3.5 Lin Homogeneous Constant Coeff

▶ The n^{th}-order homogeneous linear differential equation with constant co-efficient is $a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = 0$. Here a_i's are constants.
▶ Take $y = e^{mx}$ is a solution $\Rightarrow e^{mx}(a_nm^n + \cdots + a_1m + a_0) = 0 \Rightarrow a_nm^n + \cdots + a_1m + a_0$. It is called auxiliary or, characteristic equation of the Ode.
▶ Rule 1: Auxiliary equation has n distinct real roots $\alpha_1, \alpha_2, \cdots, \alpha_n \Rightarrow$ The general solution is $y = c_1e^{\alpha_1x} + c_2e^{\alpha_2x} + \cdots + c_ne^{\alpha_nx}$.
▶ Rule 2: Auxiliary equation has p equal and $n-p$ distinct real roots $\alpha, \cdots, \alpha, \alpha_1, \cdots, \alpha_{n-p}$ $\Rightarrow$ The general solution is $y = e^{\alpha x}(c_1 + c_2x + \cdots + c_px^{p-1}) + d_1e^{\alpha_1x} + \cdots + d_{n-p}e^{\alpha_{n-p}x}$.
▶ Rule 3: Auxiliary equation has p complex conjugate roots $\alpha_1 \pm i\beta_1, \cdots \alpha_p \pm i\beta_p \Rightarrow$ The general solution is $y = e^{\alpha_1x}[c_1\sin(\beta_1x) + d_1\cos(\beta_1x)] + \cdots + e^{\alpha_px}[c_p\sin(\beta_px) + d_p\cos(\beta_px)]$.
▶ Rule 4: Auxiliary equation has p equal complex conjugate roots $\alpha \pm i\beta \Rightarrow$ The general solution is $y = e^{\alpha x}[(c_1 + c_2x + \cdots + c_px^{p-1})\sin(\beta x) + (d_1 + d_2x + \cdots + d_px^{p-1})\cos(\beta x)]$.

Example 3.10. *Find the general solution of $4y'' - 12y' + 5y = 0$.*
$\Rightarrow$ *Let $y = e^{mx}$ be a trial solution of the equation.*
So the auxiliary equation is: $4m^2 - 12m + 5 = 0 \Rightarrow (2m-1)(2m-5) = 0 \Rightarrow m = 1/2, 5/2$.
Therefore the general solution is $y = c_1e^{x/2} + c_2e^{5x/2}$, where c_1, c_2 are arbitrary constants.

[Do It Yourself] 3.56. *Solve the Ode's: i) $y'' - 4y' + 4y = 0$, ii) $y'' + 6y' + 11y = 0$, iii) $16y'' + 32y' + 25y = 0$, iv) $y''' - 3y'' - y' + 3y = 0$, v) $y''' - 6y'' + 12y' - 8y = 0$, vi) $8y''' + 12y'' + 6y' + y = 0$, vii) $y^{(iv)} = 0$, viii) $y^{(iv)} - y = 0$, ix) $y^{(iv)} + 8y'' + 16y = 0$, x) $y^{(v)} - 2y^{(iv)} + y''' = 0$, xi) $y^{(v)} + 5y^{(iv)} + 10y''' + 10y'' + 5y' + y = 0$, xii) $y^{(iv)} + 64y = 0$.*

[Do It Yourself] 3.57. *Given that $m^4 + 2m^3 + 5m^2 + 4m + 4 = (m^2 + m + 2)^2$, find the general solution of $y^{(iv)} + 2y''' + 5y'' + 4y' + 4y = 0$.*

[Do It Yourself] 3.58. *The roots of the auxiliary equation are i) $4, 4, 4, 4, 2 + 3i, 2 - 3i, 2 + 3i, 2 - 3i, 2 + 3i, 2 - 3i, 4 + 5i, 4 - 5i$ and ii) $2, 2, 2, -1, -1, 4, 3 + 4i, 3 - 4i, 3 + 4i, 3 - 4i, 3 + 4i, 3 - 4i$. Write the general solution in each case.*

3.5.1 Lin Non-Homogeneous Constant Coeff

▶ Consider n^{th}-order non-homogeneous linear differential equation $a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = b(x)$. The solution $\boxed{y = y_c + y_p}$ is called the general solution of this non-homogeneous equation.
▶ Suppose $a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = b(x)$ has a solution ψ and a particular solution ψ_p. Then $[a_2D^2(\psi - \psi_p) + a_1D(\psi - \psi_p) + a_0]y = 0$ i.e. $\psi - \psi_p$ is a solution of the homogeneous equation $\Rightarrow \psi = Homogeneous\ solution + \psi_p$.
▶ We already studied how to find complementary function y_c from the corresponding homogeneous equation. Now we will study some techniques to find the particular integral y_p.

- The equation $y'' + 2y' + y = x$ can be written as $(D^2 + 2D + 1)y = x$, where $D \equiv \frac{d}{dx}$.
- Now the solution of $(D^2 + 2D + 1)y = 0$ gives C.F. y_c.
- The P.I. y_p of $(D^2 + 2D + 1)y = x$ can be found by $y_p = \frac{1}{D^2+2D+1}x$.
- **Rule 1**: $\dfrac{1}{D-a}f(x) = e^{ax}\displaystyle\int e^{-ax}f(x)dx$.
- **Rule 2**: $\dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax}$, provided $f(a) \neq 0$.
- **Rule 3**: $\dfrac{1}{(D-a)^r}e^{ax} = \dfrac{x^r}{r!}e^{ax}$.
- **Rule 4**: $\dfrac{1}{D^2+D+1}6 = \dfrac{1}{D^2+D+1}6.e^{0x} = 6\dfrac{1}{D^2+D+1}e^{0x} = 6\dfrac{1}{0^2+0+1}e^{0x} = 6$.
- **Rule 5**: $\dfrac{1}{f(D)}\cos ax = \dfrac{1}{\phi(D^2)}\cos ax = \dfrac{1}{\phi(-a^2)}\cos ax$, provided $\phi(-a^2) \neq 0$.
- If $f(D) = \phi(D^2) = D^2 + a^2 \Rightarrow \phi(-a^2) = 0$, then we will use next rule.
- **Rule 6**: $\dfrac{1}{D^2+a^2}\cos ax = \Re[\dfrac{1}{D^2+a^2}e^{iax}] = \Re[e^{iax}\dfrac{1}{(D+ia)^2+a^2}(1)] = \Re[e^{iax}\dfrac{1}{D^2+2Dia}(1)]$

$= \Re[e^{iax}\dfrac{1}{2Dia}\dfrac{1}{(1+D/2ia)}(1)] = \Re[e^{iax}\dfrac{1}{2Dia}(1 - D/2ia + \cdots)(1)] = \Re[e^{iax}\dfrac{1}{2Dia}(1)]$

$= \Re[e^{iax}\dfrac{1}{2ia}x] = \Re[\dfrac{\cos ax + i\sin ax}{2ia}x] = \dfrac{x}{2a}\sin ax$.

- **Rule 7**: $\dfrac{1}{D^2+a^2}\sin ax = -\dfrac{x}{2a}\cos ax$. [Easy Proof]
- **Rule 8**: $\dfrac{1}{f(D)}e^{ax}g(x) = e^{ax}\dfrac{1}{f(D+a)}g(x)$.

Example 3.11. *Find P.I. of* $(D^3 - D^2 - 6D)y = x^2 + 1$.

$\Rightarrow P.I. = \frac{1}{D^3-D^2-6D}(x^2+1) = -\frac{1}{6D}\frac{1}{[1+(\frac{D}{6}-\frac{D^2}{6})]}(x^2+1) = -\frac{1}{6D}[1+(\frac{D}{6}-\frac{D^2}{6})]^{-1}(x^2+1)$

$= -\frac{1}{6D}[1-(\frac{D}{6}-\frac{D^2}{6})+(\frac{D}{6}-\frac{D^2}{6})^2 - \cdots](x^2+1) = -\frac{1}{6D}[1-(\frac{D}{6}-\frac{D^2}{6})+\frac{D^2}{36}](x^2+1)$

$= -\frac{1}{6}[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}]$.

[Do It Yourself] 3.61. *Find the general solution of :* $y'' + 4y' + 4y = 4x^2 + 6e^x$, $y'' - 3y' + 2y = 2xe^{3x} + 3\sin x$, $y'' - 3y' + 2y = e - x$, $y(0) = 1$, $y'(0) = -1$, $y'' + y = \sin^2 x$. [$Ans: y = (c_1 + c_2x)e^{-2x} + x^2 - 2x + \frac{3}{2} + \frac{2}{3}e^x$, $y = c_1e^x + c_2e^{2x} + xe^{3x} - \frac{3}{2}e^{3x} + \frac{3}{10}\sin x + \frac{9}{10}\cos x$, $6y = -10e^{2x} + 15e^x + e^{-x}$].

[Do It Yourself] 3.62. *Find the general solution of :* $y'' + y = \sin x + e^{-x}$, $y'' + y = 4x\sin x$, $y'' + 3y' + 2y = e^{-2x} + x^2$.

[Do It Yourself] 3.66. *Which of the following Ode is satisfied by functions* $y_1(x) = e^{(-1+\sqrt{3})x}$ *and* $y_2(x) = e^{-2x}$*?*
(A) $(D^2 + 5D + 6)y = 0$. *(B)* $(D^3 + 6D^2 + 11D + 6)y = 0$. *(C)* $(D^2 + D - 2)y = 0$.
(D) $(D^3 + 4D^2 + 2D - 4)y = 0$.

[Do It Yourself] 3.67. *Let* $\alpha(t), \beta(t)$ *be differentiable functions on* $\mathbb{R}$ *such that* $\alpha(0) = 2$, $\beta(0) = 1$. *If* $\alpha(t) + \beta'(t) = 1$, $\alpha'(t) + \beta(t) = 1$ *for all* $t \in [0, \infty)$, *find the value of* $\alpha(\ln 2)$.
[$Hint: \ Find \ CF + PI, \ Ans: \ 7/2$]

3.5.2 Method Of Undetermined Coefficients

▶ Consider n^{th}-order non-homogeneous linear differential equation $a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = b(x)$. The solution $\boxed{y = y_c + y_p}$ is called the general solution of this non-homogeneous equation.

▶ We already studied the methods to find $P.I$ in previous section.

▶ The method of undetermined coefficients (UC) is an another method to find $P.I$ *i.e.* y_p. This methods can be applied if the function $b(x)$ has certain forms.

▶ Rule 1: $a_nx^n \twoheadrightarrow A_0 + A_1x + \cdots + A_nx^n$.

▶ Rule 2: $e^{ax} \twoheadrightarrow Ae^{ax}$. If fails, then use Axe^{ax}.

▶ Rule 3: $x^ne^{ax} \twoheadrightarrow e^{ax}(A_0 + A_1x + \cdots + A_nx^n)$.

▶ Rule 4: $\sin ax \twoheadrightarrow A\sin ax + B\cos ax$. If fails, then use $x(A\sin ax + B\cos ax)$.

▶ Rule 5: $e^{ax}\sin bx \twoheadrightarrow e^{ax}(A\sin bx + B\cos bx)$.

▶ Here we have to find these coefficients to obtain y_p.

Example 3.12. *Find P.I. of* $(D^3 - D^2 - 6D)y = x^2 + 1$.

$\Rightarrow$ *Let* $y_p = A_0 + A_1x + A_2x^2$. *Therefore,* $Dy_p = A_1 + 2A_2x, D^2y_p = 2A_2,\ D^3y_p = 0$.
Since y_p *is a solution of the given equation. It implies*
$-2A_2 - 6(A_1 + 2A_2x) = x^2 + 1$, *Now its impossible to find the coefficients.*
Let $y_p = A_1x + A_2x^2 + A_3x^3$.
Therefore, $Dy_p = A_1 + 2A_2x + 3A_3x^2, D^2y_p = 2A_2 + 6A_3x,\ D^3y_p = 6A_3$.
Since y_p *is a solution of the given equation. It implies*
$6A_3 - 2A_2 - 6A_3x - 6(A_1 + 2A_2x + 3A_3x^2) = x^2 + 1 \Rightarrow -18A_3x^2 - (12A_2 + 6A_3)x + 6A_3 - 2A_2 - 6A_1 = x^2 + 1$.
So $A_3 = -\frac{1}{18},\ 12A_2 + 6A_3 = 0 \Rightarrow A_2 = \frac{1}{36},\ 6A_3 - 2A_2 - 6A_1 = 1 \Rightarrow A_1 = -\frac{25}{108}$.
Therefore, $y_p = -\frac{1}{6}[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}]$.

[Do It Yourself] 3.68. *Find the general solution of :* $y'' + 2y' + 2y = 10\sin 4x,\ y'' - 2y' - 8y = 4e^{2x},\ y'' - 4y = 16xe^{2x},\ y''' + 4y'' + y' - 6y = -18x^2 + 1$.

3.5.3 Variation of Constants Method

▶ This is more general method than the above two. Here we need to know the solution of corresponding homogeneous equation.

▶ We will study through an example.

Example 3.13. *Solve the Ode:* $y'' + y = \tan x$.

$\Rightarrow y_c = c_1\sin x + c_2\cos x$.
Let $y_p = v_1\sin x + v_2\cos x$, *the functions* $v_1(x), v_2(x)$ *will be determined such that this is a P.I. of the given system.*
Now, $y_p' = v_1'\sin x + v_2'\cos x + v_1\cos x - v_2\sin x$, *we impose the condition,* $v_1'\sin x + v_2'\cos x = 0$.
So $y_p' = v_1\cos x - v_2\sin x \Rightarrow y_p'' = v_1'\cos x - v_2'\sin x - v_1\sin x - v_2\cos x \Rightarrow y_p'' + y_p = v_1'\cos x - v_2'\sin x$.

So we obtain two equations, $v_1' \sin x + v_2' \cos x = 0, \quad v_1' \cos x - v_2' \sin x = \tan x.$

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \sin x \Rightarrow v_1 = -\cos x + c_3.$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \cos x - \sec x \Rightarrow v_2 = \sin x - \ln|\sec x + \tan x| + c_4.$$

Therefore, $y_p(x) = c_3 \sin x + c_4 \cos x - \cos x \ln|\sec x + \tan x|.$
The general solution is $y = y_c + y_p \Rightarrow y = A \sin x + B \cos x - \cos x \ln|\sec x + \tan x|.$

[Do It Yourself] 3.69. *Find the general solution of :* $y'' + y = \sec x,\ y'' + y = \tan^3 x,\ y'' + 3y' + 2y = \frac{e^{-x}}{x}.$