TWO STAGE ~~SAMPLING~~ SAMPLING

Sampling Scheme : Advantages & Disadvantages :

Though cluster sampling is economical under circumstances, it is generally less efficient than sampling of individual units directly. A compare between direct sampling and cluster sampling units can be achieved by selecting a sample of clusters and surveying only a sample of units within each sample cluster instead of complete enumerating all the units in the sample cluster. Such a procedure is known as two-stage sampling, since the units is selected in two stages. Here clusters are turned as first stage unit (fsu) and the ultimate observational unit are turned as second stage unit (ssu). It may b noted that this procedure can be generalized to multistage ~~sampling~~ sampling where the sampling units at each stage are the clusters of units of the next stage.

This procedure being a compromise between ~~unit or~~ direct sampling and cluster sampling is expected to be

i) more efficient than uni-stage sampling, less efficient than cluster sampling from consideration of operational convenience and cost.

ii) less efficient than uni-stage sampling, more efficient than cluster sampling from the view point of variability, when the sample size in terms of the number of unit is fixed.

In practice, it usually happens that we have more information for group of sampling units from individual units. Hence, the groups are taken as fsu, the information available for them can be used in effecting good stratification or arrangement and in selection of the sample of fsu. Further since the ssu's are selected only from sample fsus, it would be practical to collect information about the ssu at the time of listing them and use the information for obtaining a better sample of ssu. Because of this, it may be possible that a multistage design where the information available at each stage is properly utilised is more efficient than ~~unique~~ uni-stage sampling even from the point of view of sampling variability.

Unbiased estimation of population total / mean :

Notations:

N : Number of FSUs

M_i : Number of SSUs within i^{th} FSU $(i=1(1)N)$

Y_{ij} : j^{th} SSU within the i^{th} FSU of the population $(i=1(1)N,\ j=1(1)M_i)$

n : Number of sample FSUs

m_i : Number of SSUs selected from i^{th} sample FSU $(i=1(1)n)$

y_{ij} : j^{th} selected SSU from the i^{th} sample FSU $\left(\begin{matrix} i=1(1)n \\ j=1(1)m_i \end{matrix}\right)$

If the total values of the selected FSUs ~~where~~ are Known, it can be seen that it could be possible to get an estimator of the population total Y with the help of the probability scheme at the first stage as in cluster sampling. But in two stage sampling the actual total of the selected FSUs are not known, and hence they also have to be estimated on the basis of the selected SSUs using the probability scheme adopted in selecting them.

- y_i = total of the i^{th} sample FSU, $i=1(1)n$

In case of cluster sampling, Y is estimated by $\sum_{i=1}^{n} a_i y_i$, where a_i = inflation factor.

But in case of two stage sampling, y_i's are also unknown and they also have to be estimated, ~~There~~ Their estimate is of the form

$\hat{y}_i = \sum_{j=1}^{m_i} a_{ij} y_{ij}$, where a_{ij} is the inflation factor.

Ultimately, $\hat{Y} = \sum_{i=1}^{n} a_i \hat{y}_i = \sum_{i=1}^{n} a_i \left[\sum_{j=1}^{m_i} a_{ij} y_{ij} \right]$

$\bar{y}_i = \frac{1}{m_i} \sum_{j=1}^{m_i} y_{ij}$ = Mean of the sampled units of the i^{th} selected FSU. $(i=1(1)n)$

$\cancel{\frac{\sum_{j=1}^{M_i} Y_{ij}}{M_i} = \frac{Y_i}{M_i}}$ ~~= Actual Mean of all observations~~

$$\frac{y_i}{M_i} = \frac{\sum_{j=1}^{M_i} y_{ij}}{M_i} = \text{Actual mean of all observations in the } i\text{th selected FSU } (i=1(1)n)$$

Under SRS, $\frac{y_i}{M_i}$ is unbiasedly estimated by $\frac{1}{m_i}\sum_{j=1}^{m_i} y_{ij} = \bar{y}_i$

Thus, y_i is unbiasedly estimated by $\frac{M_i}{m_i}\sum_{j=1}^{m_i} y_{ij} = M_i\bar{y}_i = \hat{y}_i$

$, i = 1(1)n.$

Estimated true mean of all units of the sample FSUs is $\frac{1}{n}\sum_{i=1}^{n}\hat{y}_i$.

$\frac{1}{n}\sum_{i=1}^{n}\hat{y}_i$ unbiasedly estimates the population mean $\frac{Y}{N}$.

$$\therefore \hat{Y} = \frac{N}{n}\sum_{i=1}^{n}\hat{y}_i = \frac{N}{n}\sum_{i=1}^{n}\frac{M_i}{m_i}\sum_{j=1}^{m_i} y_{ij} = \frac{N}{n}\sum_{i=1}^{n} M_i\bar{y}_i$$

▸ Expectation and Variance:

$$E(\hat{Y}) = E_1E_2(\hat{Y})$$

$$Var(\hat{Y}) = E_1V_2(\hat{Y}) + V_1E_2(\hat{Y})$$

where, E_2 & V_2 : Conditional expectation / variance over the SSUs for given sample FSU

E_1 & V_1 : Unconditional mean / variance of the FSUs.

Notation: $M = \frac{1}{N}\sum_{i=1}^{N} M_i$ = Average number of SSUs per FSUs.

$$S_b^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{M_i\bar{Y}_i}{M} - \bar{Y}\right)^2 = \text{Between FSU variance}$$

$$S_{W_i}^2 = \frac{1}{M_i-1}\sum_{j=1}^{M_i}(Y_{ij} - \bar{Y}_i)^2 = i\text{th FSU variance } (i=1(1)N)$$

Now,

$$E(\hat{Y}) = E_1E_2(\hat{Y}\mid FSU) = E_1\left[E_2\left(\frac{N}{n}\sum_{i=1}^{n} M_i\bar{y}_i \mid FSU\right)\right]$$

$$= E_1\left[\frac{N}{n}\sum_{i=1}^{n} M_i E_2(\bar{y}_i \mid FSU)\right]$$

$$= E_1\left[\frac{N}{n}\sum_{i=1}^{n} M_i\bar{Y}_i\right] = E_1\left[\frac{N}{n}\sum_{i=1}^{n} M_i\frac{y_i}{M_i}\right]$$

$$= E_1\left[\frac{N}{n}\sum_{i=1}^{n} y_i\right]$$

$\left[\because E_2(\bar{y}_i \mid FSU) = \frac{1}{M_i}\sum_{j=1}^{M_i} y_{ij} = \frac{y_i}{M_i}\right.$ = Actual Mean of the ith Sample FSU $\left., i=1(1)n\right]$

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$$= N \; E_1 \left[\frac{1}{n} \sum_{i=1}^{n} y_i \right] = N \cdot \frac{1}{N} \sum_{i=1}^{N} Y_i = \sum_{i=1}^{N} Y_i = Y$$

$$= N \bar{Y} \quad \left[\because E_1 \left[\frac{1}{n} \sum_{i=1}^{n} y_i \right] = \frac{1}{N} \sum_{i=1}^{N} Y_i = \bar{Y} = \frac{Y}{N} \right]$$

$$= Y.$$

$$\therefore \quad E(\hat{Y}) = Y$$

Hence, $\hat{Y} = \frac{N}{n} \sum_{i=1}^{n} M_i \bar{y}_i$ is an unbiased estimator of Y.

$$Var(\hat{Y}) = E_1 V_2 (\hat{Y} \mid FSU) + V_1 E_2 (\hat{Y} \mid FSU)$$

$$E_2 (\hat{Y} \mid FSU) = \frac{N}{n} \sum_{i=1}^{n} y_i$$

$$V_1 E_2 (\hat{Y} \mid FSU) = V_1 \left(\frac{N}{n} \sum_{i=1}^{n} y_i \right) = N^2 V_1 \left(\frac{1}{n} \sum_{i=1}^{n} y_i \right)$$

$$= N^2 \cdot \frac{1}{n} Var(y_i) \left(\frac{N-n}{N-1} \right)$$

~~$E(y_i)$~~

$$E(y_i) = \frac{1}{N} \sum_{i=1}^{N} Y_i = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{M_i} Y_{ij} = \frac{NM\bar{Y}}{N} = M\bar{Y} \quad \left[\because M = \frac{1}{N} \sum_{i=1}^{N} M_i \right]$$

$$Var(y_i) = \frac{1}{N} \sum_{i=1}^{N} (Y_i - M\bar{Y})^2$$

$$= \frac{1}{N} \sum_{i=1}^{N} (M_i \bar{Y}_i - M\bar{Y})^2$$

$$= \frac{M^2}{N} \sum_{i=1}^{N} \left(\frac{M_i \bar{Y}_i}{M} - \bar{Y} \right)^2$$

$$= \frac{M^2}{N} (N-1) S_b^2$$

$$\therefore \quad V_1 E_2 (\hat{Y} \mid FSU) = \frac{N^2}{n} \frac{(N-n)}{(N-1)} \cdot \frac{M^2}{N} (N-1) S_b^2$$

$$= N^2 M^2 (1-f) \frac{S_b^2}{n} \quad ; \text{where } f = \frac{n}{N}$$

$$V_2(\hat{Y} \mid FSU) = V_2\left[\frac{N}{n}\sum_{i=1}^{n}\frac{M_i}{m_i}\sum_{j=1}^{m_i} y_{ij} \,\Big|\, FSU\right]$$

$$= V_2\left[\frac{N}{n}\sum_{i=1}^{n} M_i \bar{y}_i \,\Big|\, FSU\right]$$

$$= \frac{N^2}{n^2}\sum_{i=1}^{n} M_i^2 \, V_2(\bar{y}_i \mid FSU) \quad \left[\because Cov(\bar{y}_i, \bar{y}_{i'}) = 0 \quad i \neq i' = 1(1)n\right]$$

$$= \frac{N^2}{n^2}\sum_{i=1}^{n} M_i^2 \, S_{w_i}^2\left(\frac{1}{m_i} - \frac{1}{M_i}\right)$$

$$= \frac{N^2}{n^2}\sum_{i=1}^{n} M_i^2 \, \frac{S_{w_i}^2}{m_i}(1 - f_i) \quad \left[\text{where } f_i = \frac{m_i}{M_i}, i = 1(1)n\right]$$

$$E_1 V_2(\hat{Y} \mid FSU) = \frac{N^2}{n} E_1\left[\frac{1}{n}\sum_{i=1}^{n} M_i^2 \frac{S_{w_i}^2}{m_i}(1 - f_i)\right]$$

$$= \frac{N^2}{n}\,\frac{1}{N}\sum_{i=1}^{N} M_i^2 \frac{S_{w_i}^2}{m_i}(1 - f_i)$$

$$= \frac{N}{n}\sum_{i=1}^{N}\frac{M_i^2 S_{w_i}^2}{m_i}(1 - f_i) \quad \left[\text{where } f_i = \frac{m_i}{M_i}, i = 1(1)N\right]$$

$$\therefore Var(\hat{Y}) = \frac{N}{n}\sum_{i=1}^{N}\frac{M_i^2 S_{w_i}^2}{m_i}(1 - f_i) + N^2 M^2 (1 - f)\frac{S_b^2}{n}$$

Note: Unbiased Estimator of $\bar{Y}$ is $\hat{\bar{Y}} = \dfrac{\hat{Y}}{NM} = \dfrac{1}{Mn}\sum_{i=1}^{n}\dfrac{M_i}{m_i}\sum_{j=1}^{m_i} y_{ij}$

$$\therefore Var(\hat{\bar{Y}}) = \frac{Var(\hat{Y})}{N^2 M^2}$$

$$= \frac{1}{nNM^2}\sum_{i=1}^{N}\frac{M_i^2 S_{w_i}^2}{m_i}(1 - f_i) + (1 - f)\frac{S_b^2}{n}$$

$\frac{1}{N}\sum_{i=1}^{N}\frac{M_i^2}{m_i}S_{w_i}^2(1-f_i)$ is estimated unbiasedly by $\frac{1}{n}\sum_{i=1}^{n}\frac{M_i^2}{m_i}s_{w_i}^2(1-f_i)$

$\frac{N}{n}\sum_{i=1}^{N}\frac{M_i^2 S_{w_i}^2}{m_i}(1-f_i)$ is estimated unbiasedly by $\frac{N^2}{n^2}\sum_{i=1}^{n}\frac{M_i^2 s_{w_i}^2}{m_i}(1-f_i)$

▸ **Estimation of S_b^2:**

$$S_b^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{M_i\bar{Y}_i}{M}-\bar{Y}\right)^2$$

$$= \frac{1}{M^2(N-1)}\sum_{i=1}^{N}(M_i\bar{Y}_i - M\bar{Y})^2$$

$$= \frac{1}{M^2(N-1)}\sum_{i=1}^{N}\left[M_i^2\bar{Y}_i^2 + M^2\bar{Y}^2 - 2M\bar{Y}M_i\bar{Y}_i\right]$$

$$= \frac{1}{M^2(N-1)}\left[\sum_{i=1}^{N}M_i^2\bar{Y}_i^2 + NM^2\bar{Y}^2 - 2M\bar{Y}\sum_{i=1}^{N}M_i\bar{Y}_i\right]$$

$$= \frac{1}{M^2(N-1)}\left[\sum_{i=1}^{N}M_i^2\bar{Y}_i^2 + NM^2\bar{Y}^2 - 2NM^2\bar{Y}^2\right]$$

$$= \frac{1}{M^2(N-1)}\left[\sum_{i=1}^{N}M_i^2\bar{Y}_i^2 - NM^2\bar{Y}^2\right]$$

$$= \frac{1}{M^2(N-1)}\left[\sum_{i=1}^{N}M_i^2\bar{Y}_i^2 - NM^2\left(\frac{Y}{NM}\right)^2\right]$$

$$= \frac{1}{M^2(N-1)}\left[\sum_{i=1}^{N}M_i^2\bar{Y}_i^2 - \frac{Y^2}{N}\right]$$

Need to estimate $\sum_{i=1}^{N}M_i^2\bar{Y}_i^2$ & Y^2 separately.

Estimation of Y^2: $E(\hat{Y}) = Y$ &

$$Var(\hat{Y}) = E(\hat{Y}^2) - \{E(\hat{Y})\}^2 = E(\hat{Y}^2) - Y^2$$

ie. $Y^2 = E(\hat{Y}^2) - Var(\hat{Y})$

Suppose, v be an unbiased estimator of $Var(\hat{Y})$ ie.

$E(v) = Var(\hat{Y})$

$\therefore\ Y^2 = E(\hat{Y}^2 - v)$

$\therefore\ \frac{Y^2}{N}$ is unbiasedly estimated by $\frac{\hat{Y}^2 - v}{N}$

Estimation of $\sum_{i=1}^{N} M_i^2 \bar{Y}_i^2$

$\frac{1}{N}\sum_{i=1}^{N} M_i^2 \bar{Y}_i^2$ is unbiasedly estimated by $\frac{1}{n}\sum_{i=1}^{n} M_i^2 \hat{\bar{Y}}_i^2$.

We have, $E\left(\frac{1}{m_i}\sum_{j=1}^{m_i} y_{ij}\right) = \bar{Y}_i$

ie. $E(\bar{y}_i) = \bar{Y}_i \quad , i = 1(1)n$

& $Var(\bar{y}_i) = \frac{S_{w_i}^2}{m_i}(1-f_i) \quad , i = 1(1)n$

$\therefore \quad E(\bar{y}_i^2) - \{E(\bar{y}_i)\}^2 = \frac{S_{w_i}^2}{m_i}(1-f_i) \quad , i = 1(1)n$

$\Rightarrow \quad E(\bar{y}_i^2) - \bar{Y}_i^2 = \frac{S_{w_i}^2}{m_i}(1-f_i)$

$\Rightarrow \quad \bar{Y}_i^2 = E(\bar{y}_i^2) - \frac{S_{w_i}^2}{m_i}(1-f_i) \quad , i = 1(1)n$

$= E(\bar{y}_i^2) - E\left(\frac{s_{w_i}^2 (1-f_i)}{m_i}\right)$

$\therefore \quad \bar{Y}_i^2 = E\left[\bar{y}_i^2 - \frac{s_{w_i}^2}{m_i}(1-f_i)\right] \quad , i = 1(1)n$

$\therefore \quad \hat{\bar{Y}}_i^2 = \bar{y}_i^2 - \frac{s_{w_i}^2}{m_i}(1-f_i) \quad , i = 1(1)n$

& hence $\frac{1}{N}\sum_{i=1}^{N} M_i^2 \bar{Y}_i^2$ is unbiasedly estimated by

$$\frac{1}{n}\sum_{i=1}^{n} M_i^2\left(\bar{y}_i^2 - \frac{s_{w_i}^2}{m_i}(1-f_i)\right) = \frac{1}{n}\sum_{i=1}^{n} M_i^2 \bar{y}_i^2 - \frac{1}{n}\sum_{i=1}^{n}\frac{M_i^2 s_{w_i}^2}{m_i}(1-f_i)$$

$\therefore \quad S_b^2 = \frac{1}{M^2(N-1)}\left[\sum_{i=1}^{N} M_i^2 \bar{Y}_i^2 - \frac{Y^2}{N}\right]$ is unbiasedly estimated by

$$s_b^2 = \frac{1}{M^2(N-1)}\left[\frac{N}{n}\sum_{i=1}^{n} M_i^2 \bar{y}_i^2 - \frac{N}{n}\sum_{i=1}^{n}\frac{M_i^2 s_{w_i}^2}{m_i}(1-f_i) - \frac{\hat{Y}^2 - v}{N}\right]$$

$\therefore \quad Var(\hat{Y}) = N^2M^2(1-f)\frac{S_b^2}{n} + \frac{N}{n}\sum_{i=1}^{N}\frac{M_i^2 S_{w_i}^2}{m_i}(1-f_i)$

is unbiasedly estimated by

$$v = \frac{N^2M^2(1-f)}{n}s_b^2 + \frac{N^2}{n^2}\sum_{i=1}^{n}\frac{M_i^2 s_{w_i}^2}{m_i}(1-f_i)$$

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ie. $v = \frac{N^2 \cancel{M^2} (1-f)}{n} \cdot \frac{1}{\cancel{M^2}(N-1)} \left[\frac{N}{n}\sum_{i=1}^{n} M_i^2 \bar{y}_i^2 - \frac{N}{n}\sum_{i=1}^{n} \frac{M_i^2 s_{wi}^2}{m_i}(1-f_i) - \frac{\hat{Y}^2 - v}{N} \right] + \left[\frac{N^2}{n^2} \sum_{i=1}^{n} \frac{M_i^2 s_{wi}^2}{m_i} (1-f_i) \right]$

$$v = \frac{N^2(1-f)}{n(N-1)} \cdot \frac{N}{n}\sum_{i=1}^{n} M_i^2 \bar{y}_i^2 - \frac{N^2(1-f)}{n(N-1)} \frac{N}{n}\sum_{i=1}^{n} \frac{M_i^2 s_{wi}^2}{m_i}(1-f_i) + \left(-\frac{\hat{Y}^2}{N} + \frac{v}{N}\right)\frac{N^2(1-f)}{n(N-1)} + \frac{N^2}{n^2}\sum_{i=1}^{n}\frac{M_i^2 s_{wi}^2}{m_i}(1-f_i)$$

$$v\left[1 - \frac{N(1-f)}{n(N-1)}\right] = \frac{N^2(N-n)}{n^2(N-1)}\sum_{i=1}^{n} M_i^2 \bar{y}_i^2 - \frac{\hat{Y}^2}{N} \cdot \frac{N^2(1-f)}{n(N-1)} + \left(\frac{N^2}{n^2}\sum_{i=1}^{n}\frac{M_i^2 s_{wi}^2}{m_i}(1-f_i)\right)\left(1 - \frac{N(1-f)}{N-1}\right)$$

$$v\left[\frac{nN - \cancel{n} - N + \cancel{n}}{n(N-1)}\right] = \frac{N^2(N-n)}{n^2(N-1)}\sum_{i=1}^{n} M_i^2 \bar{y}_i^2 - \hat{Y}^2 \left(\frac{N-n}{n(N-1)}\right) + \left(\frac{N^2}{n^2}\sum_{i=1}^{n}\frac{M_i^2 s_{wi}^2}{m_i}(1-f_i)\right)\left(\frac{\cancel{N}-1-\cancel{N}+n}{N-1}\right)$$

$$v \, \frac{N(n-1)}{n(N-1)} = \frac{N^2(N-n)}{n^2(N-1)}\sum_{i=1}^{n} M_i^2 \bar{y}_i^2 - \hat{Y}^2\left(\frac{N-n}{n(N-1)}\right) + \frac{N^2(n-1)}{n^2(N-1)}\sum_{i=1}^{n}\frac{M_i^2 s_{wi}^2}{m_i}(1-f_i)$$

$$v = \frac{N(N-n)}{n(n-1)}\sum_{i=1}^{n} M_i \bar{y}_i^2 - \hat{Y}^2\frac{(N-n)}{N(n-1)} + \frac{N}{n}\sum_{i=1}^{n}\frac{M_i^2 s_{wi}^2}{m_i}(1-f_i)$$

$$\therefore \widehat{Var}(\hat{Y}) = v = \left(\frac{N-n}{n-1}\right)\left[\frac{N}{n}\sum_{i=1}^{n} M_i^2 \bar{y}_i^2 - \frac{\hat{Y}^2}{N}\right] + \frac{N}{n}\sum_{i=1}^{n}\frac{M_i^2 s_{wi}^2}{m_i}(1-f_i)$$

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Optimal Allocation in two stage sampling with equal SSUs in the population ~~[struck word]~~ & the sample FSU : $\left(\text{ie. } M_i = M, \; i = 1(1)N \;\&\; m_i = m, \; i = 1(1)n\right)$

We minimize $Var(\hat{\bar{Y}})$ subject to a given cost when $M_i = M$ $\forall i = 1(1)N$ and $m_i = m$ $\forall i = 1(1)n$.

$$Var(\hat{\bar{Y}}) = \frac{S_b^2}{n}\left(1 - \frac{n}{N}\right) + \frac{1}{N}\sum_{i=1}^{N} \frac{S_{wi}^2}{mn}\left(1 - \frac{m}{M}\right)$$

$$= \frac{S_b^2}{n} - \frac{S_b^2}{N} + \frac{S_w^2}{mn}\left(1 - \frac{m}{M}\right) \quad \left[\text{where, } S_w^2 = \frac{1}{N}\sum_{i=1}^{N} S_{wi}^2\right]$$

$$= \frac{S_b^2}{n} - \frac{S_b^2}{N} + \frac{S_w^2}{mn} - \frac{S_w^2}{nM}$$

$$Var(\hat{\bar{Y}}) = A_0 + \frac{A_1}{n} + \frac{A_2}{mn} \quad (\text{say})$$

Where, $A_0 = -\frac{S_b^2}{N}$, $A_1 = S_b^2 - \frac{S_w^2}{M}$, $A_2 = S_w^2$

Consider the following cost function $C = a + c_1 n + c_2 mn$

a = overhead cost, c_1 = cost of selecting a FSU, c_2 = per unit cost of selecting a SSU

We will minimize $Var(\hat{\bar{Y}})$ ~~subh~~ subject to $C = C_0$

Consider the Lagrangian function $Z = Var(\hat{\bar{Y}}) + \lambda(C - C_0)$

where, λ = the Lagrangian multiplier.

$$\therefore Z = \left(A_0 + \frac{A_1}{n} + \frac{A_2}{mn}\right) + \lambda\left(a + c_1 n + c_2 mn - C_0\right)$$

Minimize Z with respect to m and n,

$$\frac{\partial Z}{\partial m} = 0 \Rightarrow -\frac{A_2}{m^2 n} + \lambda c_2 n = 0 \Rightarrow \frac{A_2}{c_2 m^2} = \lambda n^2 \quad \text{--- (1)}$$

$$\frac{\partial Z}{\partial n} = 0 \Rightarrow -\frac{A_1}{n^2} - \frac{A_2}{mn^2} + \lambda(c_1 + c_2 m) = 0$$

$$\Rightarrow \left(A_1 + \frac{A_2}{m}\right) = \lambda n^2 (c_1 + c_2 m)$$

$$\Rightarrow \left(A_1 + \frac{A_2}{m}\right) = \frac{A_2}{c_2 m^2}(c_1 + c_2 m) \quad [\text{using (1)}]$$

$$\Rightarrow A_1 + \cancel{\frac{A_2}{m}} = \frac{A_2 c_1}{c_2 m^2} + \cancel{\frac{A_2}{m}}$$

$$\Rightarrow m^2 = \frac{A_2 c_1}{A_1 \ldots}$$

$$\therefore \boxed{m = \sqrt{\frac{A_2/c_2}{A_1/c_1}}}$$

Now, $C_0 = a + c_1 n + c_2 m n$

ie. $(C_0 - a) = n(c_1 + c_2 m)$

ie. $$n = \frac{C_0 - a}{c_1 + c_2 m} = \frac{C_0 - a}{c_1 + c_2 \sqrt{\frac{A_2/c_2}{A_1/c_1}}}$$

$$\therefore \boxed{n = \frac{C_0 - a}{c_1 + c_2 \sqrt{\frac{A_2/c_2}{A_1/c_1}}}} = \left(\frac{C_0 - a}{\sqrt{A_1 c_1} + \sqrt{A_2 c_2}} \right) \sqrt{\frac{A_1}{c_1}}$$

Assignment → m_i's are not equal, then what is the optimal allocation.