3.3 1^{st} Order Higher Degree Equation

▶ $p = \frac{dy}{dx}$ and we will solve equations involving function of p.

▶ $\boxed{\text{Lagrange's Equation}}$: Form: $\boxed{y = xf(p) + g(p)}$. Now, $y = xf(p) + g(p) \Rightarrow p = f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx} \Rightarrow p - f(p) = [xf'(p) + g'(p)]\frac{dp}{dx} \Rightarrow \frac{dx}{dp} - [\frac{f'(p)}{p-f(p)}]x = \frac{g'(p)}{p-f(p)}$.

▶ $\boxed{\text{Clairaut's Equation}}$: Form: $\boxed{y = px + g(p)}$. Now, It has two types of solution: Complete Primitive or, General Solution: $y = cx + g(c)$ and Singular Solution: Through $\boxed{p - disc = 0}$ and $\boxed{c - disc = 0}$.

▶ $\boxed{\text{Equation Solvable for } p}$: Ex. $x^2p^2 - 2xyp + y^2 = x^2y^2 + x^4$.

▶ $\boxed{\text{Equation Solvable for } y}$: Ex. $y = px + p^2x \Rightarrow p = p + x\frac{dp}{dx} + p^2 + 2xp\frac{dp}{dx}$.

▶ $\boxed{\text{Equation Solvable for } x}$: Ex. $x = py - p^2 \Rightarrow \frac{1}{p} = p + y\frac{dp}{dy} - 2p\frac{dp}{dy}$.

Example 3.7. *Find the general solution: i) $y = xp^2 + \ln(p)$ ii) $y = px + f(p)$.*

$\Rightarrow$ *The given Ode is* $y = xp^2 + \ln(p) \Rightarrow p = p^2 + 2xp\frac{dp}{dx} + \frac{1}{p}\frac{dp}{dx} \Rightarrow p - p^2 = (2xp + \frac{1}{p})\frac{dp}{dx} \Rightarrow \frac{dx}{dp} + \frac{2p}{p^2-p}x = \frac{1}{p(p-p^2)} \Rightarrow \frac{dx}{dp} + \frac{2}{p-1}x = \frac{1}{p(p-p^2)}$.

[Note: Here we lost solution $p = 0, p = 1$ i.e. $y = 0, y = x$. It leads to singular solution].

Now I.F. $= exp[\int \frac{2}{p-1}dp] = (p-1)^2$. *Therefore*

$(p-1)^2\frac{dx}{dp} + 2(p-1)x = \frac{1-p}{p^2} \Rightarrow \frac{d}{dp}[(p-1)^2x] = \frac{1-p}{p^2} \Rightarrow (p-1)^2x = -\frac{1}{p} - \ln(p) + c \Rightarrow x = \frac{c-\frac{1}{p}-\ln(p)}{(p-1)^2}$. *Again,* $y = \frac{cp^2-p-p^2\ln(p)}{(p-1)^2} + \ln(p) = \frac{cp^2-p-(2p-1)\ln(p)}{(p-1)^2}$.

So the general solution in parametric form is: $x = \frac{c-\frac{1}{p}-\ln(p)}{(p-1)^2}$, $y = \frac{cp^2-p-(2p-1)\ln(p)}{(p-1)^2}$.

Note: Eliminating p from these equations we get the general solution in form of $f(x, y) = 0$. *Although it is not easy.*

☐ *The given Ode is* $y = px + f(p) \Rightarrow p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \Rightarrow (x + f'(p))\frac{dp}{dx} = 0 \Rightarrow \frac{dp}{dx} = 0 \Rightarrow p = c$.

So the general solution is $y = cx + f(c)$.

[Do It Yourself] 3.47. *Find the general solution: i)* $x^2p^2 - 2xyp + y^2 = x^2y^2 + x^4$ *ii)* $y = px + p^2x$, *iii)* $x = py - p^2$.

3.4 Higher Order Linear ODE

▶ 2^{nd} order linear ODE: $a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$ with $a_2(x) \neq 0$.

▶ 2^{nd} order linear Homogeneous ODE: $a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$ with $a_2(x) \neq 0$.

▶ 2^{nd} order linear Homogeneous ODE with Constant Coefficients: $a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = 0$ with $a_2 \neq 0$.

▶ 3rd order linear ODE: $a_3(x)\frac{d^3y}{dx^3} + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$ with $a_3(x) \neq 0$.
▶ 3rd order linear Homogeneous ODE: $a_3(x)\frac{d^3y}{dx^3} + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$ with $a_3(x) \neq 0$.
▶ n^{th} order linear ODE: $a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y = b(x)$ with $a_n(x) \neq 0$.
▶ All $a_i(x), b(x)$ are continuous on $x \in [\alpha, \beta]$.

[Do It Yourself] 3.49. *Determine the type of the Ode's: i)* $y''+3xy'+x^3y = e^x$, *ii)* $y'''+xy''+3x^2y'-5y = \sin(x)$, *iii)* $y'''+2y''+4xy'+x^2y = 0$, *iv)*$y'''-2y''-y'+2y = 0$.

3.4.1 Higher Order Linear ODE & Its Solution

▶ If $f_1, f_2, \cdots, f_n$ be any n solutions of the n^{th}-order homogeneous linear differential equation $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \Rightarrow c_1f_1 + c_2f_2 + \cdots + c_nf_n$ is also a solution of that DE, where c_i's are arbitrary constants.
▶ The n^{th}-order homogeneous linear differential equation always possesses n solutions that are linearly independent. Here the set of n solutions $f_1, f_2, \cdots, f_n$ is called a fundamental set of solutions. The function $f(x) = c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x)$ is called general solution, where c_i's are arbitrary constants.

Theorem 3.3. *The n solutions $f_1, f_2, \cdots, f_n$ of the n^{th}-order homogeneous linear differential equation are linearly independent on $a \le x \le b$ if and only if the Wronskian of $f_1, f_2, \cdots, f_n$ is either identically zero on $a < x < b$ or, else is never zero on $a < x < b$.*

$$\text{The Wronskian is } W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

[Do It Yourself] 3.50. *Consider the differential equation $y''-2y'+y = 0$. i) Show that e^x and xe^x are linearly independent solutions of this equation on the interval $-\infty < x < \infty$. ii) Write the general solution of the given equation. iii) Find the solution that satisfies the condition $y(0) = 1, y'(0) = 4$. Explain why this solution is unique. Over what interval is it defined?*

[Do It Yourself] 3.51. *Consider the differential equation $x^2y''+xy'-4y = 0$. i) Show that x^2 and $1/x^2$ are linearly independent solutions of this equation on the interval $0 < x < \infty$. ii) Write the general solution of the given equation. iii) Find the solution that satisfies the condition $y(2) = 3, y'(2) = -1$. Explain why this solution is unique. Over what interval is it defined?*

Theorem 3.4. *Reducing Order: Let $f(x)$ be a nontrivial solution of the 2^{nd}-order homogeneous linear DE $\boxed{a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0}$. Then the transformation $\boxed{y = f(x)v}$ reduces the equation to a 1^{st}-order homogeneous linear DE $\boxed{b_1(x)\frac{dz}{dx} + b_0(x)z = 0}$, where $z = \frac{dv}{dx}$. The new solution $g(x) = f(x)v$ and $f(x)$ are linearly independent. Hence the general solution is $c_1 f(x) + c_2 g(x)$.*

Example 3.9. *Given that $y = x$ is a solution of $(x^2+1)y'' - 2xy' + 2y = 0$, find a linearly independent solution by reducing the order.*
$\Rightarrow$ Here $y = x$ is a solution of $(x^2+1)y'' - 2xy' + 2y = 0$ [show].
Let, $y = xv \Rightarrow y' = v + xv' \Rightarrow y'' = v' + v'' + v'$. Put these values in the given equation we get, $x(x^2+1)v'' + 2v' = 0$.
Let, $z = v' \Rightarrow z' = v''$. Therefore $x(x^2+1)z' + 2z = 0 \Rightarrow \frac{dz}{z} + \frac{2}{x(x^2+1)}dx \Rightarrow zx^2 = c(x^2+1)$.
So $dv = c(1 + \frac{1}{x^2})dx \Rightarrow v = c(x - \frac{1}{x}) \Rightarrow y = c(x^2-1)$.
So the new solution $g(x) = x^2 - 1$ is linearly independent to the previous solution. Hence the general solution is $y = c_1 x + c_2(x^2-1)$, where c_1, c_2 are arbitrary constants.

[Do It Yourself] 3.52. *Let $y_1(x), y_2(x)$ be the linearly independent solutions of $xy'' + 2y' + xe^x y = 0$. If $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ with $W(1) = 2$ then find $W(5)$. [$Hint:\ W'(x) = y_1y_2'' - y_2y_1'',\ Now\ try\ to\ remove\ y\ term\ from\ ode$]*

3.4.2 Solution of n^{th} Order Linear System

► Consider n^{th}-order homogeneous linear differential equation $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$. The general solution of the homogeneous equation is called the complementary function and denoted by y_c for the corresponding non-homogeneous equation.

► Consider n^{th}-order non-homogeneous linear differential equation $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$. Any particular solution of involving no arbitrary constants is called a particular integral and denoted by y_p. The solution $\boxed{y = y_c + y_p}$ is called the general solution of this non-homogeneous equation.

► The ode $(a_2D^2 + a_1D + a_3)y = b(x)$ has solution $y = y_c + y_p \Rightarrow (a_2y_c^{(2)} + a_1y_c^{(1)} + a_3y_c) + (a_2y_p^{(2)} + a_1y_p^{(1)} + a_3y_p) = 0 + b(x) = b(x)$

[Do It Yourself] 3.53. *Given that $y = x+1$ is a solution of $(x+1)^2y'' - 3(x+1)y' + 3y = 0$, find a linearly independent solution by reducing the order. Write the general solution.*

[Do It Yourself] 3.54. *Given that $y = e^{2x}$ is a solution of $(2x+1)y'' - 4(x+1)y' + 4y = 0$, find a linearly independent solution by reducing the order. Write the general solution.*

[Do It Yourself] 3.55. *Consider the nonhomogeneous differential equation $y'' - 3y' + 2y = 4x^2$. i) Show that e^x and e^{2x} are linearly independent solutions of the corresponding homogeneous equation $y'' - 3y' + 2y = 0$. ii) What is the complementary function of the given non-homogeneous equation? iii) Show that $2x^2 + 6x + 7$ is a particular integral of the given equation. iv) What is the general solution of the given equation?*