Chapter 2

Game Theory & Networking

2.1 Game theory

In the world of business, conflicts of interest often arise between competitors operating in the same field. To illustrate this, consider two businessmen, A and B, who are players in a game of business. Each has several executives, with A having A_1, A_2, and A_3, and B having B_1, B_2, B_3, and B_4. To control their respective businesses, both players may only utilize the services of one executive at a time. The selection of a particular executive is a strategy, and choosing one strategy, ignoring the strategy taken by the opponent, is a pure strategy. There are a total of seven executives to choose from, so there are seven possible pure strategies for each player. Here we make two assumptions: (i) A is a maximizing player, while B is a minimizing player; and (ii) the total gain of one player is exactly equal to the total loss of the other, resulting in a net gain of zero i.e. zero sum game.

2.1.1 Rectangular game

The possible outcomes of the game can be represented in a matrix, known as a payoff matrix, where each cell represents the gain or loss for each player, depending on the strategies chosen. To illustrate this, let us assume that the gain or loss for each player is measured in terms of rupee. Suppose that the payoff matrix is as follows:

	B_1	B_2	B_3	B_4
A_1	$10, -10$	$-5, 5$	$-2, 2$	$-3, 3$
A_2	$-5, 5$	$8, -8$	$-4, 4$	$-1, 1$
A_3	$-2, 2$	$-4, 4$	$6, -6$	$-3, 3$

In this matrix, the first number in each cell represents the gain for player A, while the second number represents the loss for player B. For example, if A chooses A_1 and B chooses B_1, then A gains Rs. 10, while B loses Rs. 10.

■ In general, if the player A takes m pure strategies and B takes n pure strategies, then the game is called two person zero sum game or, $m \times n$ rectangular game (zero sum as the total gain of one player is exactly equal to the total loss of the other). If $m = n$ then the game is called a square game.

■ As the pay-off matrix of the player B (minimizing payer) is the negative of the pay-off matrix of A (maximizing player). So we can write the pay-off matrix (*say* M) in terms of A (maximizing player) is:

	B_1	B_2	B_3	B_4
A_1	10	-5	-2	-3
A_2	-5	8	-4	-1
A_3	-2	-4	6	-3

► Similarly the pay-off matrix of B is $-M$.

► In general, the pay-off matrix of A is

	B_1	B_2	$\cdots$	B_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{12}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

There are $m \times n$ elements a_{ij} in the pay-off matrix which are the gains obtained by A from the pure moves of A and $B's$ i.e. A_i and B_j respectively for $[i = 1, \cdots, m;\ j = 1, \cdots, n]$.

■ **Pay-off Matrix**: A pay-off matrix is a real matrix (a_{ij}) indicates the gain of the maximization player (Row player) for using the i^{th} and i^{th} move of the row (A) and column (B) players respectively.

2.1.2 Minimax-Maximin Principle

■ To find the optimal strategies for each player, we use minimax theorem, which states that in a zero-sum game, the optimal strategy for a maximizing player (i.e. A) is to choose the strategy that maximizes their minimum gain, while the optimal strategy for a minimizing player is to choose the strategy that minimizes their maximum loss.

► When the equality condition holds, we say that the game problem is solved and the value of the game is: 'maximum of the minimum gains' for A = 'minimum of the maximum losses' for B. Assuming the existence of the value of the game, if the value of the game be the element at of the pay-off matrix, the point (k, l) is called the saddle point of the pay-off matrix.

► There may be more than one saddle point in a pay-off matrix of a game with pure strategy. The principle of determination of the value of a game and saddle point or points is known as the maximin minimax principle. Again if the value of the game be V_{kl} for A and B's pure moves A_k and B_l respectively then the stratigies A_k and B_l taken by both players are called the optimal strategies for the players.

Example 2.1. *Find the value of the following game (pure strategy) for a 3 × 4 pay-off matrix*

	B_1	B_2	B_3	B_n
A_1	4	6	−2	1
A_2	3	3	4	2
A_3	4	5	5	1

for the maximizing player A by using maximin-minimax principle.

⇒ *Here A's pure moves are A_1, A_2 and A_3 and B's pure moves are B_1, B_2, B_3 and B_4 Now each player has every right to select a pure move according to his own suitability.*

Now for A's pure move A_1, the minimum gain is −2 units (whatever strategy may be taken by B). Similarly for A's pure move A_2 and A_3 minimum possible gains are 2 and 1 unit respectively. Therefore, as A has every right to select a move which will maximize his minimum gains, he will definitely select the pure move (strategy) A_2 and for that his gain will be 2 units. His gain will not be less than 2 units at any case.

Now considering B's point of view, if B selects his pure move B_1 the maximum loss will be 4 units (independent of A's move). Similarly, for B's pure moves B_2, B_3 and B_4 the maximum amount of losses will be 6, 5 and 2 units respectively. Therefore, as the ultimate aim of B is to minimize his maximum losses, then B will definitely select the pure strategy B_4 and the minimum loss will be 2 units. His amount of loss cannot be increased any further. But it is interesting to note that the 'maximum of the minimum gains' for A is equal to the 'minimum of the maximum losses' for B. Hence in this pay-off matrix or, game the value of game exists and is 2 units. The optimal strategies are A_2 and B_4 for A and B respectively and $(2,4)^{th}$ position of the pay-off matrix is the saddle point of the pay-off matrix.

▶ In general, there exists no saddle point and the value of the game cannot be determined by the above principle.
▶ There may be more than one saddle point in a particular game and in that case optimal strategies are not unique.

[Do It Yourself] 2.1. *Solve the given games: i)*

	B_1	B_2	B_3
A_1	4	−2	1
A_2	3	4	2
A_3	−3	4	0

and ii)

	B_1	B_2	B_3
A_1	6	3	−3
A_2	−2	1	2
A_3	5	4	6

by using maximum and minimum principle.

[Do It Yourself] 2.2. *Show that the game*

	B_1	B_2	B_3
A_1	6	0	−4
A_2	−3	2	−1
A_3	4	−3	5

does not have any saddle point.

[Do It Yourself] 2.3. *Show that the game*

	B_1	B_2	B_3	B_4
A_1	4	2	3	5
A_2	−2	−1	4	−3
A_3	5	2	3	3
A_4	4	0	0	1

have multiple saddle points. [CU 90,93]

[Do It Yourself] 2.4. *Each of two players A and B shows one, two and three fingers simultaneously. The player B pays to A an amount equal to the two times total number of fingers shown, on the other hand A pays to B equal to the product of the numbers of finger shown. Form the pay-off matrix.*

▪ **Fair Game**: If the value of the game be zero, i.e., no loss or gain for any player then the game is called a fair game.

■ Strictly Determinate Game : If the value of game is a non-zero quantity then the game is called as strictly determinate game. If the value of game is positive (negative) then the game is in favour of the player A (B).

Theorem 2.1. *Suppose $f(x,y)$ be a real valued function of x, y defined for $x \in A$, $y \in B$, where $A, B \subseteq \mathbb{R}$. Now if both* $\max_{x \in A} \min_{y \in B} f(x,y)$ *and* $\min_{y \in B} \max_{x \in A} f(x,y)$ *exist then show that* $\max_{x \in A} \min_{y \in B} f(x,y) \leq \min_{y \in B} \max_{x \in A} f(x,y)$.

Proof. As $f(x,y) \leq \max_{x \in A} f(x,y)$ and $\min_{y \in B} f(x,y) \leq f(x,y) \Rightarrow \min_{y \in B} f(x,y) \leq \max_{x \in A} f(x,y)$.
Now, $\min_{y \in B} f(x,y)$ is independent of $y \Rightarrow \min_{y \in B} f(x,y) \leq \min_{y \in B} \max_{x \in A} f(x,y)$ ... (1).
Now from (1) we have, $\max_{x \in A} \min_{y \in B} f(x,y) \leq \min_{y \in B} \max_{x \in A} f(x,y)$. □

[Do It Yourself] 2.5. *Let $[a_{ij}]_{m \times n}$ be the pay-off matrix for a two-person zero-sum game. Then using Theorem 2.1 prove that,* $\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}$, *where* $i = 1(1)m$, $j = 1(1)n$. [Hint : $A = \{1, 2, \cdots, m\}$, $B = \{1, 2, \cdots, n\}$, $f(i,j) = a_{ij}$]

2.1.3 Mixed Strategy

■ Generally, a game problem cannot be solved by using pure strategies. However, it can be solved by using another technique known as mixed strategy. This technique can solve all two-person zero-sum games when ∄ saddle point.

■ In pure strategy, both players A and B select only a single move at a time say A_i and B_j, at their discretion, irrespective of the move taken by other. In mixed strategy both players A and B select their m and n moves simultaneously.

► Define a set of m positive quantities $p_1, p_2, \cdots, p_m$ with $\sum_{i=1}^{m} p_i = 1$. Now to obtain a least possible gain A may utilize the service of the moves $A_1, A_2, \ldots, A_m$ in such way that A_i performs p_i times of the total work will be done by the moves $A_1, A_2, \ldots, A_m$. Similarly if $q_1, q_2, \cdots, q_n$ be a set of positive quantities such that $\sum_{j=1}^{n} q_j = 1$ then B may utilize the services of the moves $B_1, B_2, \cdots, B_n$ in such way that B_j performs q_j times of the total work will be done by the moves $B_1, B_2, \ldots, B_n$.

► The quantities p_i and q_j associated with the i^{th}, j^{th} move of A and B respectively are called the probabilities of the respective moves.

► We now define two variable vectors $\underset{\sim}{p} = (p_1, p_2, \cdots, p_m)$ in E^m and $\underset{\sim}{q} = (q_1, q_2, \cdots, q_n)$ in E^n. It is always possible to determine some particular value of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ say p^* and q^* such that the value of the game can be determined. Here it is possible to determine the same value of 'maximum of the minimum expected gain for A and the minimum of the maximum expected loss for B'.

■ Pay off function : Let $[a_{ij}]_{m \times n}$ be the pay-off matrix for a two-person zero-sum game. Then the pay-off function or, mathematical expectation of a game which is defined as $E(\underset{\sim}{p}, \underset{\sim}{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$, where $\underset{\sim}{p}, \underset{\sim}{q}$ are the mixed strategies for A and B respectively.

► In matrix notation, $E(\underset{\sim}{p}, \underset{\sim}{q}) = \underset{\sim}{p}' A \underset{\sim}{q}$, where $A = [a_{ij}]$ the pay-off matrix.

► If B takes his pure j^{th} move only then the expected gain of A is given by $E_j(\underset{\sim}{p}) = \sum_{i=1}^{m} a_{ij}p_i,\ j = 1, 2, \cdots, n$.
► Similarly for particular i^{th} pure move of A only, the expected loss of B is given by $E_i(\underset{\sim}{q}) = \sum_{j=1}^{n} a_{ij}q_j,\ i = 1, 2, \cdots, m$.
■ For any $\underset{\sim}{p}$, A is sure that his expected winning will be at least $\min\limits_{q} E(p, q)$. He then maximizes the expression over $\underset{\sim}{p}$, so that his expected winnings will be at least $\max\limits_{p}\min\limits_{q} E(p, q)$.

Example 2.2. *Find the value of the game*

	B_1	B_2
A_1	2	3
A_2	4	−1

algebraically by using mixed strategies.
⇒ *The problem has no saddle point for pure strategy in the pay-off matrix for A.*
Let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_1, p_2)$ *and* $\underset{\sim}{q} = (q_1, q_2)$ *with* $p_1 + p_2 = 1,\ p_1, p_2 > 0$ *and* $q_1 + q_2 = 1,\ q_1, q_2 > 0$ *for A and B respectively. Assuming the existence of the value of game we have*
$E_1(p) = 2p_1 + 4p_2 = 2p_1 + 4(1 - p_1),\ E_2(p) = 3p_1 - p_2 = 3p_1 - (1 - p_1)$.
To determine the optimal values of p_1, p_2 *we have,* $2p_1 + 4(1 - p_1) = v = 3p_1 - (1 - p_1)$.
Solving we get, $p_1^* = 5/6 (> 0)$. *So,* $p_2^* = 1 - p_1^* = 1/6$ *and value of the game is* $v = 2p_1^* + 4(1 - p_1^*) = 7/3$.
Again considering from the B's point of view we have,
$E_1(q) = 2q_1 + 3q_2 = 2q_1 + 3(1 - q_1),\ E_2(q) = 4q_1 - q_2 = 4q_1 - (1 - q_1)$.
To determine the optimal values of q_1, q_2, *we have,* $2q_1 + 3(1 - q_1) = v = 4q_1 - (1 - q_1)$.
Solving we get $q_1^* = 2/3 (> 0)$ *and* $q_2^* = 1 - q_1^* = 1/3$ *and the value of the game is* $v = 2q_1^* + 3(1 - q_1^*) = 7/3$.
Hence the optimal strategies are $p^* = (5/6, 1/6)$ *and* $q^* = (2/3, 1/3)$ *and* $v = 7/3$.

► It can be verified that $E(q^*, p^*) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_i^*q_j^* = 2 * \frac{5}{6} * \frac{2}{3} + 4 * \frac{1}{6} * \frac{2}{3} + 3 * \frac{5}{6} * \frac{1}{6} - 1 * \frac{1}{6} * \frac{1}{3} = \frac{7}{3}$.

[Do It Yourself] 2.6. *Find the value of the following* 2×2 *games by using mixed strategies.*

	B_1	B_2
A_1	4	2
A_2	1	5

	B_1	B_2
A_1	a	$-b$
A_2	$-c$	d

with $a, b, c, d > 0$.

2.1.4 Graphical Method

■ We already solved the 2×2 game without any saddle point through algebraically. Although it is not possible to easily solve any $m \times n$ game algebraically. However, by using graph, it is possible to reduce any rectangular game of order $2 \times n$ or, $m \times 2$ to a

2×2 game and then solve it by algebraic method.

► A $2 \times n$ games looks like

	B_1	B_2	$\cdots$	B_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}

.

Without any saddle point, let the mixed strategies used by A be $p = (p_1, p_2)$ and B be $q = (q_1, q_2, \cdots, q_n)$. Then the net expected gain of a A when B plays his pure strategy B_j is given by $E_j(p) = a_{1j}p_1 + a_{2j}p_2 = a_{1j}p_1 + a_{2j}(1 - p_1),\ j = 1, 2, \cdots, n$. As $p_1 + p_2 = 1$, so both p_1 and p_2 must lie in the open interval $(0, 1)$ [because if either p_1 or $p_2 = 1$ the game reduces to a game of pure strategy which is against our assumption]. Hence $E_j(p)$ is a linear function of either p_1 or p_2. Considering $E_j(p)$ as a linear function of p_1 (say), we have from the limiting values $(0, 1)$ of p, $E_j(p) = a_{2j}\ if\ p_1 = 0$ and $E_j(p) = a_{1j}\ if\ p_1 = 1$. Therefore, Hence $E_j(p)$ represents a line segment joining the points $(0, a_{2j})$ and $(1, a_{1j})$.

► Steps: 1. Draw two parallel vertical lines with distance one unit length. Left one represents the line $p_1 = 0$ and the right one is $p_1 = 1$.

2. Draw n line segments joining the points $(0, a_{2j})$ and $(1, a_{1j})$, $j = 1, 2, \cdots, n$. The lower envelope of these line segments is the minimum expected gain of A as a function of p_1 and the highest point of the lower envelope will give the maximum of minimum gain of A.

3. The line segments passing through the point corresponding to B's two pure moves say B_k and B_l are the critical moves for B which will maximize the minimum expected gain of A.

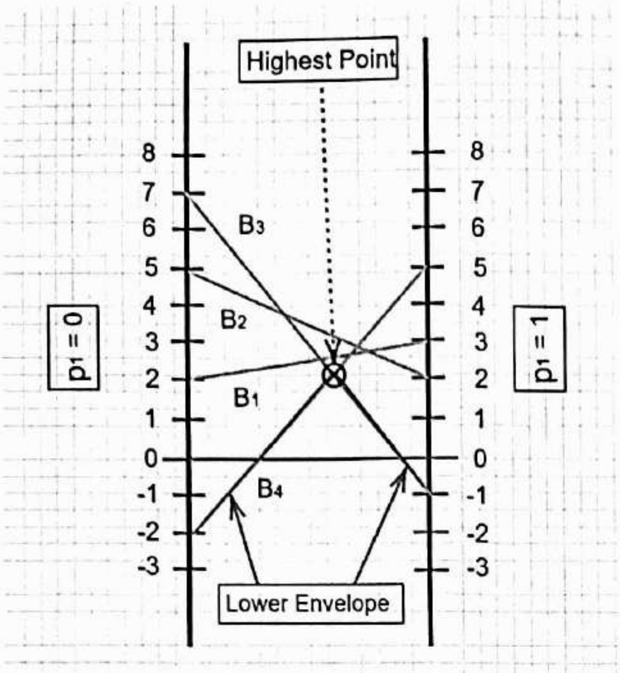
4. Now after finding 2×2 pay-off matrix corresponding to A's moves A_1 and A_2 and B's moves B_k and B_l, we can solve the pay-off matrix algebraically and find the value of the game.

Example 2.3. *Reduce the following $2 \times n$ game*

	B_1	B_2	B_3	B_4
A_1	3	2	−1	5
A_2	2	5	7	−2

into a 2×2 game using graphical method and hence solve it algebraically.

⇒ *The given game does not have any saddle point, let the mixed strategies used by A be $p = (p_1, p_2)$ with $p_1 + p_2 = 1$ and both p_1, p_2 lie in the open interval $(0, 1)$. Also the mixed strategies used by B be $q = (q_1, q_2, q_3, q_4)$ with $q_1 + q_2 + q_3 + q_4 = 1$.*



Draw two vertical lines $p_1 = 0,\ p_1 = 1$ at unit distance apart. Draw a line segment joining $(0,2)$, $(1,3)$. This line correspomding to strategy B_1. It represents the expected gain of A due to B's pure moves B_1. Similarly, we can draw lines B_2, B_3, B_4. The lower envelop is the cyan colour segment and the highest point here is the intersection of B_3 and B_4. Therefore, the given 2×4 game can be solved by solving the 2×2 game with A_1, A_2 and B_3, B_4. So the new 2×2 game with strategies of B are $q = (0, 0, q_3, q_4)$ is

	B_3	B_4
A_1	-1	5
A_2	7	-2

. The problem has no saddle point for pure strategy in the pay-off matrix for A.

Let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_1, p_2)$ *and* $\underset{\sim}{q} = (q_3, q_4)$ *with* $p_1 + p_2 = 1,\ p_1, p_2 > 0$ *and* $q_3 + q_4 = 1,\ q_3, q_4 > 0$ *for A and B respectively. Assuming the existence of the value of game we have*

$E_1(p) = -1p_1 + 7p_2 = -p_1 + 7(1 - p_1),\ E_2(p) = 5p_1 - 2p_2 = 5p_1 - 2(1 - p_1)$.

To determine the optimal values of p_1, p_2 *we have,* $-p_1 + 7(1 - p_1) = v = 5p_1 - 2(1 - p_1)$. *Solving we get,* $p_1^* = 3/5(> 0)$. *So,* $p_2^* = 1 - p_1^* = 2/5$ *and value of the game is* $v = -p_1^* + 7(1 - p_1^*) = 11/5$.

Again considering from the B's point of view we have,

$E_1(q) = -1q_3 + 5q_4 = -q_3 + 5(1 - q_3),\ E_2(q) = 7q_3 - 2q_4 = 7q_3 - 2(1 - q_3)$.

To determine the optimal values of q_3, q_4, *we have,* $-q_3 + 5(1 - q_3) = v = 7q_3 - 2(1 - q_3)$. *Solving we get* $q_3^* = 7/15(> 0)$ *and* $q_4^* = 1|q_1^* = 8/15$ *and the value of the game is* $v = -q_3 + 5(1 - q_3) = 11/5$.

Hence the optimal strategies are $p^* = (3/5, 2/5)$ *and* $q^* = (0, 0, 7/15, 8/15)$ *and* $v = 11/5$.

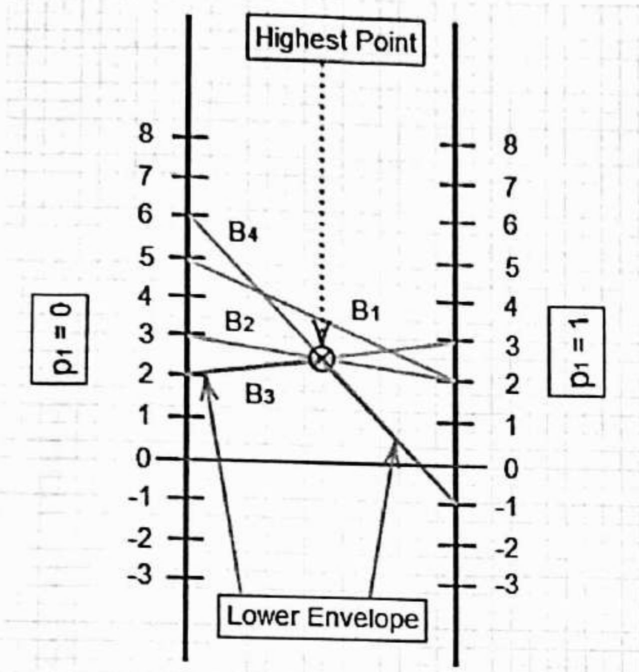
Example 2.4. *Reduce the following* $2 \times n$ *game*

	B_1	B_2	B_3	B_4
A_1	2	2	3	-1
A_2	5	3	2	6

into a 2×2 *game using graphical method and hence solve it algebraically.*

$\Rightarrow$ *The given game does not have any saddle point, let the mixed strategies used by A be*

$p = (p_1, p_2)$ with $p_1 + p_2 = 1$ and both p_1, p_2 lie in the open interval $(0, 1)$. Also the mixed strategies used by B be $q = (q_1, q_2, q_3, q_4)$ with $q_1 + q_2 + q_3 + q_4 = 1$.



Draw two vertical lines $p_1 = 0,\ p_1 = 1$ at unit distance apart. Draw a line segment joining $(0,5)$, $(1,2)$. This line correspomding to strategy B_1. It represents the expected gain of A due to B's pure moves B_1. Similarly, we can draw lines B_2, B_3, B_4. The lower envelop is the cyan colour segment and the highest point here is the intersection of B_2, B_3 and B_4. Therefore, the given 2×4 game can be solved by solving the 2×2 game with A_1, A_2 and B_2, B_3; B_2, B_4; B_3, B_4. We can discard B_2, B_4 as same sign slope. So the two new 2×2 game with strategies of B are $q = (0, q_2, q_3, 0)$ and $q = (0, 0, q_3, q_4)$ are respectively

	B_2	B_3
A_1	2	3
A_2	3	2

and

	B_3	B_4
A_1	3	-1
A_2	2	6

. The problem has no saddle point for pure strategy in the pay-off matrix for A.

For the First matrix, let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_1, p_2)$ and $\underset{\sim}{q} = (q_2, q_3)$ with $p_1 + p_2 = 1,\ p_1, p_2 > 0$ and $q_2 + q_3 = 1,\ q_2, q_3 > 0$ for A and B respectively. Assuming the existence of the value of game we have
$E_1(p) = 2p_1 + 3p_2 = 2p_1 + 3(1 - p_1),\ E_2(p) = 3p_1 + 2p_2 = 3p_1 + 2(1 - p_1)$.
To determine the optimal values of p_1, p_2 we have, $2p_1 + 3(1 - p_1) = v = 3p_1 + 2(1 - p_1)$. Solving we get, $p_1^* = 1/2 (> 0)$. So, $p_2^* = 1 - p_1^* = 1/2$ and value of the game is $v = 2p_1^* + 3(1 - p_1^*) = 5/2$.
Again considering from the B's point of view we have,
$E_1(q) = 2q_2 + 3q_3 = 2q_2 + 3(1 - q_2),\ E_2(q) = 3q_2 + 2q_3 = 3q_2 + 2(1 - q_2)$.
To determine the optimal values of q_2, q_3, we have, $2q_2 + 3(1 - q_2) = v = 3q_2 + 2(1 - q_2)$. Solving we get $q_2^* = 1/2 (> 0)$ and $q_3^* = 1 | q_1^* = 1/2$ and the value of the game is $v = 2q_2 + 3(1 - q_2) = 5/2$.
Hence the optimal strategies are $p^* = (1/2, 1/2)$ and $q^* = (0, 1/2, 1/2, 0)$ and $v = 5/2$.

For the second matrix, let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_1, p_2)$ and $\underset{\sim}{q} = (q_3, q_4)$ with $p_1 + p_2 = 1,\ p_1, p_2 > 0$ and $q_3 + q_4 = 1,\ q_3, q_4 > 0$ for A and B respectively. Assuming the existence of the value of game we have

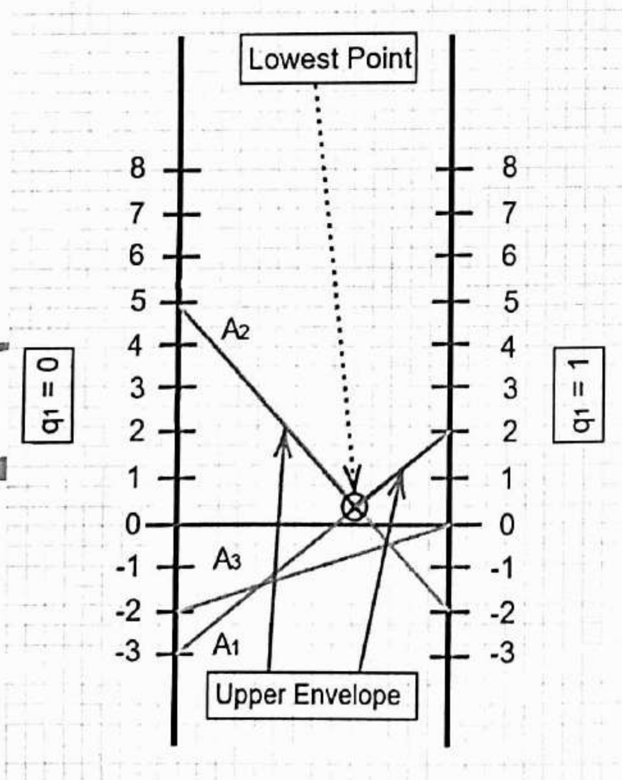
$E_1(p) = 3p_1 + 2p_2 = 3p_1 + 2(1 - p_1),\ E_2(p) = -1p_1 + 6p_2 = -p_1 + 6(1 - p_1)$.
To determine the optimal values of p_1, p_2 *we have,* $3p_1 + 2(1-p_1) = v = -p_1 + 6(1-p_1)$.
Solving we get, $p_1^* = 1/2 (> 0)$. *So,* $p_2^* = 1 - p_1^* = 1/2$ *and value of the game is* $v = 3p_1^* + 2(1 - p_1^*) = 5/2$.
Again considering from the B's point of view we have,
$E_1(q) = 3q_3 - 1q_4 = 3q_3 - (1 - q_3),\ E_2(q) = 2q_3 + 6q_4 = 2q_3 + 6(1 - q_3)$.
To determine the optimal values of q_3, q_4, *we have,* $3q_3 - (1 - q_3) = v = 2q_3 + 6(1 - q_3)$.
Solving we get $q_3^* = 7/8 (> 0)$ *and* $q_4^* = 1|q_1^* = 1/8$ *and the value of the game is* $v = 3q_3 - (1 - q_3) = 5/2$.
Hence the optimal strategies are $p^* = (1/2, 1/2)$ *and* $q^* = (0, 0, 7/8, 1/8)$ *and* $v = 5/2$.
Here multiple optimal solution exists.

■ We can solve $m \times 2$ games in a similar way.

Example 2.5. *Reduce the following* 3×2 *game*

	B_1	B_2
A_1	2	−3
A_2	−2	5
A_3	0	−2

into a 2×2 *game using graphical method and hence solve it algebraically.*
⇒ *The given game does not have any saddle point, let the mixed strategies used by B be* $q = (q_1, q_2)$ *with* $q_1 + q_2 = 1$ *and both* q_1, q_2 *lie in the open interval* $(0, 1)$. *Also the mixed strategies used by A be* $p = (p_1, p_2, p_3)$ *with* $p_1 + p_2 + p_3 = 1$.



Draw two vertical lines $q_1 = 0,\ q_1 = 1$ *at unit distance apart. Draw a line segment joining* $(0, -3), (1, 2)$. *This line correspomding to strategy* A_1. *It represents the expected gain of B due to A's pure moves* A_1. *Similarly, we can draw lines* A_2, A_3. *The upper envelop is the cyan colour segment and the lowest point here is the intersection of* A_1 *and* A_2. *Therefore, the given* 3×2 *game can be solved by solving the* 2×2 *game with* B_1, B_2 *and* A_1, A_2. *So the new* 2×2 *game with strategies of A are* $p = (p_1, p_2, 0)$ *is*

	B_3	B_4
A_1	2	−3
A_2	−2	5

. The problem has no saddle point for pure strategy in the pay-off matrix for A.

Let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_1, p_2)$ *and* $\underset{\sim}{q} = (q_1, q_2)$ *with* $p_1 + p_2 = 1,\ p_1, p_2 > 0$ *and* $q_1 + q_2 = 1,\ q_1, q_2 > 0$ *for A and B respectively. Assuming the existence of the value of game we have*

$E_1(p) = 2p_1 - 2p_2 = 2p_1 - 2(1-p_1),\ E_2(p) = -3p_1 + 5p_2 = -3p_1 + 5(1-p_1)$.

To determine the optimal values of p_1, p_2 *we have,* $2p_1 - 2(1-p_1) = v = -3p_1 + 5(1-p_1)$.

Solving we get, $p_1^* = 7/12(>0)$. *So,* $p_2^* = 1 - p_1^* = 5/12$ *and value of the game is* $v = 2p_1^* - 2(1-p_1^*) = 1/3$.

Again considering from the B's point of view we have,

$E_1(q) = 2q_1 - 3q_2 = 2q_1 - 3(1-q_1),\ E_2(q) = -2q_1 + 5q_2 = -2q_1 + 5(1-q_1)$.

To determine the optimal values of q_1, q_2, *we have,* $2q_1 - 3(1-q_1) = v = -2q_1 + 5(1-q_1)$.

Solving we get $q_1^* = 2/3(>0)$ *and* $q_2^* = 1|q_1^* = 1/3$ *and the value of the game is* $v = 2q_1 - 3(1-q_1) = 1/3$.

Hence the optimal strategies are $p^* = (7/12, 5/12, 0)$ *and* $q^* = (2/3, 1/3)$ *and* $v = 1/3$.

[Do It Yourself] 2.7. *Reduce the games i)*

	B_1	B_2	B_3	B_4
A_1	2	2	3	−1
A_2	4	3	2	6

, *ii)*

	B_1	B_2	B_3	B_4
A_1	1	2	−3	7
A_2	2	5	4	−6

into a 2×2 *game using graphical method and hence solve it algebraically. [CU 92, 98]*

[Do It Yourself] 2.8. *Reduce the games i)*

	B_1	B_2
A_1	2	7
A_2	3	5
A_3	11	2

, *ii)*

	B_1	B_2
A_1	1	−3
A_2	3	5
A_3	−1	6
A_4	4	1
A_5	2	2
A_6	−5	0

into a 2×2 *game using graphical method and hence solve it algebraically. [CU 88, 86]*

1.1.5 Dominance Property

■ The principle of dominance in Game Theory states that if one strategy of a player dominates over the other strategy in all conditions then the later strategy can be ignored.
► The concept of dominance is useful in two-person zero-sum games when there ∄ a saddle point.
► Row Dominance: If all the elements of a row (i) are less than or equal to the corresponding elements of any other row j, then the row (i) is dominated by row (j) and can be deleted from the matrix.
► Column Dominance: If all the elements of a column (k) are greater than or equal to the corresponding elements of any other column (l), then the column k is dominated by the column j and can be deleted from the pay-off matrix.
► Using this property the game may be reduced to 2×2 or, $2 \times n$ or, $m \times 2$ game so that it can be solved easily.

Example 1.6. *Reduce the given 4×5 game*

	B_1	B_2	B_3	B_4	B_5
A_1	10	5	5	20	4
A_2	11	15	10	17	25
A_3	7	12	8	9	8
A_4	5	13	9	10	5

by using dominance property and hence solve it. [CU 94]

⇒ *As row A_2 dominates both the A_3 and A_4. Therefore, by using dominance property we can discard rows A_3, A_4 and the resulting matrix is*

	B_1	B_2	B_3	B_4	B_5
A_1	10	5	5	20	4
A_2	11	15	10	17	25

The column B_3 dominates B_1, B_2, B_4. Therefore, by using dominance property we can discard columnsB_1, B_2, B_4 and the resulting matrix is

	B_3	B_5
A_1	5	4
A_2	10	25

Again row A_2 dominates A_1. Therefore, by using dominance property we can discard *rows A_1 and the resulting matrix is*

	B_3	B_5
A_2	10	25

In this matrix, column B_3 dominates B_5. So the value of the game is 10 at (A_2, B_3) and $(2,3)$ is the saddle point.

1.1.6 Modified Dominance Property

► Row Modified Dominance: If all the elements of a row (i) are less than or equal to the corresponding elements of the convex combination of some other rows, then the row (i) is dominated by all those rows and can be deleted from the pay-off matrix.
► Column Modified Dominance: If all the elements of a column (k) are greater than or equal to the corresponding elements of the convex combination of some other columns, then column k is dominated by all those columns and can be deleted from the pay-off matrix.

Example 1.7. *Reduce the given 4×4 game*

	B_1	B_2	B_3	B_4
A_1	1	2	−1	2
A_2	3	1	2	3
A_3	0	3	2	1
A_4	−2	1	1	−1

by using dominance/ modified dominance property and hence solve it.

⇒ *As row A_2 dominates $(\geq)$ A_4. Therefore, by using dominance property we can discard row A_4 and the resulting matrix is*

	B_1	B_2	B_3	B_4
A_1	1	2	−1	2
A_2	3	1	2	3
A_3	0	3	2	1

The column B_1 dominates $(\leq)$ B_4. Therefore, by using dominance property we can discard column B_4 and the resulting matrix is

	B_1	B_2	B_3
A_1	1	2	-1
A_2	3	1	2
A_3	0	3	2

Again convex combination of rows i.e. $\frac{1}{2}A_2 + \frac{1}{2}A_3$ *dominates* $(\geq)$ A_1. *Therefore, by using modified dominance property we can discard row* A_1 *and the resulting matrix is*

	B_1	B_2	B_3
A_2	3	1	2
A_3	0	3	2

Also convex combination of columns i.e. $\frac{1}{2}B_1 + \frac{1}{2}B_2$ *dominates* $(\leq)$ B_3. *Therefore, by using modified dominance property we can discard column* B_3 *and the resulting matrix is*

	B_1	B_2
A_2	3	1
A_3	0	3

The problem has no saddle point for pure strategy in the pay-off matrix for A.
For the First matrix, let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_2, p_3)$ *and* $\underset{\sim}{q} = (q_1, q_2)$ *with* $p_2 + p_3 = 1,\ p_2, p_3 > 0$ *and* $q_1 + q_2 = 1,\ q_1, q_2 > 0$ *for A and B respectively. Assuming the existence of the value of game we have*
$E_1(p) = 3p_2 + 0p_3 = 3p_2,\ E_2(p) = 1p_2 + 3p_3 = p_2 + 3(1 - p_2)$.
To determine the optimal values of p_2, p_3 *we have,* $3p_2 = v = p_2 + 3(1 - p_2)$.
Solving we get, $p_2^* = 3/5(> 0)$. *So,* $p_3^* = 1 - p_2^* = 2/5$ *and value of the game is* $v = 3p_2^* = 9/5$.
Again considering from the B's point of view we have,
$E_1(q) = 3q_1 + 1q_2 = 3q_1 + (1 - q_1),\ E_2(q) = 0q_1 + 3q_2 = 3(1 - q_1)$.
To determine the optimal values of q_2, q_3, *we have,* $3q_1 + (1 - q_1) = v = 3(1 - q_1)$.
Solving we get $q_1^* = 2/5(> 0)$ *and* $q_2^* = 1|q_1^* = 3/5$ *and the value of the game is* $v = 3q_2^* = 9/5$.
Hence the optimal strategies are $p^* = (0, 3/5, 2/5, 0)$ *and* $q^* = (2/5, 3/5, 0, 0)$ *and* $v = 9/5$.

[Do It Yourself] 1.9. *Solve the game problem by reducing it into* 2×2 *problem using dominance property. [CU 87]*

	B_1	B_2	B_3	B_4	B_5
A_1	0	0	0	0	0
A_2	4	2	0	2	1
A_3	4	3	1	3	2
A_4	4	3	4	-1	2

[Do It Yourself] 1.10. *Reduce the following game to 2×2 game by modified dominance property and then solve it. [CU 90]*

	B_1	B_2	B_3	B_4
A_1	3	2	4	0
A_2	3	4	2	4
A_3	4	2	4	0
A_4	0	4	0	8

[Do It Yourself] 1.11. *Use dominance and modified dominance property to reduce the pay-off matrix given by*

	B_1	B_2	B_3	B_4
A_1	3	-1	1	2
A_2	-2	3	2	3
A_3	2	-2	-1	2

into a 2×2 and find the mixed strategies for A and B and the value of the game. [CU 84]

1.1.7 Game theory to LPP

Theorem 1.2. *Show that every two person zero sum game problem can be converted into to a LPP.*

Proof. Hi □

Theorem 1.3. *If we add a fixed number (positive/ negative) to each element of a pay-off matrix the optimal strategies rematn same, while the value of the game will be increased by that number.*

Proof. Suppose the pay-off matrix is $[\ a_{ij}\ ]_{n\times m}$ and let the mixed strategies taken by A and B are $p = (p_1, p_2, \cdots, p_m)$ with $\sum_{i=1}^{m} p_i = 1,\ with\ p_i \geq\ 0\ \forall i$ and $q = (q_1, q_2, \cdots, q_n)$ with $\sum_{j=1}^{m} q_j = 1,\ q_j \geq 0,\ \forall j$ respectively.

So, the pay-off function of the game is given by $E(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_iq_j$.
So after adding a number k to each element of the pay-off matrix, the pay-off function of the new game is given by: $E^k(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+k)p_iq_j = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_iq_j + k\sum_{i=1}^{m}\sum_{j=1}^{n} p_iq_j = E(p,q) + k$ as $\sum p_i = 1, \sum q_j = 1$ ... (1)
WLOG we can assume that $(a_{ij}+k) > 0\ \forall i,j$. Then the value of the second game exist and unique if it is optimal strategies be p^*q^* for the second game, then its value of the game is $E^k(p^*,q^*) = \max_p \min_q E(p,q) = E^k(p^*,q^*) + k$
Again, $\max_p \min_q E^k(p,q) = \max_p \min_q E(p,q) + k$.
Therefore, it implies $\max_p \min_q E(p,q) = E^k(p^*,q^*)$ ... (2)
Similarly, we can get $\min_q \max_p E(p,q) = E^k(p^*,q^*)$ ... (3)
So, from (2) and (3) we can say that the value of original game also exists and unique, the optimal strategies for both games remain same and from (1) we can say that value of the second game is only increased by the number k.
In a similar way, we can also show that the relation is also valid if $a_{ij} + k \leq 0,\ \forall i,j$. □

Example 1.8. *Solve the game problem*

	B_1	B_2
A_1	2	1
A_2	−1	3

by converting it into LPP.

⇒ *The value of the game may not be positive. Adding 2 to each element we get a pay-off matrix whose values will be essentially positive and solving the new problem we can find the value of original problem. So the pay off matrix transform to*

	B_1	B_2
A_1	4	3
A_2	1	5

Let the optimal strategies for A is $p^* = (p_1^*, p_2^*)$ and B is $q^* = (q_1^*, q_2^*)$. Now considering B's problem it can be reduced to

$$\begin{aligned} &Maximize,\ q_0 = q_1' + q_2' \\ &Subject\ to, \\ &4q_1' + 3q_2' \leq 1 \\ &q_1' + 5q_2' \leq 1 \\ &q_1', q_2' \geq 0. \end{aligned}$$

If $Max\ q_0 = \frac{1}{v^*}$, then the value of the game of the original problem will be $v^* - 2$ and the optimal solution $q_j^* = q_j'^* v^*,\ p_i^* = p_i'^* v^*,\ i,j = 1,2.$

Introducing the slack variables q_3', q_4' one in each constraint, we get the following converted equations.

$$\begin{aligned} 4q_1' + 3q_2' + q_3' \quad &= 1 \\ q_1' + 5q_2' \quad + q_4' &= 1 \end{aligned}$$

Here the coefficient matrix contain a unit basis. The adjusted objective function z is given by: Maximize, $q_0 = q_1' + q_2' + 0.q_3' + 0.q_4'$.
Now we will construct the simplex table to solve the given LPP.

		c_j	1	1	0	0	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	*Min Ratio*
a_3	0	1	4	3	1	0	1/3 = 0.33
a_4	0	1	1	(5)	0	1	1/5 = 0.2 ←
$z_j - c_j$		0	−1	*-1*	0	0	
a_3	0	2/5	(17/5)	0	1	−3/5	(2/5)/(17/5) = 2/17 ←
a_2	1	1/5	1/5	1	0	1/5	(1/5)/(1/5) = 1
$z_j - c_j$		1/5	*-4/5*	0	0	1/5	
a_1	1	2/17	1	0	5/17	−3/17	
a_2	1	3/17	0	1	−1/17	4/17	
$z_j - c_j$		5/17	0	0	4/17	1/17	

So Max $q_0 = \frac{5}{17} = \frac{1}{v^*}$ *(say) at* $q_1'^* = \frac{2}{17}$, $q_2'^* = \frac{3}{17}$.
The value of the original game is $v^* - 2 = \frac{17}{5} - 2 = \frac{7}{5}$ *at* $q_1^* = q_1'^* v^* = \frac{2}{5}$ *and* $q_2^* = q_2'^* v^* = \frac{3}{5}$.
Now using the duality theory, we have $p_1'^* = \frac{4}{17}$ *and* $p_2'^* = \frac{1}{17}$. *So,* $p_1^* = p_1'^* v^* = \frac{4}{5}$ *and* $p_2^* = p_2'^* v^* = \frac{1}{5}$.
Therefore, the optimal strategies are $p^* = (\frac{4}{5}, \frac{1}{5})$, $q^* = (\frac{2}{5}, \frac{3}{5})$ *and the value of game is* $v = \frac{7}{5}$.

[Do It Yourself] 1.12. *Determine the optimum strategies of A and B from the pay-off matrix*

	B_1	B_2
A_1	2	7
A_2	3	5
A_3	11	2

using LPP. [CU 90]

[Do It Yourself] 1.13. *Solve the game problem by LPP [CU 91]*

	B_1	B_2	B_3
A_1	1	−1	−1
A_2	−1	−1	3
A_3	−1	2	−1