}_{rd}\right) \simeq -\beta \bar{X}\, E\left[(e-e')(1+e_1-e_2)\right]$$

$$= -\beta \bar{X} \quad E_1 E_2 \left[\left[(e-e')\,(1+e_1-e_2) \right] \Big| \bar{x}_m \right]$$

Now, $E_2\left[(e-e') \mid \bar{x}_m\right] = 0$

$E_2\left[e'(e_1-e_2) \mid \bar{x}_m\right] = 0$ [$\because$ Given, $\bar{x}_m$, e' is also given and the covariance term becomes 0]

Need to find $E_2(ee_1 \mid \bar{x}_m)$ & $E_2(ee_2 \mid \bar{x}_m)$

$$E_2(ee_1 \mid \bar{x}_m) = \text{Cov}(e, e_1 \mid \bar{x}_m)$$

$$= \text{Cov}\left(\frac{\bar{x}_n - \bar{X}}{\bar{X}}, \frac{s_{xy} - S_{XY}}{S_{XY}} \Big| \bar{x}_m \right)$$

$$= \frac{1}{\bar{X}\, S_{XY}} \text{Cov}\left(\bar{x}_n, s_{xy} \mid \bar{x}_m\right)$$

$\cancel{s_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x}_m)}$

<u>Result</u> (based on Large sample)

$$\text{Cov}\left(m'_{rs}, m_{r_1 s_1}\right) \approx \left(\frac{1}{n} - \frac{1}{N}\right) \mu_{r+r_1,\, s+s_1}$$

Thus, $\text{Cov}\left(\bar{x}_n, s_{xy} \mid \bar{x}_m\right) \approx \left(\frac{1}{n} - \frac{1}{m}\right) . m_{21}$ (1st sample)

$$\therefore E_2(ee_1 \mid \bar{x}_m) \approx \frac{\left(\frac{1}{n} - \frac{1}{m}\right)}{\bar{X}\, S_{XY}} m_{21} \quad \text{(1st sample)}$$

Similarly,

$$E_2(ee_2 \mid \bar{x}_m) = \text{Cov}(e, e_2 \mid \bar{x}_m)$$

$$= \frac{1}{\bar{X}\, S_X^2} \text{Cov}\left(\bar{x}_n, s_x^2 \mid \bar{x}_m\right)$$

$$\approx \frac{1}{\bar{X}\, S_X^2} \left(\frac{1}{n} - \frac{1}{m}\right) m_{30} \quad \text{(1st sample)}$$

Finally,

$$\text{Bias}\left(\hat{\bar{Y}}_{rd}\right) \simeq -\beta\bar{x}\left(\frac{1}{n}-\frac{1}{m}\right) E_1\left[\frac{m_{21}\,(\text{1st sample})}{\bar{x}\,S_{xy}} - \frac{m_{30}\,(\text{1st sample})}{\bar{x}\,S_x^2}\right]$$

$$= -\beta\bar{x}\left(\frac{1}{n}-\frac{1}{m}\right)\left[\frac{\mu_{21}}{\bar{x}S_{xy}} - \frac{\mu_{30}}{\bar{x}S_x^2}\right]$$

$$= -\beta\left(\frac{1}{n}-\frac{1}{m}\right)\left[\frac{\mu_{21}}{S_{xy}} - \frac{\mu_{30}}{S_x^2}\right]$$

Here, $\mu_{rs} = \frac{1}{N-1}\sum_{\alpha=1}^{N}(x_\alpha-\bar{x})^r(y_\alpha-\bar{Y})^s$

$$m_{rs}\,(\text{1st sample}) = \frac{1}{(m-1)}\sum_{i=1}^{m}(x_i-\bar{x}_m)^r(y_i-\bar{y}_m)^s$$

If either m or n is large, the bias is negligible.

$$\text{Var}\left(\hat{\bar{Y}}_{rd}\right) = E_1 V_2\left(\hat{\bar{Y}}_{rd}\,|\,\bar{x}_m\right) + V_1 E_2\left(\hat{\bar{Y}}_{rd}\,|\,\bar{x}_m\right)$$

$$E_2\left(\hat{\bar{Y}}_{rd}\,|\,\bar{x}_m\right) = E_2\left[\bar{y}_n - b_n(\bar{x}_n-\bar{x}_m)\,|\,\bar{x}_m\right]$$

$$= \bar{y}_m$$

$$V_1E_2\left(\hat{\bar{Y}}_{rd}\,|\,\bar{x}_m\right) = V_1(\bar{y}_m) = S_y^2\left(\frac{1}{m}-\frac{1}{N}\right)$$

$$V_2\left(\hat{\bar{Y}}_{rd}\,|\,\bar{x}_m\right) = V_2\left(\bar{y}_n - b_n(\bar{x}_n-\bar{x}_m)\,|\,\bar{x}_m\right)$$

$$= s_u'^2\left(\frac{1}{n}-\frac{1}{m}\right)$$, where $s_u'^2$ is the variance of $(y-\beta x)$ in the 1st sample.

Thus,

$$Var(\hat{\bar{Y}}_{rd}) = E_1\left[s_u'^2\left(\frac{1}{n}-\frac{1}{m}\right)\right] + V_1[\bar{y}_m]$$

$$= S_u^2\left(\frac{1}{n}-\frac{1}{m}\right) + S_Y^2\left(\frac{1}{m}-\frac{1}{N}\right)$$

, where S_u^2 is the variance of $y-\beta x$ in the population.

$S_u^2 = S_Y^2(1-\rho^2)$ = Residual Variance

$$\therefore Var(\hat{\bar{Y}}_{rd}) = \left(\frac{1}{n}-\frac{1}{m}\right)S_Y^2(1-\rho^2) + S_Y^2\left(\frac{1}{m}-\frac{1}{N}\right)$$

$$= \frac{(1-\rho^2)S_Y^2}{n} + \frac{S_Y^2\rho^2}{m} - \frac{S_Y^2}{N}$$

For single sampling of size m,

$$Var(\hat{\bar{Y}}_{r}) = \frac{1-f}{m}S_Y^2(1-\rho^2) \quad \text{where } f=\frac{m}{N}$$

$$= \left(\frac{1}{m}-\frac{1}{N}\right)S_Y^2(1-\rho^2)$$

$$= \frac{(1-\rho^2)S_Y^2}{m} + \frac{S_Y^2\rho^2}{N} - \frac{S_Y^2}{N}$$

Usually, $n<m<N$

hence, $\frac{1}{n} > \frac{1}{m} > \frac{1}{N}$ $\therefore Var(\hat{\bar{Y}}_{rd}) > Var(\hat{\bar{Y}}_{r})$

<u>Note</u>: Variance of the regression estimator in double sampling is larger than the direct estimator in uniphase sampling when the sample size is m. However, ~~the~~ collection of information on Y for all of the m-units in the 1st phase may be too expensive and double sampling may be preferred on cost consideration.

Unbiased Estimator of $Var(\hat{\bar{Y}}_{rd})$ is

$$\widehat{Var}(\hat{\bar{Y}}_{rd}) = \left(\frac{1}{n} - \frac{1}{m}\right)(1-r^2)s_y^2 + \left(\frac{1}{m} - \frac{1}{N}\right)s_y^2$$

where, $r = \dfrac{s_{xy}}{s_x s_y}$

Allocation Problem :

Cost Function $C = a + c_1 m + c_2 n$.

Where, $a =$ overhead cost

$c_1 =$ Cost / 1st phase sampling unit

$c_2 =$ cost / 2nd phase sampling unit

The quantity to be minimized is

$Z = Var(\hat{\bar{Y}}_{rd}) + \lambda(C - C_0)$ [Assuming the cost to be fixed at C_0]

$$= \frac{S_Y^2 \rho^2}{m} + \frac{S_Y^2(1-\rho^2)}{n} - \frac{S_Y^2}{N} + \lambda(a + c_1 m + c_2 n - C_0)$$

We solve the equations

$$\frac{\partial z}{\partial m} = 0 \Rightarrow -\frac{S_Y^2 \rho^2}{m^2} + \lambda c_1 = 0$$

$$\Rightarrow \frac{\lambda}{S_Y^2} = \frac{\rho^2}{m^2 c_1}$$

$$\frac{\partial z}{\partial n} = 0 \Rightarrow -\frac{S_Y^2(1-\rho^2)}{n^2} + \lambda c_2 = 0$$

$$\Rightarrow \frac{\lambda}{S_Y^2} = \frac{1-\rho^2}{n^2 c_2}$$

$$\therefore \quad \frac{m^2 c_1}{\rho^2} = \frac{n^2 c_2}{1-\rho^2}$$

or, $$\frac{m\sqrt{c_1}}{\rho} = \frac{n\sqrt{c_2}}{\sqrt{1-\rho^2}} = \frac{mc_1 + nc_2}{\rho\sqrt{c_1} + \sqrt{c_2(1-\rho^2)}}$$

(Componendo & dividendo)

Now, $C_0 = a + c_1 m + c_2 n$

$\Rightarrow c_1 m + c_2 n = C_0 - a$

$$\therefore \quad \frac{m\sqrt{c_1}}{\rho} = \frac{n\sqrt{c_2}}{\sqrt{1-\rho^2}} = \frac{C_0 - a}{\rho\sqrt{c_1} + \sqrt{c_2(1-\rho^2)}}$$

$$\therefore \quad m = \frac{\rho}{\sqrt{c_1}} \cdot \frac{(C_0 - a)}{\left(\rho\sqrt{c_1} + \sqrt{c_2(1-\rho^2)}\right)}$$

$$n = \frac{\sqrt{1-\rho^2}}{\sqrt{c_2}} \cdot \frac{(C_0 - a)}{\left(\rho\sqrt{c_1} + \sqrt{c_2(1-\rho^2)}\right)}$$