Chapter 3

Ordinary Differential Equations

3.1 Differential Equations

▶ An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.
▶ Here we do not include in the class of differential equations those equations that are actually derivative identities. For ex. $\frac{d}{dx}x^2 = 2x$, $\frac{d}{dx}e^{ax} = ae^x$.
▶ A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation or, ODE.
▶ A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation or, PDE.
▶ The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.
▶ The power of the highest ordered derivative(no derivatives involved any fractional power) involved in a differential equation is called the degree of the differential equation.

[Do It Yourself] **3.1.** *Find the type, order and the degree of the following differential equations:* (A) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 1$. (B) $\frac{d^5y}{dx^5} + 3\frac{d^3y}{dx^3} + \frac{dy}{dx} = e^x$. (C) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$.
(D) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial w^2} = 0$. (E) $\frac{\partial^2 z}{\partial x^2} = 2\frac{\partial z}{\partial x}$.
$[Hint: Ode(2),\ Ode(5),\ Pde(1),\ Pde(2),\ Pde(2)]$

3.1.1 Linear and Nonlinear ODE

▶ A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x, is an equation that can be expressed in the form:

$$\boxed{a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)} \text{ with } a_n(x) \neq 0.$$

▶ In linear ODE: 1) Dependent variable y and its various derivatives occur to the first degree only. 2) No products of y and/or any of its derivatives are present. 3) No transcendental functions of y and/or its derivatives occur.
▶ A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.
▶ A linear Ode is said to be linear with constant coefficients if a_i's are all constants.
▶ A linear Ode is said to be linear with variable coefficients if at least one a_i is a function of the independent variable.

[Do It Yourself] **3.2.** *Find the type and the order of the following differential equations. Also classify the linear Ode's in terms of their coefficients.*

(A) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 1$. (B) $\frac{d^5y}{dx^5} + 3\frac{d^3y}{dx^3} + \frac{dy}{dx} = e^x$. (C) $\frac{d^5y}{dx^5} + 3\frac{d^3y}{dx^2} + \frac{dy}{dx} = e^y$.

(D) $\frac{d^2y}{dx^2} + 2\Big(\frac{dy}{dx}\Big)^2 + 3y = 1$. (E) $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + 3y = 1$. (F) $2\frac{dy}{dx} + 3y = 1$.

(G) $2\frac{dy}{dx} + 3xy = 1$. (H) $2\sin(\frac{dy}{dx}) + 3y = 1$. (I) $2\frac{dy}{dx} + 3y = \sin(x+y)$. (J) $\sqrt[3]{\frac{dy}{dx} + 3y} = \frac{d^2y}{dx^2}$.

[$Hint: L(2),\ L(5),\ NL,\ NL,\ NL,\ L(1),\ L(1),\ NL,\ NL,\ L(3)$]

[Do It Yourself] **3.3.** *Find Ode's by eliminating the arbitrary constants.*
(A) $ax + by = c$. (B) $ax + by = 1$. (C) $ax^2 + by^2 = c$. (D) $x^2 + ay = b$.

[Do It Yourself] **3.4.** *Classify each of the following differential equations as ordinary/partial differential equations, also state the order, degree of each equation and determine whether the equation under consideration is linear or nonlinear. Also classify the linear Ode's in terms of their coefficients.*

(A) $\frac{dy}{dx} + x^2y = xe^x$. (B) $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = \sin(x)$. (C) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(D) $x^2dy + y^2dx = 0$. (E) $\frac{d^4y}{dx^4} + 3\Big(\frac{d^2y}{dx^2}\Big)^5 + 3y = 1$. (F) $\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0$.

(G) $\frac{d^2y}{dx^2} + y\sin(x) = 0$. (H) $\frac{d^2y}{dx^2} + x\sin(y) = 0$. (I) $\frac{d^6x}{dt^6} + \frac{d^4x}{dt^4}\frac{d^3x}{dt^3} + x = t$.

(J) $\Big(\frac{dy}{dx}\Big)^3 = \sqrt{\frac{d^2y}{dx^2} + 1}$. (K) $\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 3y = x$.

3.1.3 Various Types of Solution of ODE

▶ Consider the Ode in the form: $F(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}) = 0$. Here F is a real valued function with $(n+2)$ arguments.

▶ A solution $\boxed{y = f(x)}$ is said to be an explicit solution of the ode $F(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}) = 0$ if $F(x, f(x), f'(x), \cdots, f^{(n)}(x)) = 0$.

▶ For example, $y = 2\sin(x) + 3\cos(x)$ is an explicit solution of the ode $\frac{d^2y}{dx^2} + y = 0$.

▶ For explicit solution, $f(x)$ is defined with all n order derivatives. Also $f(x)$ and its derivatives has to be defined for all x.

▶ A solution $\boxed{f(x,y) = 0}$ is said to be an implicit solution of $F(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}) = 0$ if $f(x,y) = 0$ satisfies the ode.

▶ For example, $y^2 - x = 0,\ x > 0$ is an implicit solution of the ode $2y\frac{uy}{dx} = 1$.

▶ For example, $y^2 + x = 0,\ x > 0$ is a formal solution of the ode $2y\frac{dy}{dx} + 1 = 0$. Since $y^2 + x = 0,\ x > 0$ does not define any real function on any interval for $x > 0$.

▶ Both explicit and implicit solutions are called solutions.

▶ The Ode $\frac{dy}{dx} = 3x^2$ has solution $y = x^3$ as well as $y = x^3 + c$, where c is arbitrary parameter or, constant and the solutions $y = x^3 + c$ is known as one-parameter family of solutions.
▶ There are various methods exists to solve Ode: Exact method, Series solution, Approximate methods such as numerical solution, graphical method.

3.1.4 Geometric Interpretations

Suppose we have an Ode: $\frac{dy}{dx} = 2x$. This ode may be interpreted as defining the slope $2x$ at the point (x, y) for every real x.

The ode has a one-parameter family of solutions of the form $y = x^2 + c$. The solution geometrically represents (draw it) a one-parameter family of curves (parabola) in the xy-plane. These parabolas are the integral curves of the differential equation $\frac{dy}{dx} = 2x$.

[Do It Yourself] 3.5. *Show that $x^3 + 3xy^2 = 1$ is an implicit solution of the differential equation $2xy\frac{dy}{dx} + x^2 + y^2 = 0$ on the interval $0 < x < 1$.*

[Do It Yourself] 3.6. *Show that every function f defined by $f(x) = 2 + ce^{-2x^2}$, where c is an arbitrary constant, is a solution of the differential equation $\frac{dy}{dx} + 4xy = 8x$.*

[Do It Yourself] 3.7. *Show that every function f defined by $f(x) = c_1e^{2x} + c_2e^{-x}$, where c is an arbitrary constant, is a solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.*

[Do It Yourself] 3.8. *For certain values of the constant m the function f defined by $f(x) = e^{mx}$ is a solution of the differential equation $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 12y = 0$. Determine all such values of m.*

3.1.5 Initial & Boundary Value Problems

▶ The ode $y'' = F(x, y, y')$, with $y(c) = a,\ y'(c) = b$ is known as Initial value problem (IVP). IVP's are evaluated at the same point has unique solution.
▶ The ode $y'' = F(x, y, y')$, with $y(c_1) = a_1,\ y(d_1) = b_1$ or, $y'(c_2) = a_2,\ y'(d_2) = b_2$ is known as Boundary value problem (BVP). BVP's are evaluated at different points and generally doesn't have unique solution. Sometimes it doesn't possess any solution at all.
▶ Example: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, with $y(0) = 1,\ y'(0) = 2$ is an IVP.

▶ Example: $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$, with $y(0) = 1,\ y'(0) = 2,\ y''(0) = 2.5$ is an IVP.

▶ Example: $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$, with $y(2) = 1,\ y'(2) = 3,\ y''(2) = 5$ is an IVP.

▶ Example: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, with $y(0) = 1,\ y(1) = 2$ is a BVP.

▶ Example: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, with $y'(0) = 1.5,\ y'(1) = 2$ is a BVP.

▶ Example: $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$, with $y(0) = 1,\ y(1.5) = 2,\ y(3) = 2.8$ is a BVP.

[Do It Yourself] 3.12. *Given that every solution of* $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$ *may be written in the form* $y = c_1e^{4x} + c_2e^{-3x}$ *for some choice of the arbitrary constants* c_1 *and* c_2*, solve the IVPs: i)* $y(0) = 5,\ y'(0) = 6$*. ii)* $y(0) = -2,\ y'(0) = 6$.

[Do It Yourself] 3.13. *Given that every solution of* $\frac{d^2y}{dx^2} + y = 0$ *may be written in the form* $y = c_1 \sin(x) + c_2 \cos(x)$ *for some choice of the arbitrary constants* c_1 *and* c_2*, solve the BVPs: i)* $y(0) = 0,\ y(\pi/2) = 1$*. ii)* $y(0) = 0,\ y(\pi) = 1$*. iii)* $y(0) = 1,\ y'(\pi/2) = -1$.

3.1.6 Existence & Uniqueness

Theorem 3.1. *The IVP:* $\boxed{y' + p(x)y = g(x) \text{ with } y(x_0) = y_0}$ *has a* <u>*unique solution*</u> *in the interval* $I = \{x : a < x < b\} = (a, b)$ *if*

1. $p(x), g(x)$ *is continuous on* I.
2. I *contains the point* $x = x_0$ *corresponding to the initial value.*

Example 3.1. *Find an interval in which the IVP:* $xy' + 2y = 4x^2;\ y(1) = 2$ *has unique solution. Also discuss when* $y(0) = 0$.
$\Rightarrow$ *Given IVP has the form:* $\boxed{y' + p(x)y = g(x);\ y(1) = 2}$*, where* $p(x) = \frac{2}{x},\ g(x) = 4x$.
We need to find an interval of x: i) On which p, g *are continuous and ii) Containing* $x = 1$*. It implies the interval is* $I = \{x : 0 < x < \infty\} = (0, \infty)$.
So the IVP has unique solution on I.
☐ *If* $y(0) = 0$ *then i) demands* $I = (0, \infty)$ *but it violates ii) as 0 doesn't belongs to that interval. So we can't conclude anything using the above theorem.*

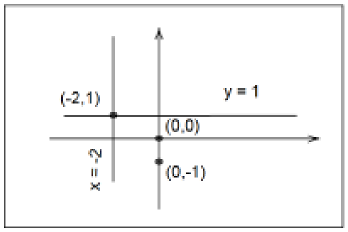
Theorem 3.2. <u>*Picards Theorem (A General Approach)*</u>*: Consider the IVP:*

$$\boxed{\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0}.$$

If $f(x, y)$ *and* $\frac{\partial f}{\partial y}$ *be continuous in some rectangle* $a < x < b,\ c < y < d$ *containing the point* (x_0, y_0)*. Then* $\exists$ *a non-trivial interval* $x_0 - h < x < x_0 + h$ *contained in* $a < x < b$ *such that the IVP has unique solution in* $x_0 - h < x < x_0 + h$.

Example 3.2. *Find an interval in which the IVP:* $\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)};\ y(0) = -1$ *has unique solution.*
$\Rightarrow$ <u>*Note that*</u>*: The given ODE can't be written in the form* $y' + p(x)y = g(x)$*. So we use Picards Theorem.*
Given Ode has the form: $\boxed{\frac{dy}{dx} = f(x, y)}$*, where* $f(x, y) = \frac{3x^2+4x+2}{2(y-1)} \Rightarrow \frac{\partial f}{\partial y} = -\frac{3x^2+4x+2}{2(y-1)^2}$.
Both $f(x, y)$ *and* $\frac{\partial f}{\partial y}$ *are continuous except* $y = 1$.
Now $y(0) = -1 \Rightarrow (0, -1)$ *does not lie on the line* $y = 1$.
So by Picard's Theorem, $\exists$ *a rectangle about* $(0, -1)$ *in which* $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)};\ y(0) = -1$ *has unique solution.*

Solving we get, $y^2 - 2y = x^3 + 2x^2 + 2x + c$. *Using* $y(0) = -1$ *we get,* $c = 3$. *Therefore, we get* $y^2 - 2y = x^3 + 2x^2 + 2x + 3 \Rightarrow (y-1)^2 = (x^2+2)(x+2) \Rightarrow y = 1 \pm \sqrt{(x^2+2)(x+2)}$. *Using* $y(0) = -1$ *we get,* $y = 1 - \sqrt{(x^2+2)(x+2)}$. *Therefore,* $x > -2$.



So, the x *range of the rectangle is*
$-2 < x < \infty$
Also, the y *range of the rectangle is*
$-\infty < y < 1$
So the unique solution in the rectangle is
$\{(x, y) :\ x > -2,\ y < 1\}$.

[Do It Yourself] 3.14. *Show that each of the following IVPs has unique solution defined on some sufficiently small interval* $|x - 1| \leq h$ *about* $x_0 = 1$*: i)* $\frac{dy}{dx} = x^2 \sin(y),\ y(1) = -2$. *ii)* $\frac{dy}{dx} = \frac{y^2}{x-2},\ y(1) = 0$.