Chapter 4

Partial Differential Equations

4.1 Preliminaries

▶ $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$, $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$.
▶ In PDE of z is a dependent variable and x, y are independent variables.
▶ Form of PDE of 1^{st} order is $f(x, y, z, p, q) = 0$.
▶ $\phi\big[u(x,y,z), v(x,y,z)\big] = 0 \Rightarrow Differentiate$
★ W.r.t. $x \Rightarrow \frac{\partial \phi}{\partial u}\Big[\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\Big] + \frac{\partial \phi}{\partial v}\Big[\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\Big] = 0$
$\Rightarrow \frac{\partial \phi}{\partial u}\Big[\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\Big] + \frac{\partial \phi}{\partial v}\Big[\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\Big] = 0.$
★ W.r.t. $y \Rightarrow \frac{\partial \phi}{\partial u}\Big[\frac{\partial u}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\Big] + \frac{\partial \phi}{\partial v}\Big[\frac{\partial v}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\Big] = 0$
$\Rightarrow \frac{\partial \phi}{\partial u}\Big[\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\Big] + \frac{\partial \phi}{\partial v}\Big[\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\Big] = 0.$
★ Note: If $u = x^2 + y^2 + z^2 \Rightarrow \frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = 2y,\ \frac{\partial u}{\partial z} = 2z$, and $v = xy^2 z \Rightarrow$ $\frac{\partial v}{\partial x} = y^2 z,\ \frac{\partial v}{\partial y} = 2xyz,\ \frac{\partial v}{\partial z} = xy^2$.

[Do It Yourself] **4.1.** *Eliminate arbitrary constants and form the PDE:*
i) $z = 2(x-\alpha)^2 - 3(y-\beta)^2$, *ii)* $z = (x+a)(y+b)$, *iii)* $2z = (ax+y)^2 + b$, *iv)* $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$,
v) $\ln(az-1) = x + ay + b$, *vi)* $ax^2 + by^2 + cz^2 = 1$.
[*Ans* : *i)* $z = \frac{p^2}{8} - \frac{q^2}{12}$, *ii)* $z = pq$, *iii)* $px = q(q-y)$, *iv)* $2z = px + qy$, *v)* $p(q+1) = zq$, *vi)* $zpx + zqy - z^2 = -\frac{1}{c}$, *now differentiate*]

4.1.1 Order and Degree of a PDE

▶ Order : The order of highest derivative.
▶ Degree : Power of highest order derivative.

▶ *i)* $x\frac{\partial z}{\partial x} = \Big(\frac{\partial^2 z}{\partial x^2}\Big)^2$: Order 2 and Degree 2. *ii)* $\frac{\partial^2 z}{\partial x^2} = xy\frac{\partial^3 z}{\partial x^3}$: Order 3 and Degree 1.

4.2 Classification of 1^{st} Order (p,q) PDE

■ Linear: $\boxed{P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)}$,
Ex. $(xy)p + (x + y^2)q = (x^3y)z + (2 + 3x)$. Write down two more.

■ Semi-linear: $\boxed{P(x,y)p + Q(x,y)q = R(x,y,z)}$,
Ex. $(xy)p + (x + y^2)q = x^3yz^2$. Write down two more.

■ Quasi-linear: $\boxed{P(x,y,z)p + Q(x,y,z)q = R(x,y,z)}$,
Ex. $(xyz^2)p + (xz + y^2)q = x^3yz^3$. Write down two more.

■ Non-linear: $\boxed{f(x,y,z,p,q) = 0}$ and not in the form of above three,
Ex. $(p^2 + q^2)y = qz$. Write down two more.

4.2.1 Lagrange's Method

▶ The equation is: $Pp + Qq = R$ where P, Q, R are functions of x, y, z.
▶ Lagrange's auxiliary equation: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.
▶ Working Rule: Step i): Using AE: $u(x,y,z) = c_1,\ v(x,y,z) = c_2$.
▶ Working Rule: Step ii): General Solution: $\phi(u,v) = 0,\ or,\ u = \phi(v),\ or,\ v = \phi(u)$.
▶ Working Rule: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R}$. Now if $Denom = 0 \Rightarrow Num = 0$.

Example 4.1. *Solve $xzp - yzq = y^2 - x^2$.*
$\Rightarrow$ *The given equation is $xzp - yzq = y^2 - x^2$.*
So Lagrange's AE is $\frac{dx}{xz} = \frac{dy}{-yz} = \frac{dz}{y^2-x^2} = \frac{xdx+ydy+zdz}{x^2z-y^2z+y^2z-x^2z} = \frac{xdx+ydy+zdz}{0}$.
So $xdx + ydy + zdz = 0 \Rightarrow x^2 + y^2 + z^2 = c_1$.
Again $\frac{dx}{xz} = \frac{dy}{-yz} \Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0 \Rightarrow xy = c_2$.
Therefore, the general solution is $f(x^2 + y^2 + z^2, xy) = 0$.

Example 4.2. *Find the integral surface of the linear PDE $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the line $x + y = 0,\ z = 1$.*
$\Rightarrow$ *The given equation is $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$.*
So Lagrange's AE is $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0} = \frac{xdx+ydy-dz}{0}$.
So $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \Rightarrow xyz = c_1$.
Again $xdx + ydy - dz = 0 \Rightarrow x^2 + y^2 - 2z = c_2$.
Now $z = 1 \Rightarrow c_1 = xy,\ c_2 = x^2 + y^2 - 2$.
Also $x = -y, \Rightarrow y^2 = -c_1,\ c_2 = 2y^2 - 2 \Rightarrow c_2 + 2c_1 + 2 = 0 \Rightarrow x^2 + y^2 - 2z + 2xyz + 2 = 0$
Therefore, the integral surface is $x^2 + y^2 - 2z + 2xyz + 2 = 0$.

[Do It Yourself] 4.4. *Find the equation of the integral surface of the differential equation* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ *which passes through the line* $x = 1,\ y = 0$.

4.2.2 Integral Surface Orthogonal to a Given Surface

▶ Suppose $f(x, y, z) = c$ ... (1) is a one parameter family of surface. Then the surface orthogonal to (1) is $\boxed{p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}}$.

Example 4.3. *Find surface which is orthogonal to one parameter system* $z = cx(x^2 - y^2)$ *and passes through the circle* $x^2 + y^2 = 1,\ z = 0$.
$\Rightarrow$ *The given equation is* $\frac{x(x^2-y^2)}{z} = \frac{1}{c}$ ... (1).
Let $f(x, y, z) = \frac{x(x^2-y^2)}{z}$, *So* $f_x = \frac{3x^2-y^2}{z},\ f_y = \frac{-2xy}{z},\ f_z = -\frac{x(x^2-y^2)}{z^2}$.
So the surface orthogonal to (1) *is* $pf_x + qf_y = f_z$.
Now Lagrange's AE is $\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{dz}{f_z} \Rightarrow \frac{dx}{\frac{3x^2-y^2}{z}} = \frac{dy}{-\frac{2xy}{z}} = \frac{dz}{-\frac{x(x^2-y^2)}{z^2}}$.
From $i), ii)$ *we get,* $\frac{dx}{dy} = \frac{y^2-3x^2}{2xy} \Rightarrow y^3(y^2 - 5x^2) = c_1$
From $i), ii), iii)$ *we get,* $\frac{dx}{3x^2-y^2} = \frac{dy}{-2xy} = \frac{zdz}{xy^2-x^3} = \frac{xdx+ydy+3zdz}{0} \Rightarrow x^2 + y^2 + 3z^2 = c_2$
So any surface which is orthogonal to (1) *is of the form* $x^2 + y^2 + 3z^2 = f[y^3(y^2 - 5x^2)]$, *where* f *is an arbitrary function.*
Now $x^2+y^2 = 1,\ z = 0$ *implies* $f[y^3(y^2-5x^2)] = 1$, *so the required surface is* $x^2+y^2+3z^2 = 1$.

[Do It Yourself] 4.22. *Find the surfaces orthogonal to the given surfaces:*
i) $z(x + y) = c(3z + 1)$ *passes through* $x^2 + y^2 = 1, z = 1$, *ii)* $z = cxy(x^2 + y^2)$ *passes through* $x^2 - y^2 = a^2, z = 0$.
[*Ans* : *i)* $x^2 + y^2 - 2z^3 - z^2 + 2 = 0$, *ii)* $(x^2 - y^2)^2(x^2 + y^2 + 4z^2) = a^4(x^2 + y^2)$]

4.2.3 Solutions/Integrals & Compatibility

▶ Complete Integral: Complete integral solution is solution of a partial differential equation of the first order that contains as many arbitrary constants as there are independent variables. $f(x,y,z,p,q)=0 \Rightarrow \boxed{g(x,y,z,a,b)=0} \Rightarrow g$ is a complete integral of f.
▶ Particular Integral: Particular integral solution is a solution free from arbitrary constants i.e the solution obtained from complete integral by giving particular values to the arbitrary constants.
▶ General Integral: Assume $b=\phi(a)$ then $g(x,y,z,a,b)=0 \Rightarrow g(x,y,z,a,\phi(a))=0$. Now eliminating a from $g(x,y,z,a,\phi(a))=0$ and $\frac{\partial g}{\partial a}=0$ we get the general integral.
▶ Singular Integral: Singular integral is obtained by eliminating a,b from complete integral $g(x,y,z,a,b)=0$ and $\frac{\partial g}{\partial a}=0$, $\frac{\partial g}{\partial b}=0$.
■ Two PDE's $f_1(x,y,z,p,q)$ and $f_2(x,y,z,p,q)$ are compatible (i.e. every solution of one is a solution of the other) iff $[f_1,f_2]=0$, where $[f_1,f_2]=\frac{\partial(f_1,f_2)}{\partial(x,p)}+p\frac{\partial(f_1,f_2)}{\partial(z,p)}+\frac{\partial(f_1,f_2)}{\partial(y,q)}+q\frac{\partial(f_1,f_2)}{\partial(z,q)}$.

4.2.4 Charpit's Method

▶ $f(x,y,z,p,q)=0 \Rightarrow \frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0}$.

Example 4.4. *Find a complete, singular and general integrals of* $(p^2+q^2)y=qz$.
$\Rightarrow$ *The given equation is* $(p^2+q^2)y-qz=0$ ... (A).
So Charpit's AE is $\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$.
$\Rightarrow \frac{dp}{-pq}=\frac{dq}{p^2}=\frac{dz}{-2p^2y+qz-2q^2y}=\frac{dx}{-2py}=\frac{dy}{-2qy+z}$.
Using $(1),(2)$ *we get,* $p^2+q^2=a$.
So by (A), $ay=qz \Rightarrow q=\frac{ay}{z}$. *Again,* $p^2=a-\frac{a^2y^2}{z^2} \Rightarrow p=\frac{\sqrt{az^2-a^2y^2}}{z}$.
Now putting these values of p,q *in* $pdx+qdy=dz$ *implies* $\frac{\sqrt{az^2-a^2y^2}}{z}dx+\frac{ay}{z}dy=dz \Rightarrow zdz-aydy=\sqrt{az^2-a^2y^2}dx \Rightarrow \frac{zdz-aydy}{\sqrt{az^2-a^2y^2}}=dx \Rightarrow 2a\frac{zdz-aydy}{\sqrt{az^2-a^2y^2}}=2adx \Rightarrow \sqrt{az^2-a^2y^2}=ax+b \Rightarrow (az^2-a^2y^2)=(ax+b)^2$ ... (3).
The complete integral is $(az^2-a^2y^2)=(ax+b)^2$, *where* a,b *are arbitrary constants.*
★ *Singular Integral: Differentiate* (3) *w.r.t.* a *and* b *we get,* $z^2-2ay^2=2ax^2+2xb$ ... (4) *and* $2xa+2b=0$ ... (5). *Eliminating* a,b *from* $(3),(4),(5)$ *we get the singular solution.*
★ *General Integral: Replacing* b *by* $\phi(a)$ *in* (3), *we get* $(az^2-a^2y^2)=(ax+\phi(a))^2$ ... (6). *Differentiate* (6) *partially w.r.t.* a, *we get* $z^2-2ay^2=2(ax+\phi(a))(x+\phi'(a))$ ... (7). *General integral is obtained by eliminating* a *from* $(6),(7)$.

Example 4.5. *Find a complete, singular and general integrals of* $2xz - px^2 - 2qxy + pq = 0$.
$\Rightarrow$ *The given equation is* $2xz - px^2 - 2qxy + pq = 0$ ... (A).
So Charpit's AE is $\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$.
$\Rightarrow \frac{dp}{2z-2qy} = \frac{dq}{0} = \frac{dz}{px^2+2xyq-2pq} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p}$.
Using (2) *we get,* $q = a$.
So by (A), $p(x^2 - a) = 2x(z - ay) \Rightarrow p = \frac{2x(z-ay)}{x^2-a}$.
Now putting these values of p, q *in* $pdx + qdy = dz$ *implies* $\frac{2x(z-ay)}{x^2-a}dx + ady = dz \Rightarrow \frac{2x}{x^2-a}dx = \frac{d(z-ay)}{z-ay} \Rightarrow z - ay = b(x^2 - a) \Rightarrow z = ay + b(x^2 - a)$ ... (3).
The complete integral is $z = ay + b(x^2 - a)$, *where* a, b *are arbitrary constants.*
★ *Singular Integral: Differentiate* (3) *w.r.t.* a *and* b *we get,* $y - b = 0$ ... (4) *and* $x^2 - a = 0$ ... (5). *Eliminating* a, b *from* (3), (4), (5) *we get the singular solution* $z = x^2 y$.
★ *General Integral: Replacing* b *by* $\phi(a)$ *in* (3), *we get* $z - ay = \phi(a)(x^2 - a)$ ... (6).
Differentiate (6) *partially w.r.t.* a, *we get* $-y = \phi'(a)(x^2 - a) - \phi(a)$ ... (7).
General integral is obtained by eliminating a *from* (6), (7).

[**Do It Yourself**] **4.23.** *Find CI for i)* $pxy + pq + qy - yz = 0$, *ii)* $p^2x + q^2y = z$, *iii)* $z^2 = pqxy$.
[Ans : *i)* $(z - ax)(y + a)^a = be^y$, *ii)* $\sqrt{(1 + a)z} = \sqrt{ax} + \sqrt{y} + b$, *iii)* $z = x^a y^{1/a} b$].

[**Do It Yourself**] **4.24.** *Consider the first order PDE:* $p + q = pq$, *where* $p \equiv \frac{\partial z}{\partial x}$, $q \equiv \frac{\partial z}{\partial y}$. *Then which of the following are correct?*

(A) The Charpit's equation for the above Pde reduce to $\frac{dx}{1-q} = \frac{dy}{1-p} = \frac{dz}{-pq} = \frac{dp}{p+q} = \frac{dq}{0}$. *(B) A solution of the Charpit's equation is* $q = b =$ *constant. (C) The corresponding value of* p *is* $p = \frac{b}{b-1}$. *(D) A solution of the equation is* $z = \frac{bx}{b-1} + by + a$.

[**Do It Yourself**] **4.25.** *The Charpit's equation for the Pde:* $up^2 + q^2 + x + y = 0$, *where* $p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y}$ *are given by*
(A) $\frac{dx}{-1-p^3} = \frac{dy}{-1-qp^2} = \frac{du}{2p^2u+2q^2} = \frac{dp}{2pu} = \frac{dq}{2q}$. *(B)* $\frac{dx}{2pu} = \frac{dy}{2q} = \frac{du}{2p^2u+2q^2} = \frac{dp}{-1-p^3} = \frac{dq}{-1-qp^2}$.