Syllabus

Unit - I : *Introduction to Operations Research, phases of O.R., model building, various types of O.R. problems. Linear Programming Problem, Mathematical formulation of the L.P.P. graphical solutions of a L.P.P., Simplex method for solving L.P.P., Charne's M-technique for solving L.P.P. involving artificial variables. Special cases of L.P.P. Concept of Duality in L.P.P: Dual simplex method. Post-optimality analysis.*

Unit - II : *Transportation Problem: Initial solution by North West corner rule, Least cost method and Vogel's approximation method (VAM), MODI's method to find the optimal solution, special cases of transportation problem. Assignment problem: Hungarian method to find optimal assignment, special cases of assignment problem.*

Unit - III : *Game theory: Rectangular game, minimax-maximin principle, solution to rectangular game using graphical method, dominance and modified dominance property to reduce the game matrix and solution to rectangular game with mixed strategy. Networking: Shortest route and minimal spanning tree problem.*

Unit - IV : *Inventory Management: ABC inventory system, characteristics of inventory system. EOQ Model and its variations, with and without shortages, Quantity Discount Model with price breaks.*

Unit - Practical : *1) Mathematical formulation of L.P.P and solving the problem using graphical method, Simplex technique and Charne's Big M method involving artificial variables. 2) Identifying Special cases by Graphical and Simplex method and interpretation a. Degenerate solution b. Unbounded solution c. Alternate solution d. Infeasible solution. 3) Post-optimality a. Addition of constraint b. Change in requirement vector c. Addition of new activity d. Change in cost vector. 4) Allocation problem using Transportation model. 5) Allocation problem using Assignment model. 6) Networking problem a. Minimal spanning tree problem b. Shortest route problem. 7) Problems based on game matrix a. Graphical solution to mx2 / 2xn rectangular game b. Mixed strategy. 8) To find optimal inventory policy for EOQ models and its variations. 9) To solve all-units quantity discounts model.*

Chapter 1

Operations Research

1.1 Introduction

The first formal activities of Operations Research (OR) were initiated to the military services in England during World War II. Because of the war effort, there was an urgent need to allocate scarce resources to the various military operations and to the activities within each operation in an effective manner. Therefore, the British and then the U.S. military management called upon a large number of scientists to apply a scientific approach to dealing with this and other strategic and tactical problems. In effect, they were asked to do research on (military) operations. These teams of scientists were the first OR teams. By developing effective methods of using the new tool of radar, these teams were instrumental in winning the Air Battle of Britain. Through their research on how to better manage convoy and antisubmarine operations, they also played a major role in winning the Battle of the North Atlantic. Similar efforts assisted the Island Campaign in the Pacific.

When the war ended, the success of OR in the war effort spurred interest in applying OR outside the military as well. As the industrial boom following the war was running its course, the problems caused by the increasing complexity and specialization in organizations were again coming to the forefront. It was becoming apparent to a growing number of people, including business consultants who had served on or with the OR teams during the war, that these were basically the same problems that had been faced by the military but in a different context. By the early $1950's$, these individuals had introduced the use of OR to a variety of organizations in business, industry, and government. The rapid spread of OR soon followed.

1.1.1 Development of OR

Two factors played vital role in the rapid growth of OR during that period. One was the substantial progress that was made early in improving the techniques of OR. After the war, many of the scientists who had participated on OR teams or who had heard about this work were motivated to pursue research relevant to the field, important advancements in the state of the art resulted. A prime example is the simplex method for solving linear programming problems, developed by George Dantzig in 1947. Many of the standard tools of OR, such as linear programming, dynamic programming, queueing theory, and inventory theory, were relatively well developed before the end of the $1950's$.

A second factor that gave great impetus to the growth of the field was the onslaught of the computer revolution. A large amount of computation is usually required to deal most effectively with the complex problems typically considered by OR. Doing this by hand would often be out of the question. Therefore, the development of electronic digital computers, with their ability to perform arithmetic calculations thousands or even millions of times faster than a human being can, was a tremendous boon to OR.

1.1.2 Characteristics of OR

▶ OR adopts an organizational (broader) point of view. Thus, it attempts to resolve the conflicts of interest among the components of the organization in a way that is best for the organization as a whole.
▶ OR frequently attempts to find a best solution (multiple solutions may exists) known as an optimal solution for the problem under consideration.

▶ It is evident that no single individual should be expected to be an expert on all the many aspects of OR work or the problems typically considered, this would require a group of individuals having diverse backgrounds and skills. Therefore, when a full-fledged OR study of a new problem is undertaken, it is usually necessary to use a team approach. Such an OR team typically needs to include individuals who collectively are highly trained in mathematics, statistics and probability theory, economics, business administration, computer science, engineering and the physical sciences, the behavioral sciences, and the special techniques of OR. The team also needs to have the necessary experience and variety of skills to give appropriate consideration to the many ramifications of the problem throughout the organization.

1.1.3 Phases of OR

There are mainly six phases of OR which are presented below:

1. Define the problem of interest and gather relevant data.
2. Formulate a mathematical model to represent the problem.
3. Develop a computer-based procedure for deriving solutions to the problem from the model.
4. Test the model and refine it as needed.
5. Prepare for the ongoing application of the model as prescribed by management.
6. Implement

A) Defining The Problem and Collect Data:

► Problem definition involves defining the scope of the problem under investigation. This function should be carried out by the entire OR team. The aim is to identify three principal elements of the decision problem: (1) description of the decision alternatives, (2) determination of the objective of the study, and (3) specification of the limitations under which the modeled system operates.

► OR teams spend a large amount of time gathering relevant data about the problem. Much data usually are needed both to gain an accurate understanding of the problem and to provide the needed input for the mathematical model being formulated in the next phase of study. Frequently, much of the needed data will not be available when the study begins, either because the information never has been kept or because what was kept is outdated or in the wrong form. Therefore, it often is necessary to install a new computer-based management information system to collect the necessary data on an ongoing basis and in the needed form. The OR team normally needs to enlist the assistance of various other key individuals in the organization to track down all the vital data. Even with this effort, much of the data may be quite 'soft', i.e., rough estimates based only on educated guesses. Typically, an OR team will spend considerable time trying to improve the precision of the data and then will make do with the best that can be obtained.

B) Formulating A Mathematical Model:

► After the decision maker's problem is defined, the next phase is to reformulate this problem in a form that is convenient for analysis. The conventional OR approach for doing this is to construct a mathematical model that represents the essence of the problem. Suppose there are n related quantifiable decisions to be made, they are represented as **decision variables** (say, $x_1, x_2, \cdots, x_n$) whose respective values are to be determined. The appropriate measure of performance (e.g., profit) is then expressed as a mathematical function of these decision variables (for example, $P = c_1x_1 + c_2x_2 + \cdots + c_nx_n$). This function is called the **objective function**. Any restrictions on the values that can be assigned to these decision variables are also expressed mathematically, typically by means of inequalities or equations (for example, $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b_1$, $c_1x_1 + c_2x_1x_2 + \cdots + c_nx_3^2x_n \leq b_2$). Such mathematical expressions for the restrictions often are called **constraints**. The constants (namely, the coefficients and right-hand sides) in the constraints and the objective function are called the parameters of the model. The mathematical model might then say that the problem is to choose the values of the decision variables to maximize the objective function, subject to the specified constraints. We refer an OR problem as a **linear programming model**, if the mathematical functions appearing in both the objective function and the constraints are all linear functions.

► Determining the appropriate values to assign to the parameters of the model (one value per parameter) is both a critical and a challenging part of the model-building process. In contrast to textbook problems where the numbers are given to you, determining parameter values for real problems requires gathering relevant data. Since, gathering accurate data frequently is difficult, it implies the value assigned to a parameter often is only a rough estimate. Because of the uncertainty about the true value of the parameter, it is important to analyze how the solution derived from the model would change (if at all) if the value assigned to the parameter were changed to other plausible values. This process is known as **sensitivity analysis**.

► In developing the model, a good approach is to begin with a very simple version and then move in evolutionary fashion toward more elaborate models that more nearly reflect the complexity of the real problem. This process of model enrichment continues only as long as the model remains tractable (solvable). The basic trade-off under constant consideration is between the precision and the tractability of the model.

► A crucial step in formulating an OR model is the construction of the objective function. This requires developing a quantitative measure of performance relative to each of the decision maker's ultimate objectives that were identified while the problem was being defined. If there are multiple objectives, their respective measures commonly are then transformed and combined into a composite measure, called the **overall measure of performance**.

◘ C) Solution Finding from The Model :

► After a mathematical model is formulated for the problem under consideration, the next phase in an OR study is to develop a procedure (usually a computer-based procedure) for deriving solutions to the problem from this model. A common theme in OR is the search for an optimal or, best solution.

► Since a model is an approximate representation of the real problem, it does not imply that the optimal solution for the model is the best possible solution for the real problem. However, if the model is well formulated and tested, the resulting solution is a good approximation to the real problem.

► Sometimes we will use the term **satisficing** i.e. satisfactory and optimizing. The distinction between optimizing and satisficing reflects the difference between theory and the realities frequently faced in trying to implement that theory in practice. The goal of an OR study is to conduct the study in an optimal manner, regardless of whether this involves finding an optimal solution for the model. The team also consider the cost of the study and the disadvantages of delaying its completion, and then attempt to maximize the net benefits resulting from the study. In recognition of this concept, OR teams occasionally use only **heuristic procedures** (i.e., intuitively designed procedures that do not guarantee an optimal solution) to find a good **suboptimal solution**.

► An optimal solution for the original model may be far from ideal for the real problem, so additional analysis is needed. Therefore, **post-optimality analysis** (analysis done after finding an optimal solution) is a very important part of most OR studies. This analysis also known as **what-if analysis** because it involves addressing some questions about what would happen to the optimal solution if different assumptions are made about future conditions. These questions often are raised by the managers who will be making the ultimate decisions rather than by the OR team.

► An important aspect of the model solution phase is **sensitivity analysis**. It deals with obtaining additional information about the behavior of the optimum solution when the model undergoes some parameter changes. Sensitivity analysis is particularly needed when the parameters of the model cannot be estimated accurately. In these cases, it is important to study the behavior of the optimum solution in the neighborhood of the estimated parameters.

D) Test and Improve The Model:

► Before use the model, it must be thoroughly tested to try to identify and correct as many flaws as possible. Usually, after a long succession of improved models, the OR team concludes that the current model now is giving reasonably valid results. Although some minor flaws undoubtedly remain hidden in the model (and may never be detected), the major flaws have been sufficiently eliminated that the model now can be reliably used. This process of testing and improving a model to increase its validity is known as **model validation**. Also, the mathematical expressions has to be **dimensionally consistent**.

► A systematic approach to test the model is to use a **retrospective test**. When it is applicable, this test involves using historical data to reconstruct the past and then determining how well the model and the resulting solution would have performed if they had been used. However, there is no assurance that future performance will continue to duplicate past behavior. Also, because the model is usually based on careful examination of past data, the proposed comparison is usually favorable. If the proposed model represents a new (nonexisting) system, no historical data would be available. In such cases, we may use simulation as an independent tool for verifying the output of the mathematical model.

E) Preparing To Apply The Model:

► After the testing phase has been completed and an acceptable model has been developed. The next step is to install a well-documented system for applying the model as prescribed by management. This system will include the model, solution procedure (including post-optimality analysis), and operating procedures for implementation. Then, even as personnel changes, the system can be called on at regular intervals to provide a specific numerical solution.

F) Implementation:

▶ After a system is developed for applying the model, the last phase of an OR study is to implement this system as prescribed by management. The success of the implementation phase depends on the support of both top management and operating management. Throughout the entire period during which the new system is being used, it is important to continue to obtain feedback on how well the system is working and whether the assumptions of the model continue to be satisfied. When significant deviations from the original assumptions occur, the model should be revisited to determine if any modifications should be made in the system. The post-optimality analysis can be helpful in guiding this review process. It is also appropriate for the OR team to document its methodology clearly and accurately enough so that the work is reproducible.

1.1.4 Various Types of OR Problems

◻ The general OR model can be organized in the following general format:

1) Maximize or minimize Objective Function. 2) subject to Constraints

▶ A solution of the model is feasible if it satisfies all the constraints. It is optimal if, in addition to being feasible, it yields the best (maximum or minimum) value of the objective function.
▶ The most prominent OR technique is **linear programming**. It is designed for models with linear objective and constraint functions.
▶ Other techniques include **integer programming** ⇒ the variables assume integer values, **dynamic programming** ⇒ the original model can be decomposed into more manageable subproblems, **network programming** ⇒ the problem can be modeled as a network and **nonlinear programming** ⇒ functions of the model are nonlinear.

◻ Queuing and Simulation Models: Queuing and simulation is the study of waiting lines. They are not optimization techniques, they determine **measures of performance** of the waiting lines, such as average waiting time in queue, average waiting time for service, and utilization of service facilities.
▶ Queuing models utilize probability and stochastic models to analyze waiting lines, and simulation estimates the measures of performance by copying the behavior of the real system. In a way, simulation may be regarded as the next best thing to observing a real system. The main difference between queuing and simulation is that queuing models are purely mathematical, and hence are subject to specific assumptions that limit their scope of application. Simulation, on the other hand, is flexible and can be used to analyze practically any queuing situation.

◻ Remember that, because of the mathematical nature of OR models, one tends to think that an OR study is always rooted in mathematical analysis. Though mathematical modeling is a base of OR, simpler approaches should be explored first. In some cases,

a 'common sense' solution may be reached through simple observations. Sometimes, a study of the psychology of people may be key to solving the problem.
▶ For example: Responding to complaints of slow elevator service in a large office building, the OR team initially perceived the situation as a waiting-line problem that might require the use of mathematical queuing analysis or simulation. After studying the behavior of the people voicing the complaint, the psychologist on the team suggested installing full-length mirrors at the entrance to the elevators. Miraculously the complaints disappeared, as people were kept occupied watching themselves and others while waiting for the elevator.
▷ It implies that before embarking on sophisticated mathematical modeling, the OR team should explore the possibility of using 'aggressive' ideas to resolve the situation. The solution of the elevator problem by installing mirrors is rooted in human psychology rather than in mathematical modeling. It is also simpler and less costly than any recommendation a mathematical model might have produced. Perhaps this is the reason OR teams usually include the expertise of 'outsiders' from non-mathematical fields.
▶ For example: In a study of the check-in facilities at a large British airport, a United States Canadian consulting team used queuing theory to investigate and analyze the situation. Part of the solution recommended the use of well-placed signs to urge passengers who were within 20 minutes from departure time to advance to the head of the queue and request immediate service. The solution was not successful, because the passengers, being mostly British, were 'conditioned to very strict queuing behavior' and hence were reluctant to move ahead of others waiting in the queue.
▷ Here solutions are rooted in people and not in technology. Any solution that does not take human behavior into account is likely to fail. Even though the mathematical solution of the British airport problem may have been sound, the fact that the consulting team was not aware of the cultural differences between the United States and Britain (Americans and Canadians tend to be less formal) resulted in an unimplementable recommendation.

1.2 Linear Programming Problem (LPP)

□ Linear programming uses a mathematical model to describe the problem of concern. The adjective linear means that all the mathematical functions in this model are required to be linear functions. The word programming does not refer here to computer programming, it is essentially a synonym for planning. Thus, linear programming involves the planning of activities to obtain an optimal result.
▶ In real world, the available resources are limited (such as raw materials, manpower, capital, power and technical appliance etc.) and the main object of an industry is to produce different products in such a way that maximum profit may be earned by selling them at market prices. Similarly, the main aim of a housewife is to buy the goodgrains, vegetables, fruits and other food materials at a minimum cost which will satisfy the minimum need (regarding food values, calories, proteins, vitamins etc.) of the members of her family.
▶ These type of situations can be mathematically explained through **programming**

problem. Some restrictions or constraints are to be adopted to formulate the problem. The function which is to be optimized (such as profit, cost etc. either maximized or minimized) is known as the **objective function**. In most of the cases, the objective function and the constraints are of linear type which is known as the **Linear programming problems**.

1.2.1 Mathematical Formulation of the LPP

☐ Before going to discuss the mathematical formulation of LPP, we first study two simple problems which can be converted into mathematical form.

Example 1.1. *Four different metals namely, iron, copper, zinc and manganese are required to produce three commodities A, B and C. To produce one unit of A, 40 kg iron, 30 kg copper, 7 kg zinc and 4 kg manganese are needed. Similarly to produce one unit of B, 70 kg iron, 14 kg copper and 9 kg manganese are needed and for producing one unit of C, 50 kg iron, 18 kg copper and 8 kg zinc are required. The total available quantities of metals are 1500 kg iron, 1000 kg of copper, 500 kg of zinc and 200 kg manganese. The profits are Rs. 400, Rs. 300 and Rs. 100 in selling per one unit of A, B and C respectively. Formulate the problem mathematically so that the profit is maximum.*

⇒ *Let z be the total profit by selling A, B and C. Here we will maximize z. In mathematical terms z is known as objective function.*

Now to get the maximum profit, let x_1 units of A, x_2 units of B, and x_3 units of C are required.

The restriction due to iron is $40x_1 + 70x_2 + 50x_3 \leq 1500$.

The restriction due to copper is $30x_1 + 14x_2 + 18x_3 \leq 1000$.

The restriction due to zinc is $7x_1 + 0x_2 + 8x_3 \leq 500$.

The restriction due to manganese is $4x_1 + 9x_2 + 0x_3 \leq 200$.

The objective function is $z = 400x_1 + 300x_2 + 100x_3$ *which is to maximized.*

Therefore, the mathematical formulation of the problem is:

$$\begin{aligned} &Maximize,\ z = 400x_1 + 300x_2 + 100x_3 \\ &Subject\ to,\ 40x_1 + 70x_2 + 50x_3 \leq 1500 \\ &30x_1 + 14x_2 + 18x_3 \leq 1000 \\ &7x_1 + 0x_2 + 8x_3 \leq 500 \\ &4x_1 + 9x_2 + 0x_3 \leq 200 \\ &x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{aligned}$$

[Do It Yourself] 1.1. *A patient needs daily 5 mg, 20 mg and 15 mg of vitamins A, B and C respectively. The vitamins available from a mango, an orange and an apple, are 0.5 mg of A, 1 mg of B, 1 mg of C; 2 mg of B, 3 mg of C; 0.5 mg of A, 3 mg of B and 1 mg of C respectively. If the cost of a mango, an orange and an apple be Rs. 1.5, Rs. 1 and Rs. 3 respectively, find the minimum cost of collecting the fruits so that daily requirement of the patient be met. Formulate the problem mathematically.*

[Do It Yourself] 1.2. *A hospital has the following minimum requirement for nurses.*

Period	*Clock Time*	*Minimum No. of Nurses Required*
1	6 AM - 10 AM	60
2	10 AM - 2 PM	70
3	2 PM - 6 PM	60
4	6 PM - 10 PM	50
5	10 PM - 2 AM	20
6	2 AM - 6 AM	30

Nurses report to the hospital wards at the beginning of each period and work for eight consecutive hours. The hospital wants to determine the minimum number of nurses so that there may be sufficient number of nurses available for each period. Formulate this as a L.P.P. [Take x_i = Nurses reported at the begining of period i]

□ Mathematical Form of LPP: In general we will find out a set of values $x_1, x_2, \cdots, x_n$ which will optimize (either maximize or, minimize) the linear function

$$\begin{array}{l} \textit{Optimize, } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \textit{Subject to the restrictions,} \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n (\leq = \geq) b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n (\leq = \geq) b_2 \\ \vdots \quad \vdots \quad \cdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n (\leq = \geq) b_m \\ \text{With the the non-negative restrictions } x_j \geq 0,\ j = 1(1)n \end{array}$$

► Here a_{ij}, b_i and c_j, $i = 1(1)m$ are all constants and x_j's are variables. Each of the restrictions are also known as constraints. A constraint is either an equation or an inequality. The linear function z is known as the objective function.

► The above linear expressions can be written in matrix form as follows:

$$\begin{array}{c} \textit{Optimize, } z = \underline{c}\underline{x} \\ \textit{Subject to, } \quad A\underline{x} (\leq = \geq) \underline{b}, \quad \underline{x} \geq \underline{0} \end{array}$$

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}'$$

► Here $A = (a_{ij})$ is a $m \times n$ **co-efficient matrix.** $\underline{c}$ is a n − *component* row vector known as a **cost or price vector** as it determines the cost of production or the price of a commodity. $\underset{\sim}{x}$ is a $n-$*component* column vector which is known as **decision variable or legitimate variable vector** and $\underset{\sim}{b}$ is a $m-$*component* column vector which is known as **requirement vector**.

◘ Feasible Solution : A set of values of the variables, which satisfy all the constraints and all the non-negative restrictions of variables, is known as the feasible solution (F.S.) to the LPP.

◘ Optimal Solution : A feasible solution to a LPP which make the objective function an optimum is known as the optimal solution i.e. best feasible solution to the LPP.

► A particular LPP is either a problem of maximization or a problem of minimization. The problem of minimization of the objective function z is nothing but problem of maximization of the function $(-z)$ and vice versa and $\min(z) = -\max(-z)$ with same set of constraints and the same solution set etc.

[Do It Yourself] 1.3. *A manufacturer of leather belts makes three types of mobile covers A, B and C which are processed on three machines M_1, M_2 and M_3. Mobile cover A requires 2 hours on machine M_1, and 3 hours on machine M_3. Mobile cover B requires 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3 and mobile cover C requires 5 hours on machine M_2 and 4 hours on machine M_3. There are 7 hours of time per day available on machine M_1, 8 hours of time per day available on machine M_2 and 10 hours of time per day available on machine M_3. The profit per unit of A, B and C are Rs. 30, Rs. 50 and Rs. 40 respectively. Formulate a LPP to find out the daily production of each type of belts such that the profit be maximum.*

[Do It Yourself] 1.4. *A person requires 10, 12 and 12 units of chemical A, B and C respectively for his garden. A liquid product contains 3, 2 and 1 unit of A, B and C respectively per jar. A dry product contains 1, 2 and 4 units of A, B and C per packet. If the liquid product sells for Rs. 3 per jar and the dry product sells for Rs. 2 per packet then formulate the problem as a linear programming problem.*

[Do It Yourself] 1.5. *A company sells two different products A and B. The company makes a profit of Rs. 40 and Rs. 30 per unit of products A and B respectively. The two products are produced in a common production process and are sold in different markets. The production process has a capacity of 30,000 man hours. It takes 3 hours to produce one unit of A and one hour to produce one unit of B. The market has been surveyed and the company officials feel that maximum number of unit of A that can be sold is 8,000 and the maximum of B is 12,000 units. Subject to these limitations, formulate the problem as a LPP.*

[Do It Yourself] 1.6. *A factory is engaged in manufacturing three products A, B and C which involve lathe work, grinding and assembling. The cutting, grinding and assembling times required for one unit of A are 2, 1 and 1 hours respectively. Similarly they are 3, 1, 3 hours for one unit of B and 1, 3, 1 hours for one unit of C. The profits on A, B and C are Rs. 2, Rs. 2 and Rs. 4 per unit respectively. Assuming that there are available 300 hours of lathe time, 300 hours of grinder time and 240 hours of assembly time, how many units of each product should be produced to maximize profit? (Formulate mathematically).*

[Do It Yourself] 1.7. *A manufacturer of furniture makes two products; chairs and tables. Processing of the products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and 2 hours on machine B. There are 16 hours of time per day available on machine A and 22 hours on machine B. Profit gained by the manufacturer from a chair and a table is Rs. 1 and Rs. 5 respectively. Formulate a linear programming problem to maximize profit per day.*

Theorem 1.1. *The set of all feasible solutions of a LPP is a convex set.*

Theorem 1.2. *The objective function of a LPP has its optimal value at the extreme points of the convex set of feasible solutions.*

Theorem 1.3. *If the objective function has its optimal value at more than one extreme point then every convex combinations of these extreme points gives the optimal value of the objective function.*

1.2.2 Graphical Solutions

◻ The graphical procedure includes two steps:

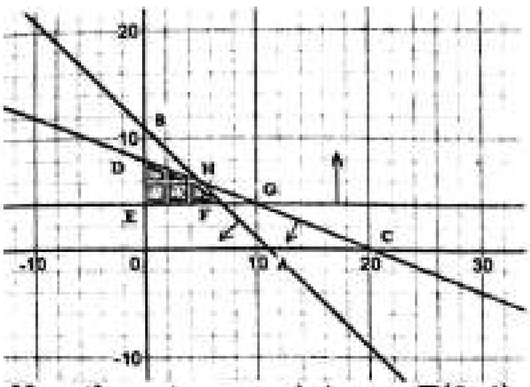
1. Determination of the feasible solution space.
2. Determination of the optimum solution from among all the feasible points in the solution space.

▶ Here you need to know the plot of a straight line and straight line inequalities.
▶ Intercept form: $\frac{x}{a} + \frac{y}{b} = 1$ can be easily plotted.
▶ $2x + 3y = 6 \Rightarrow \frac{x}{3} + \frac{y}{2} = 1$ it is a straight line and can be plotted easily.
▶ $2x + 3y \leq 6 \Rightarrow \frac{x}{3} + \frac{y}{2} \leq 1$ it is a region can be plotted by drawing the straight line first and put $(0,0)$ in $\frac{x}{3} + \frac{y}{2} - 1$. If the quantity < 0 then the region towards origin and if the quantity > 0 then the region is away from the origin.

Example 1.2. *Solve graphically the LPP*

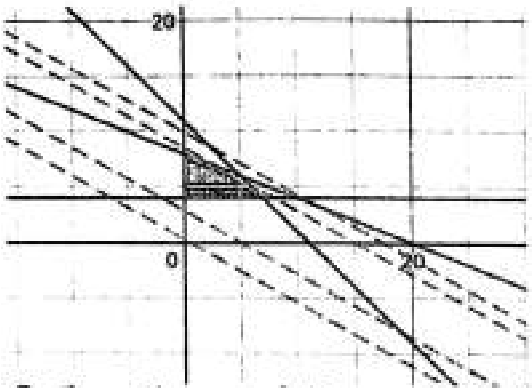
$$\begin{aligned} &\textit{Maximize, } z = 4x_1 + 7x_2 \\ &\textit{Subject to,} \\ &2x_1 + 5x_2 \leq 40 \\ &x_1 + x_2 \leq 11 \\ &x_2 \geq 4 \\ &x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

⇒ *First, we account for the non-negativity constraints $x_1 \geq 0$, $x_2 \geq 0$. The horizontal axis X_1 and the vertical axis X_2 represent the variables x_1, x_2 respectively. Thus, the nonnegativity of the variables restricts the solution space area to the first quadrant.*



Now we will draw above three region given in the constraints: $2x_1 + 5x_2 \leq 40, x_1 + x_2 \leq 11,\ x_2 \geq 4$.
Line AB in downward direction represents $2x_1 + 5x_2 \leq 40$.
Line CD in downward direction represents $x_1 + x_2 \leq 11$.
Line EF in upward direction represents $x_2 \geq 4$.
The convex set of feasible solutions of LPP is the bounded convex region $EFHDE$.

Now the extreme points are $E(0,4)$, $F(7,4)$, $H(5,6)$ *and* $D(0,8)$. *Any point within or on the boundary of the space* $EFHDE$ *is feasible.*



Because the feasible space $EFHDE$ *consists of an infinite number of points. The determination of the optimum solution requires identifying the direction in which the objective function* $z = 4x_1 + 7x_2$ *increases (as maximize). We can do so by assigning arbitrary increasing values to* z. *For example, using* $z = 2, 20, 60, 70$ *we get four dotted lines (see Figure) corresponding to the objective function.*

So the optimum solution occurs at H, *which is the point in the solution space beyond which any further increase will put* z *outside the boundaries of* $EFHDE$. *At* H, $x_1 = 5, x_2 = 6$ *and* $Max\ z = 4.5 + 7.6 = 62$.

[Do It Yourself] 1.8. *Solve graphically the LPP's*

Maximize, $z = 2x_1 + 5x_2$
Subject to,
$$2x_1 + 5x_2 \leq 40$$
$$x_1 + x_2 \leq 11$$
$$x_2 \geq 4$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Maximize, $z = 5x_1 + 4x_2$
Subject to,
$$6x_1 + 4x_2 \leq 24$$
$$x_1 + 2x_2 \leq 6$$
$$-x_1 + x_2 \leq 6$$
$$x_2 \leq 2,\ x_1 \geq 0,\ x_2 \geq 0.$$

[Do It Yourself] 1.9. *Solve graphically the LPP's*

Minimize, $z = 2x_1 + 3x_2$
Subject to,
$$2x_1 + 7x_2 \geq 22$$
$$x_1 + x_2 \geq 6$$
$$5x_1 + x_2 \geq 10$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Minimize, $z = -2x_1 + x_2$
Subject to,
$$x_1 + x_2 \geq 6$$
$$3x_1 + 2x_2 \geq 16$$
$$x_2 \leq 9$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

[Do It Yourself] 1.10. *Solve graphically the LPP's*

Minimize, $z = 4x_1 + x_2$	Maximize, $z = 2x_1 - x_2$
Subject to,	Subject to,
$x_1 + 2x_2 \leq 3$	$x_1 - x_2 \leq 1$
$4x_1 + 3x_2 = 6$	$x_1 \leq 3$
$3x_1 + x_2 \geq 3$	$x_1 \geq 0,\ x_2 \geq 0.$
$x_1 \geq 0,\ x_2 \geq 0.$	

[Do It Yourself] 1.11. *Solve graphically the LPP*

$$\begin{aligned} &\text{Maximize, } z = 500x_1 + 400x_2 \\ &\text{Subject to,} \\ &10x_1 + 8x_2 \leq 800 \\ &x_1 \leq 60 \\ &x_2 \geq 75 \\ &x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Prove that alternative optimal solutions exists. Find at least two optimal solutions where the value of the variables are all integers.

[Do It Yourself] 1.12. *Solve graphically the LPP*

$$\begin{aligned} &\text{Maximize, } z = x_1 + x_2 \\ &\text{Subject to,} \\ &x_1 + x_2 \leq 2 \\ &-4x_1 + x_2 \geq 4 \\ &x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Prove that no optimal solutions exists.

[Do It Yourself] 1.13. *A furniture company manufactures desks and chairs. The sawing department cuts the lumber for both products, which is then sent to separate assembly departments. Assembled items are sent for finishing to the painting department. The daily capacity of the sawing department is 200 chairs or 80 desks. The chair assembly department can produce 120 chairs daily and the desk assembly department 60 desks daily. The paint department has a daily capacity of either 150 chairs or 110 desks. Given that the profit per chair is Rs. 50 and that of a desk is Rs. 100, determine the optimal production mix for the company.*

[*Hint* : x_1 *Chair*, x_2 *Table*, *Max* $z = 50x_1 + 100x_2$; *Sawing* : 80 *desk* $\equiv$ 200 *chair* $\Rightarrow$ x_2 *desk* $\equiv \frac{5x_2}{2}$ *chair*, *so* $x_1 + \frac{5x_2}{2} \leq 200$; $x_1 \leq 120$; $x_2 \leq 60$; *Paint* : : 110 *desk* $\equiv$ 150 *chair* $\Rightarrow x_2$ *desk* $\equiv \frac{15x_2}{11}$ *chair*, *so* $x_1 + \frac{15x_2}{11} \leq 150$]

[Do It Yourself] 1.14. *Day Trader wants to invest a sum of money that would generate an annual yield of at least Rs. 10,000. Two stock groups are available: blue chips and high* *tech, with average annual yields of* 10% *and* 25%, *respectively. Through high-tech stocks provide higher yield, they are more risky, and Trader wants to limit the amount invested in these stocks to no more than* 60% *of the total investment. What is the minimum amount Trader should invest in each stock group to accomplish the investment goal?*

■ Note that, the simplex method is an algebraic procedure. However, its underlying concepts are geometric. Understanding these geometric concepts provides a strong intuitive feeling for how the simplex method operates and what makes it so efficient.

▶ Infeasible solution means no common region between constraints.

▶ Unbounded means common region is not bounded.

▶ Redundancy means if we eliminate some constraints then it will not effect the optimum solution.

1.2.3 Convex Set

▶ Convex Set: A set S is said to be a convex set if for any two points $x_1, x_2 \in X$, we can a point $x \in A$ such that $x = \lambda x_1 + (1-\lambda)x_2$, where $0 \leq \lambda \leq 1$. In other words, we can take any two points and join them through a straight line, then the line lies within the set.

▶ Extreme Point: An extreme point (y) of a convex set S is a point for which we can't find any y_1, y_2 such that $y = \lambda y_1 + (1-\lambda)y_2$. For example, the boundaries of a circle, corner points of a rectangle.

▶ Line Segment: Let, $y_1, y_2 \in \mathbb{R}^n$ such that $y = \lambda y_1 + (1-\lambda)y_2$, $0 \leq \lambda \leq 1$. Here y is the line segment joining y_1, y_2.

▶ Parallel Hyperplane: Two hyperplane, $c_1\underline{x} = y_1$, $c_2\underline{x} = y_2$ are called parallel hyperplane if $c_1 = \lambda c_2$.

▶ Convex Polyhedron: The set of all convex combinations of a finite number of linear independent vectors is called convex polyhedron i.e. $X = \{x : x = \sum_{i=1}^{k} \lambda_i x_i, \textit{ with } \sum_{i=1}^{k} \lambda_i = 1\}, \lambda_i \geq 0$.

▶ Convex Hull: The smallest convex set is known as convex hull.

▶ Limit Point/ Cluster Point: A point $x \in \mathbb{R}$ is said to be a limit point (cluster point) of S if $N'(x, \delta) \cap S \neq \phi$. For ex. $S = (2, 3] \Rightarrow$ each point of S is a limit point of S, 2 is also a limit point of S. Again, $\mathbb{N}$ has no limit points.

▶ Closed Set: A set S is said to be closed set if it contains all its limit points i.e. $S' \subset S$. For ex. $[1, 2]$, $\mathbb{N}$, etc. Note that, $\mathbb{Q}$, $\{\frac{1}{n}, n \in \mathbb{N}\}$ is neither open nor closed. $\mathbb{R}, \phi$ both open and closed.

Theorem 1.4. *Intersection of two convex sets is also a convex set.*

Proof. Let, X_1, X_2 be two convex sets and we will show that $X_1 \cap X_2$ is also a convex set.
Let, $x_1, x_2 \in X_1 \cap X_2 \Rightarrow x_1, x_2 \in X_1$ and $x_1, x_2 \in X_2$.
Let x_3 be a point given by $x_3 = \lambda x_1 + (1-\lambda)x_2$, $0 \leq \lambda \leq 1$.
As x_3 is a convex combination of x_1 and $x_2 \Rightarrow x_3$ is a point of X_1 as well as a point of

$X_2 \Rightarrow x_3$ is a point on $X_1 \cap X_2$ but x_3 is the convex combination of 2 distinct points of $X_1 \cap X_2$. Hence $X_1 \cap X_2$ is a convex set. □

[Do It Yourself] 1.15. *Show that intersection of a finite number of convex sets is a convex set.*

[Do It Yourself] 1.16. *Show that union of two convex sets may not be a convex set. [Hint : Consider a horizontal line $X_1 = \{x \in (-2,2)\}$; and a vertical line $X_2 = \{y \in (-2,2)\}$, both X_1, X_2 are convex. But union may not be convex.]*

Theorem 1.5. *Show that a hyperplane $c\underline{x} = k$ is a convex set.*

Proof. Let the point set X be a hyperplane given by: $X = \{x : cx = k\}$.
Let $x_1, x_2 \in X \Rightarrow cx_1 = k$ and $cx_2 = k$.
Let x_3 be a point given by $x_3 = \lambda x_1 + (1-\lambda)x_2,\ 0 \le \lambda \le 1$.
Therefore, $cx_3 = \lambda cx_1 + (1-\lambda)cx_2 = \lambda k + (1-\lambda)k = k$ which indicates that x_3 is also a point of $cx = k$ but x_3 is the convex combination of two distinct points x_1 and x_2 of X.
Hence X is a convex set. □

[Do It Yourself] 1.17. *Show that hyperplane with inequalities or, half-space (i.e. $c\underline{x} > k$) is a convex set.*

Theorem 1.6. *Show that convex polyhedron i.e. $X = \left\{x :\ x = \sum_{i=1}^{k} \lambda_i x_i,\ with\ \sum_{i=1}^{k} \lambda_i = 1\right\}$, $\lambda_i \ge 0$ is a convex set.*

Proof. Let S be a point set consisting of finite number of points $x_1, x_2, \ldots, x_k$ in $\mathbb{R}^n$.
The convex polyhedron $C(S) = X = \left\{x :\ x = \sum_{i=1}^{k} \lambda_i x_i,\ with\ \sum_{i=1}^{k} \lambda_i = 1\right\}$.
We need to show that X is a convex set.
Let u, v be nay two distinct points of X such that
$u = \sum_{i=1}^{k} a_i x_i,\ \sum_{i=1}^{k} a_i = 1$ with $a_i \ge 0$
$v = \sum_{i=1}^{k} b_i x_i,\ \sum_{i=1}^{k} b_i = 1$ with $b_i \ge 0$.
Let w be a point given by, $w = \lambda u + (1-\lambda)v,\ 0 \le \lambda \le 1$ then
$w = \lambda \sum_{i=1}^{k} a_i x_i + (1-\lambda) \sum_{i=1}^{k} b_i x_i = \sum_{i=1}^{k} \{\lambda a_i + (1-\lambda) b_i\} x_i = \sum_{i=1}^{k} c_i x_i$
where $c_i = \lambda a_i + (1-\lambda) b_i$. So, $\sum_{i=1}^{k} c_i = \lambda \sum_{i=1}^{k} a_i + (1-\lambda) \sum_{i=1}^{k} b_i = 1$
with $c_i \ge 0$ *as* $a_i \ge 0,\ b_i \ge 0,$ *and* $0 \le \lambda \le 1$.
Hence, w is also a point of X, which is a convex combination of two distinct points u and v of X. Hence, X is a convex set. □

Theorem 1.7. *The set of all feasible solutions of a LPP $A\underline{x} = \underline{b}, \underline{x} \ge \underline{0}$ is a closed convex set.*

Proof. Let X be the point set of all the feasible solutions of the problem $A\underline{x} = \underline{b}, \underline{x} \ge \underline{0}$.
If X has only one point then its trivial.
If X has at least two distinct points x_1, x_2 then $Ax_1 = b,\ x_1 \ge 0$ and $Ax_2 = b,\ x_2 \ge 0$.
Consider a point x_3 such that $x_3 = \lambda x_1 + (1-\lambda)x_2,\ 0 \le \lambda \le 1$.

So, $x_3 = \lambda A x_1 + (1-\lambda) A x_2 = \lambda b + (1-\lambda) b = b$.
Again $x_3 \geq 0$ as $x_1, x_2 \geq 0$ and $0 \leq \lambda \leq 1$.
Then x_3 is also a feasible solution to the problem $Ax = b$, $x \geq 0$ but x_3 is the convex combination of two distinct points x_1, x_2 of the set X. Thus X is a convex set.
Now the finite number of constraints represented by $Ax = b$ are closed sets and also the set of inequations (finite) represented by $x > 0$ are closed sets and therefore the intersection of finite number of closed sets which is the set of all feasible solutions is also a closed set. □

[Do It Yourself] 1.18. *The set of all feasible solutions of a LPP* $A\underline{x} \geq \underline{b}$, $\underline{x} \geq \underline{0}$ *is a closed convex set.*

Theorem 1.8. *If a LPP has at least two optimal feasible solutions, then there are infinite number of optimal solutions, which are the convex combination of the initial optimal solutions.*

Proof. Let x_1, x_2 be be two optimal FS of a LPP such that they maximize the objective function $z = \underline{cx}$ subject to $A\underline{x} \leq \underline{b}$, $\underline{x} \geq \underline{0}$.
Then $\hat{z} = cx_1$ and $\hat{z} = cx_2$ and $Ax_1 = b$, $Ax_2 = b$.
Let, $x_3 = \lambda x_1 + (1-\lambda)x_2$, $0 \leq \lambda \leq 1$, so $Ax_3 = \lambda A x_1 + (1-\lambda) A x_2 = b$
which indicates that x_3 is also a solution set of $Ax = b$.
Again, $cx_3 = \lambda c x_1 + (1-\lambda) c x_2 = \lambda \hat{z} + (1-\lambda)\hat{z}$ and $x_3 \geq 0$ as $x_1, x_2 \geq 0$ with $0 \leq \lambda \leq 1$.
Therefore, x_3 is also an optimal FS of the LPP which is the convex combination of two distinct points x_1 and x_2. It indicates that there are infinite optimal solutions. □

So, $x_3 = \lambda Ax_1 + (1-\lambda)Ax_2 = \lambda b + (1-\lambda)b = b$.
Again $x_3 \geq 0$ as $x_1, x_2 \geq 0$ and $0 \leq \lambda \leq 1$.
Then x_3 is also a feasible solution to the problem $Ax = b$, $x \geq 0$ but x_3 is the convex combination of two distinct points x_1, x_2 of the set X. Thus X is a convex set.
Now the finite number of constraints represented by $Ax = b$ are closed sets and also the set of inequations (finite) represented by $x > 0$ are closed sets and therefore the intersection of finite number of closed sets which is the set of all feasible solutions is also a closed set. □

[Do It Yourself] 1.18. *The set of all feasible solutions of a LPP $A\underline{x} \geq \underline{b}$, $\underline{x} \geq \underline{0}$ is a closed convex set.*

Theorem 1.8. *If a LPP has at least two optimal feasible solutions, then there are infinite number of optimal solutions, which are the convex combination of the initial optimal solutions.*

Proof. Let x_1, x_2 be be two optimal FS of a LPP such that they maximize the objective function $z = \underline{cx}$ subject to $A\underline{x} \leq \underline{b}$, $\underline{x} \geq \underline{0}$.
Then $\hat{z} = cx_1$ and $\hat{z} = cx_2$ and $Ax_1 = b$, $Ax_2 = b$.
Let, $x_3 = \lambda x_1 + (1-\lambda)x_2$, $0 \leq \lambda \leq 1$, so $Ax_3 = \lambda Ax_1 + (1-\lambda)Ax_2 = b$
which indicates that x_3 is also a solution set of $Ax = b$.
Again, $cx_3 = \lambda cx_1 + (1-\lambda)cx_2 = \lambda\hat{z} + (1-\lambda)\hat{z}$ and $x_3 \geq 0$ as $x_1, x_2 \geq 0$ with $0 \leq \lambda \leq 1$.
Therefore, x_3 is also an optimal FS of the LPP which is the convex combination of two distinct points x_1 and x_2. It indicates that there are infinite optimal solutions. □

Example 1.3. *Prove that in E^2, the set $X = \{(x, y) :\ x + 2y \leq 5\}$ is a convex set.*
⇒ *Let $(x_1, y_1), (x_2, y_2)$ be any two points of the set $X \Rightarrow x_1 + 2y_1 \leq 5,\ x_2 + 2y_2 \leq 5$.*
The convex combination of these two points is $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2)$, $0 \leq \lambda \leq 1$.
Now, $\lambda x_1 + (1-\lambda)x_2 + 2[\lambda y_1 + (1-\lambda)y_2] = \lambda(x_1 + 2y_1) + x_2 + 2y_2 - \lambda(x_2 + 2y_2) \leq 5\lambda + 5 - 5\lambda = 5$. So $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2)$ is a point of the given set $X \Rightarrow$ it is a convex set.

■ The Euclidean plane or, two dimensional Euclidean space (E^2) consists of all real numbers in the form (x, y) with Euclidean metric/ distance.

Example 1.4. *Prove that in E^2, the set $X = \{(x, y) :\ x^2 + y^2 \leq 4\}$ is a convex set.*
⇒ *The given set is not a null set. Let $(x_1, y_1), (x_2, y_2)$ be any two points of the set $X \Rightarrow$ $x_1^2 + y_1^2 \leq 4,\ x_2^2 + y_2^2 \leq 4$.*
The convex combination of these two points is $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2)$, $0 \leq \lambda \leq 1$.
Now, $[\lambda x_1 + (1-\lambda)x_2]^2 + [\lambda y_1 + (1-\lambda)y_2]^2 = \lambda^2(x_1^2 + y_1^2) + (1-\lambda)^2(x_2^2 + y_2^2) + 2\lambda(1-\lambda)(x_1x_2 + y_1y_2) \leq \lambda^2(x_1^2 + y_1^2) + (1-\lambda)^2(x_2^2 + y_2^2) + 2\lambda(1-\lambda)\left[\frac{x_1^2+y_1^2+x_2^2+y_2^2}{2}\right]$ $\boxed{\textit{as } AM \geq GM}$
$< 4\lambda^2 + 4(1-\lambda)^2 + 8\lambda(1-\lambda) = 4\lambda^2 + 4 + 4\lambda^2 - 8\lambda + 8\lambda - 8\lambda^2 = 4$.
So $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2)$ is a point of the given set $X \Rightarrow$ it is a convex set.

Example 1.5. *Prove that in E^2, the set $X = \{(x, y) :\ y^2 < x\}$ is a convex set.*
⇒ *The given set is not a null set. Let $(x_1, y_1), (x_2, y_2)$ be any two points of the set $X \Rightarrow$*

$y_1^2 < x_1,\ y_2^2 < x_2$.
The convex combination of these two points is $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2),\ 0 \le \lambda \le 1$.
Now, $[\lambda y_1 + (1-\lambda)y_2]^2 = \lambda^2 y_1^2 + (1-\lambda)y_2^2 + 2\lambda(1-\lambda)y_1 y_2 \le \lambda^2 y_1^2 + (1-\lambda)^2 y_2^2 + \lambda(1-\lambda)(y_1^2 + y_2^2)$
as $AM \ge GM$ $= y_2^2 - 2\lambda y_2^2 + \lambda y_1^2 + \lambda y_2^2 = \lambda y_1^2 + y_2^2(1-\lambda) \le \lambda x_1 + (1-\lambda)x_2 \quad [\because \lambda \ge 0]$.
So $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2)$ *is a point of the given set* $X \Rightarrow$ *it is a convex set.*

[Do It Yourself] 1.19. *Prove that in* E^2, $X = \{(x, y) :\ y^2 > x\}$ *is not a convex set.*
[Hint : *Use counter example.*]

1.2.4 Simplex Method Introduction

■ The simplex method is important in LPP because it provides an efficient and systematic approach to finding the optimal solution for a linear programming problem. It does this by starting at a BFS and iteratively moving to a better solution until the optimal solution is reached.

▶ The simplex method is preferred over other methods for solving LPPs because it has been shown to be very efficient in practice, and it can handle a large number of constraints and variables. Additionally, it provides information about the sensitivity of the optimal solution to changes in the problem parameters, which can be useful in decision-making processes.

▶ The behind algebraic procedure is based on solving systems of linear equations.

▶ So, in simplex method we will convert the inequality constraints to equivalent equality constraints.

■ Slack Variable : These variables are introduced when inequalities connected by '$\le$' sign and using this variable we can achieve equality sign. For example, if $x_1 - 2x_2 + x_3 - 3x_4 \le 10$ is the inequality, then we will introduce the slack variable x_5 (≥ 0) such that $x_1 - 2x_2 + x_3 - 3x_4 + x_5 = 10$.

■ Surplus Variable : These variables are introduced when inequalities connected by '$\ge$' sign and using this variable we can achieve equality sign. For example, if $x_1 - 2x_2 + x_3 - 3x_4 \ge 10$ is the inequality, then we will introduce the surplus variable x_5 (≥ 0) such that $x_1 - 2x_2 + x_3 - 3x_4 - x_5 = 10$.

■ Minimize to Maximize : Minimize z= - Maximize $(-z)$.

■ Unrestricted in Sign : If a variable x_1 is unrestricted in sign then we take $x_1 = x_{11} - x_{12}$, where $x_{11} \ge 0,\ x_{12} \ge 0$.

[Do It Yourself] 1.20. *Transform the LPP's into the form where all constraints are of* *equality type*

$$\begin{aligned} &Maximize,\ z = 4x_1 + x_2 - 3x_3 \\ &Subject\ to, \\ &x_1 + x_2 - 3x_3 \geq 3 \\ &4x_1 + 3x_2 + 5x_3 \leq 12 \\ &x_1 + x_2 - x_3 = 5 \\ &x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned}$$

$$\begin{aligned} &Maximize,\ z = 2x_1 + x_2 - 6x_3 - x_4 \\ &Subject\ to, \\ &3x_1 + x_4 \leq 25 \\ &x_1 + x_2 + x_3 + x_4 = 20 \\ &4x_1 + 6x_2 \geq 5 \\ &2 \leq 2x_1 + 3x_3 + 2x_4 \leq 30 \\ &x_j \geq 0,\ j = 1, 2, 3, 4. \end{aligned}$$

[Do It Yourself] 1.21. *Transform the LPP's into standard form where all constraints are of equality type, variables are non-negative and objective function is to be maximized.*

$$\begin{aligned} &Maximize,\ z = 4x_1 + x_2 - 3x_3 \\ &Subject\ to, \\ &x_1 + x_2 - 3x_3 \geq 3 \\ &4x_1 + 3x_2 + 5x_3 \leq 12 \\ &x_1 + x_2 - x_3 = 5 \\ &x_1 \geq 0,\ x_3 \geq 0,\ x_2\ unristricted. \end{aligned}$$

$$\begin{aligned} &Minimize,\ z = 2x_1 + x_2 - 6x_3 - x_4 \\ &Subject\ to, \\ &3x_1 + x_4 \leq 25 \\ &x_1 + x_2 + x_3 + x_4 = 20 \\ &4x_1 + 6x_2 \geq 5 \\ &x_j \geq 0,\ j = 1, 2, 3, 4. \end{aligned}$$

◘ **Basic Solution**: Given a system of m simultaneous linear equations containing n variables ($n > m$) and the set of equations be $A\underline{x} = \underline{b}$, $Rank(A) = m$. If any $m \times m$, non-singular matrix be arbitrarily selected from A and if we assume all $(n - m)$ variables zero, then solution so obtained is called a basic solution. The variables which are attached with $m - vectors$ of the non-singular matrix are called basic variables and remaining $(n - m)$ variables whose values are assumed to be zero, are called non-basic variables.

► The values of $m - components$ of the basic solution may be positive, negative or zero.

► If all components of a solution set corresponding to the basic variables are non-zero quantities then the basic solution is known as the non-degenerate basic solution.

► If some components of the solution set corresponding to the basic variables are zero, the basic solution is known as a degenerate basic solution.

► If all components of a solution set are non-negative quantities then the solution is known as a feasible solution (FS) of the LPP.

◘ **Feasible Solution**: If all components of a solution set of a LPP are non-negative then the solution is known as feasible solution (FS).

► **Basic Feasible Solution**: The solution set of a LPP which is both feasible and basic is known as the basic feasible solution of the problem i.e. all components of the solution set corresponding to the basic variables are non-negative quantities.

▶ The solution of a LPP of which all components corresponding to the basic variables are positive quantities, is known as non-degenerate BFS.

[Do It Yourself] 1.22. *Suppose the system of linear equations:* $x_1 + 2x_2 + 3x_3 + 4x_4 = 7,\ 2x_1 + x_2 + x_3 + 2x_4 = 3$, *has solutions:* $[-\frac{1}{3}, \frac{11}{3}, 0, 0], [\frac{2}{5}, 0, \frac{11}{5}, 0], [-\frac{1}{3}, 0, 0, \frac{11}{6}], [0, 2, 1, 0], [0, 0, 1, 1]$. *Write down the types of solution.*

[Do It Yourself] 1.23. *Find out all BFS of the set of linear equations:*
$2x_1 + 3x_2 - x_3 + 4x_4 = 8,\ -x_1 + 2x_2 - 6x_3 + 7x_4 = 3.$

1.2.5 Graphical To Algebraic Solution

In the graphical method, the solution space is described by the common region between constraints, and in the simplex method the solution space is represented by m simultaneous linear equations and n nonnegative variables where $m < n$. The differences are presented in a tabular form:

Graphical Method	**Algebraic Method**
1) Graph all constraints, including non-negativity restrictions	1) Represent the solution space by n variables with $m(< n)$ equations.
2) Solution space consists of infinity feasible points	2) The system has infinitely many feasible solutions.
3) Identify the feasible corner points of the solution space	3) Determine the BFS of the equations.
4) Candidates for the optimum solution are given by a finite number of corner points	4) Candidates for the optimum solution are given by a finite number of BFS.
5) Use objective function to determine the optimum corner point among the candidates	5) Use objective function to determine the optimum BFS among the candidates.

Example 1.6. *Solve the LPP by algebraic method*

$$\begin{array}{l} Maximize,\ z = x_1 + x_2 \\ Subject\ to, \\ \qquad 2x_1 + x_2 \le 4 \\ \qquad x_1 + 2x_2 \le 5 \\ \qquad x_1, x_2 \ge 0. \end{array}$$

⇒ *Here all* $b_i \ge 0, i = 1, 2$. *Also the constraints are involved with '≤' sign, so we introduced two slack variables* s_1, s_2 *one in each constraints we get the equations* $2x_1 + x_2 + s_1 = 4,\ x_1 + 2x_2 + s_2 = 5$. *So algebraically, the solution space of the LP is represented as* $2x_1 + x_2 + s_1 = 4,\ x_1 + 2x_2 + s_2 = 5,\ x_1, x_2, s_1, s_2 \ge 0$.

The system has $m = 2$ equations and $n = 4$ variables. Now we will set any two variables = 0, and solve for other two variables to get a corner point. Therefore, we have $\binom{4}{2} = 6$ corner points.
First we choose $x_1 = x_2 = 0 \Rightarrow$ the corner point is $(0, 0, 4, 5)$.
2^{nd}, we choose $x_1 = s_1 = 0 \Rightarrow$ the corner point is $(0, 4, 0, -3)$.
3^{nd}, we choose $x_1 = s_2 = 0 \Rightarrow$ the corner point is $(0, 5/2, 3/2, 0)$.
4^{rd}, we choose $x_2 = s_1 = 0 \Rightarrow$ the corner point is $(2, 0, 0, 3)$.
5^{th}, we choose $x_2 = s_2 = 0 \Rightarrow$ the corner point is $(5, 0, -6, 0)$.
6^{th}, we choose $s_1 = s_2 = 0 \Rightarrow$ the corner point is $(1, 2, 0, 0)$.
Note that, 2^{nd} and 5^{th} are not feasible $\Rightarrow$ the feasible solutions are $(0, 0), (0, 5/2), (2, 0), (1, 2)$ and the objective functions at the corner points are $0, 5/2, 2, 3$.
So the optimum value of z is 3 and attains at $x_1 = 1, x_2 = 2$.

Example 1.7. *Solve the LPP by algebraic method*

$$\begin{array}{c} \text{Maximize, } z = 5x_1 + 2x_2 + 2x_3 \\ \text{Subject to,} \\ x_1 + 2x_2 - 2x_3 + \leq 30 \\ x_1 + 3x_2 + x_3 \leq 36 \\ x_1, x_2 \geq 0. \end{array}$$

$\Rightarrow$ *Here all $b_i \geq 0, i = 1, 2$. Also the constraints are involved with '$\leq$' sign, so we introduced two slack variables s_1, s_2 one in each constraints we get the equations $x_1 + 2x_2 - 2x_3 + s_1 = 30$, $x_1 + 3x_2 + x_3 + s_2 = 36$. So algebraically, the solution space of the LP is represented as $x_1 + 2x_2 - 2x_3 + s_1 = 30$, $x_1 + 3x_2 + x_3 + s_2 = 36$, $x_1, x_2, x_3, s_1, s_2 \geq 0$.*
The system has $m = 2$ equations and $n = 5$ variables. Now we will set any three variables = 0, and solve for other two variables to get a corner point. Therefore, we have $\binom{5}{2} = 10$ corner points.
First we choose $x_1 = x_2 = x_3 = 0 \Rightarrow$ the corner point is $(0, 0, 0, 30, 36)$.
2^{nd}, we choose $x_1 = x_2 = s_1 = 0 \Rightarrow$ the corner point is $(0, 0, -15, 0, 51)$.
3^{nd}, we choose $x_1 = x_2 = s_2 = 0 \Rightarrow$ the corner point is $(0, 0, 36, 102, 0)$.
4^{rd}, we choose $x_1 = x_3 = s_1 = 0 \Rightarrow$ the corner point is $(0, 15, 0, 0, -9)$.
5^{th}, we choose $x_1 = x_3 = s_2 = 0 \Rightarrow$ the corner point is $(0, 12, 0, 6, 0)$.
6^{th}, we choose $x_2 = x_3 = s_1 = 0 \Rightarrow$ the corner point is $(30, 0, 0, 0, 6)$.
7^{th}, we choose $x_2 = x_3 = s_2 = 0 \Rightarrow$ the corner point is $(36, 0, 0, -6, 0)$.
8^{th}, we choose $x_3 = s_1 = s_2 = 0 \Rightarrow$ the corner point is $(18, 6, 0, 0, 0)$.
9^{th}, we choose $x_2 = s_1 = s_2 = 0 \Rightarrow$ the corner point is $(34, 0, 2, 0, 0)$.
10^{th}, we choose $x_1 = s_1 = s_2 = 0 \Rightarrow$ the corner point is $(9/2, 51/4, 0, 0, 0)$.
Note that, $2^{nd}, 4^{nd}$ and 7^{th} are not feasible $\Rightarrow$ the feasible solutions are $(0, 0, 0), (0, 0, 36), (0, 15, 0), (0, 12, 0), (30, 0, 0), (18, 6, 0), (34, 0, 2), (9/2, 51/4, 0)$ and the objective functions at the corner points are respectively $0, 72, 30, 24, 150, 102, 174, 48$.
So the optimum value of z is 174 and attains at $x_1 = 34, x_2 = 0, x_3 = 2$.

Example 1.8. *Solve the LPP by algebraic method*

$$\begin{array}{l} Maximize,\ z = x_1 - x_2 + 3x_3 \\ Subject\ to, \\ \quad x_1 + x_2 + x_3 \leq 10 \\ \quad 2x_1 - x_3 \leq 2 \\ \quad 2x_1 - 2x_2 + 3x_3 \leq 0 \\ \quad x_1, x_2, x_3 \geq 0. \end{array}$$

⇒ *Here all* $b_i \geq 0, i = 1, 2, 3$. *Also the constraints are involved with '≤' sign, so we introduced two slack variables* s_1, s_2, s_3 *one in each constraints we get the equations* $x_1 + x_2 + x_3 + s_1 = 10,\ 2x_1 - x_3 + s_2 = 2,\ 2x_1 - 2x_2 + 3x_3 + s_3 = 0$. *So algebraically, the solution space of the LP is represented as* $x_1 + x_2 + x_3 + s_1 = 10,\ 2x_1 - x_3 + s_2 = 2,\ 2x_1 - 2x_2 + 3x_3 + s_3 = 0,\ x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.
The system has $m = 3$ *equations and* $n = 6$ *variables. Now we will set any three variables* $= 0$, *and solve for other three variables to get a corner point. Therefore, we have* $\binom{6}{3} = 20$ *corner points.*
Now it is very time consume to do such steps also using this method will be difficult if there are so many constraints. So we will go for simplex method.

1.2.6 Algebraic to Simplex Solution

◘ The algebraic methods are useful for solving simple and small LPPs, while the simplex method is preferred for larger and more complex problems. Algebraic methods rely on algebraic techniques to solve the problem, while the simplex method uses an algorithmic approach to iteratively find the optimal solution.

► For solving the following LPP by algebraic method, the feasible solutions are: $O(0,0), A(0,5/2), B(2,0), C(1,2)$.

$$\begin{array}{l} Maximize,\ z = x_1 + 3x_2 \\ Subject\ to, \\ \quad 2x_1 + x_2 \leq 4 \\ \quad x_1 + 2x_2 \leq 5 \\ \quad x_1, x_2 \geq 0. \end{array}$$

◘ The simplex method starts at the origin $O(0,0)$ where $z = 0$. and the logical question is whether an increase in nonbasic x_1 and/or x_2 above their current zero values can improve (increase) the value of z.

1.2.7 Simplex Method and Initial BFS (IBFS)

◘ The LPP's can be categorize into <u>two forms</u>:
► a) All constraints are connected with '≤' sign and all $b_i \geq 0$: Here we will use slack

corner points.
Now it is very time consume to do such steps also using this method will be difficult if there are so many constraints. So we will go for simplex method.

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► For solving the following LPP by algebraic method, the feasible solutions are: $O(0,0), A(0,5/2), B(2,0), C(1,2)$.

$$\begin{aligned} &Maximize,\ z = x_1 + 3x_2 \\ &Subject\ to, \\ &\quad 2x_1 + x_2 \le 4 \\ &\quad x_1 + 2x_2 \le 5 \\ &\quad x_1, x_2 \ge 0. \end{aligned}$$

◻ The simplex method starts at the origin $O(0,0)$ where $z = 0$. and the logical question is whether an increase in nonbasic x_1 and/or x_2 above their current zero values can improve (increase) the value of z.

1.2.7 Simplex Method and Initial BFS (IBFS)

◻ The LPP's can be categorize into <u>two forms</u>:
► $a)$ All constraints are connected with '$\le$' sign and all $b_i \ge 0$: Here we will use slack variables to transform the inequalities into equation. Suppose we have m constraints with n variables as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ \vdots + \vdots + \cdots + \vdots &\le \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned}$$

Here we will introduce m slack variables $x_{n+1}, \cdots, x_{n+m}$ and the transformed equations are

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \quad + x_{n+1} \qquad\qquad &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad + x_{n+2} \qquad &= b_2\\
\vdots + \vdots + \cdots + \vdots + \cdots + \vdots \qquad\qquad &= \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + x_{n+m} &= b_m
\end{aligned}$$

The coefficient matrix is $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{pmatrix}_{m\times(n+m)}$

All slack vectors forms a basis matrix $B = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_m)$.

The BFS $\underset{\sim}{X}_B = \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is taken as initial BFS in solving the problem by simplex method.

► *b*) All constraints are of mixed type connected with signs '$\leq$', '$\geq$' and '$=$': Here we can adjust b_i such that $b_i \geq 0$. Suppose we have 4 constraints with 3 variables as follows:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &\leq 1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &\geq -3\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &\geq 2\\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 &= 4
\end{aligned}$$

The first step is to make all b_i's positive, so multiply 2^{nd} equation by (-1), we get

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &\leq 1\\
-a_{21}x_1 - a_{22}x_2 - a_{23}x_3 &\leq 3\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &\geq 2\\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 &= 4
\end{aligned}$$

Now we will introduce 2 slack and 1 surplus variables to transformed constraints into equation as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + x_4 \qquad\qquad &= 1\\
-a_{21}x_1 - a_{22}x_2 - a_{23}x_3 \quad + x_5 \qquad &= 3\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \qquad\qquad - x_6 &= 2\\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 \qquad\qquad &= 4
\end{aligned}$$

Now we will add two artificial variables x_7, x_8 to get a unit basis as follows:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + x_4 &= 1\\
-a_{21}x_1 - a_{22}x_2 - a_{23}x_3 \quad + x_5 &= 3\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \quad - x_6 + x_7 &= 2\\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 \quad + x_8 &= 4
\end{aligned}$$

The coefficient matrix is $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 & 0 & 0 \\ -a_{21} & -a_{22} & -a_{23} & 0 & 1 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & -1 & 1 & 0 \\ a_{41} & a_{42} & a_{43} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Here A contains a unit basis matrix $B = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4)$ corresponding to 4, 5, 7, 8 columns respectively. The fourth, fifth column vectors are slack vectors and seventh, eighth column vectors are artificial vectors. The BFS $\underset{\sim}{X}_B = \begin{pmatrix} x_4 \\ x_5 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$ is taken as <u>initial BFS</u> in solving the problem by simplex method. Also note that since surplus variable (x_6) is present we will use Charne's M or, Big-M method.

Theorem 1.7. *Fundamental Theorem of LPP: If a LPP optimize $z = \underset{\sim}{c}\underset{\sim}{x}$ subject to $A\underset{\sim}{x} = \underset{\sim}{b}$, $x \geq 0$ where $A_{m\times n}$ $(m < n)$ is coefficient matrix and $r(A) = m$, has an optimal solution then $\exists$ at least one BFS, which will be optimal.*

Proof. <u>Case I: Maximization Problem</u>: Let $x = [x_1, x_2, \dots x_n]$ be an optimal FS of the given maximization problem. Let $k(\leq n)$ components out of $x_1, x_2, \dots x_n$ be positive and remaining $(n-k)$ components are zero. We further make a assumption that the fast k components are positive and the last $(n-k)$ components are zero. So the optimal solution is $\hat{x} = [x_1, x_2, \dots x_k, 0, \dots, 0]$ and $\hat{z} = \widehat{\underset{\sim}{c}\underset{\sim}{x}} = \sum_{j=1}^{k} c_j x_j$.

If $a_1, a_2, \dots, a_k$ be the column vectors (i.e. optimal $A = [a_1, a_2, \dots, a_k, 0, \dots, 0]$) associated with the variables $x_1, x_2, \dots, x_k$ then the optimal solution will be BFS provided the vectors $a_1, a_2, \dots, a_k$ are LI which is possible only for $k \leq m$. If $a_1, a_2, \dots, a_k$ are not LI, then the solution is not a BFS.

Now, $\sum_{j=1}^{k} x_j a_j = b$, $x_j \geq 0$, also $a_1, a_2, \dots, a_k$ are not LI $\Rightarrow \sum_{j=1}^{k} \lambda_j a_j = 0$ with at least one $\lambda_j > 0$. .... (1).

Let, $\nu = \max_j \left(\frac{\lambda_j}{x_j}\right)$ is a positive quantity, so the solution set $\hat{x}' = [x'_1, x'_2, \dots, x'_k, 0, \dots, 0]$ where $x'_j = x_j - \frac{\lambda_j}{\nu} \geq 0$ for $j = 1, 2, \dots, k$ is also a FS which contains the maximum $(k-1)$ positive variables. Let $x'_k = 0$, then $\hat{x}' = [x'_1, x'_2, \dots, x'_{k-1}, 0, \dots, 0]$.

The value of the object function for the solution set $\hat{x}'$ is $\hat{z_1} = \sum_{j=1}^{k-1} c_j x'_j = \sum_{j=1}^{k} c_j x'_j$ $\boxed{\text{as } x'_k = 0}$ $= \sum_{j=1}^{k} c_j \left(x_j - \frac{\lambda_j}{\nu}\right) = \sum_{j=1}^{k} c_j x_j - \frac{1}{\nu}\sum_{j=1}^{k} c_j \lambda_j = \hat{z} - \frac{1}{\nu}\sum_{j=1}^{k} c_j \lambda_j$.

If $\sum_{j=1}^{k} c_j\lambda_j = 0$ [which is always the fact] then $\hat{z}_1 = \hat{z}$ and the solution set $\hat{x}'$ is also a optimal solution if the column vector corresponding to the variable $x'_1, x'_2, \ldots, x'_{k-1}$ are LI, then the solution is a BFS which is also optimal solution. If the column vector are not LI then repeating the process of reducing the number of non zero variables of a finite number of times ultimately an optimal solution is which is BFS and then the theorem is proved.
Now we are to prove that $\sum_{j=1}^{k} c_j\lambda_j = 0$.
If $\sum_{j=1}^{k} c_j\lambda_j \neq 0$ then $\sum_{j=1}^{k} c_j\lambda_j > 0$ or, < 0, now choose $\delta(\neq 0)$, such that $\sum_{j=1}^{k} c_j\lambda_j \geq 0$.
Hence $\sum_{j=1}^{k} c_jx_j + \delta\sum_{j=1}^{k} c_j\lambda_j > \sum_{j=1}^{k} c_jx_j \Rightarrow \sum_{j=1}^{k} c_j(x_j + \delta\lambda_j) > \hat{z}$.
Again from (1), $\sum_{j=1}^{k}(x_j + \delta\lambda_j)a_j = b$, it indicates that $x_j + \delta\lambda_j$ is a solution set of the system $Ax = b$. If $x_j + \delta\lambda_j \geq 0$ for particular values of δ (say δ^* always exists) then the solution set $x_j + \delta\lambda_j$ is a FS of $Ax = b$. Now it is clear that the set of values $(x_j + \delta\lambda_j)$ make the objective function greater than $\hat{z}$. Hence the assumption that for $\hat{x} = [x_1, x_2, \ldots, x_k, 0, \ldots, 0]$, the objective function z attains its maximum is not true. Hence $\sum_{j=1}^{k} c_j\lambda_j = 0$.
Case II: Minimization Problem: If for $\hat{x}$, the objective function attend its minimum $\hat{z}$ then also $\hat{z}_1 = \hat{z}$ where $\hat{z}_1$ is the value of the objective function for the solution set $\hat{x}'$. Proceeding similarly like the above case the theorem can be established in the problem of minimization also. □

■ Note: $max_j\left(\frac{-x_j}{\lambda_j}\right)_{(\lambda_j>0)} \leq \delta^* \leq min_j\left(\frac{-x_j}{\lambda_j}\right)_{(\lambda_j<0)}$ as for $\lambda_j > 0$ and $x_j + \delta\lambda_j \geq 0$ gives $\delta \geq \frac{-x_j}{\lambda_j}$ and for $\lambda_j < 0$ and $x_j + \delta\lambda_j \geq 0$ gives $\delta \leq \frac{-x_j}{\lambda_j}$ and for $\lambda_j = 0$, δ is unrestricted.

► The simplex method is based on the fundamental theorem of LPP. It starts from an extreme point, check the optimality conditions and move to the other adjacent extreme point.

■ Initial Simplex Table: Initial BFS i.e. IBFS: $\underset{\sim}{X}_B = \underset{\sim}{b}\ (\geq \underset{\sim}{0})$ and the initial value of the objective function is $z = \underset{\sim}{C}_B\underset{\sim}{X}_B = \underset{\sim}{C}_B\underset{\sim}{b}$, where $\underset{\sim}{C}_B$ is the coefficient vector of $\underset{\sim}{X}_B$ of z.

1.2.8 Simplex Method Steps

1. Compute IBFS. Usually we take origin as IBFS.
2. Check if IBFS satisfies the optimalily conditions or, not.
3. If the above step fails then we will move the next BFS and check the optimality condition.
4. If the above step fails then we repeat the above step.

Example 1.9.

> *Solve the LPP. described below*
> *Maximize,* $z = 5x_1 + 2x_2 + 2x_3$
> *Subject to,*
> $x_1 + 2x_2 - 2x_3 \leq 30$
> $x_1 + 3x_2 + x_3 \leq 36$
> $x_1, x_2, x_3 \geq 0.$

⇒ *The given LPP is a maximization problem with* $b_i \geq 0, \forall i$ *(i.e.* $30, 36 \geq 0$*) and the constraints has '* $\leq$ *' sign. Introducing the slack variables* x_4, x_5 *one in each constraint, we get the following converted equations.*

$$x_1 + 2x_2 - 2x_3 + x_4 \quad = 30$$
$$x_1 + 3x_2 + x_3 \quad + x_5 = 36$$

Here the coefficient matrix contain a unit basis. The adjusted objective function z *is given by: Maximize,* $z = 5x_1 + 2x_2 + 3x_3 + 0.x_4 + 0.x_5$.
Now we will construct the simplex table to solve the given LPP.

		c_j	5	2	2	0	0	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	a_5	*Min Ratio*
a_4	0	30	1	2	−2	1	0	*Basis* : a_4, a_5
a_5	0	36	1	3	1	0	1	
$z_j - c_j$		0	[-5]	−2	−2	0	0	$z_j = B_j a_{ij}$

		c_j	5	2	2	0	0	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	a_5	*Min Ratio*
a_4	0	30	(1)	2	−2	1	0	$30/1 = 30 \leftarrow$
a_5	0	36	1	3	1	0	1	$36/1 = 36$
$z_j - c_j$		0	[-5]	−2	−2	0	0	

		c_j	5	2	2	0	0	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	a_5	*Min Ratio*
a_4	0	30/1	1	2/1	−2/1	1/1	0/1	
a_5	0							
$z_j - c_j$								

Basis	B	c_j / b	a_1 (5)	a_2 (2)	a_3 (2)	a_4 (0)	a_5 (0)	*Min Ratio*
a_1	5	30	1	2	-2	1	0	*Change a_1 to*
a_5	0	6	(0)	1	3	-1	(1)	*BasisForm*
$z_j - c_j$								

Basis	B	c_j / b	a_1 (5)	a_2 (2)	a_3 (2)	a_4 (0)	a_5 (0)	*Min Ratio*
a_4	0	30	(1)	2	-2	1	0	$30/1 = 30 \leftarrow$
a_5	0	36	1	3	1	0	1	$36/1 = 36$
$z_j - c_j$		0	[-5]	-2	-2	0	0	
a_1	5	30	1	2	-2	1	0	×
a_5	0	6	0	1	(3)	-1	1	$6/3 = 2 \leftarrow$
$z_j - c_j$		150	0	8	[-12]	5	0	
a_1	5	34	1	8/3	0	1/3	2/3	
a_3	2	2	0	1/3	1	$-1/3$	1/3	
$z_j - c_j$		174	0	12	0	1	4	

As all $z_j - c_j > 0$, $\forall j$, the obtained result is an optimal solution and it is a BFS. Therefore, $max(z) = 174$ at $x_1 = 34, x_2 = 0, x_3 = 2$.

► A solution is said to be degenarate if at least one basic variable has the value 0.

1.2.9 Mathematical Form of Simplex

► Simplex Table (At any stage): Mathematical form

Basis	c_B	c_j / x_B	c_1 / a_1	c_2 / a_2	$\cdots$	c_k / a_k	$\cdots$	c_n / a_n	Min Ratio / $\frac{x_{B_i}}{y_{ik}}, y_{ik} > 0$
a_{B_1}	c_{B_1}	$x_{B_1} = y_{10}$	y_{11}	y_{12}	$\cdots$	y_{1k}	$\cdots$	y_{1n}	
a_{B_2}	c_{B_2}	$x_{B_2} = y_{20}$	y_{21}	y_{22}	$\cdots$	y_{2k}	$\cdots$	y_{2n}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
a_{B_m}	c_{B_m}	$x_{B_m} = y_{m0}$	y_{m1}	y_{m2}	$\cdots$	y_{mk}	$\cdots$	y_{mn}	
$z_j - c_j$		$z_B = c_B x_B$	$\sum c_{B_i} y_{i1} - c_1$	$\|\|$	$\cdots$	[$\sum c_{B_i} y_{ik} - c_k$]	$\cdots$	$\sum c_{B_i} y_{in} - c_n$	

Here $B_1, \cdots, B_m \in 1, \cdots, n$. Note that, at initial stage $a_i = y_i = (y_{1i}, y_{2i}, \cdots, y_{mi})$ but at any instance a_i and y_i are different.

▶ The mathematical form of an LPP after introducing slack and surplus variables is

$$\begin{array}{c} Optimize,\ z = \underline{c}\underline{x} \\ Subject\ to,\quad A_{m\times n}\underline{x} = \underline{b},\ \ \underline{x} \geq \underline{0} \end{array}$$

Here, $\underline{c}_{1\times n} = (c_1, \cdots, c_r, 0, \cdots, 0)$, $\underline{x}_{n\times 1} = (x_1, \cdots, x_r, x_{r+1}, \cdots, x_n)'$ with $x_{r+1}, \cdots, x_n$ are slack/ surplus variables. $A_{m\times n} = (a_1, \cdots, a_n)$ with an assumption that m are linearly independent i.e. A has m linearly independent columns, say $\beta_1, \cdots, \beta_m \in a_1, \cdots, a_n$. So the basis matrix $B = [\beta_1, \cdots, \beta_m] = [a_{B_1}, \cdots, a_{B_m}]$.

▶ Let $x_{B_1}, \cdots, x_{B_m}$ be the basic variables associated with the column vectors of B. Then the basic variable vector is $x_B = [x_{B_1}, \cdots, x_{B_m}]$.
▶ The solution set corresponding to the basic variables is $\boxed{x_B = B^{-1}\underline{b}}$. The solution x_B is BFS and B is known as <u>admissible basis</u>.
▶ $c_B = (c_{B_1}, \cdots, c_{B_m})$ is known as <u>associated cost vector</u> and they are the coefficients of $x_B = (x_{B_1}, \cdots, x_{B_m})$. Moreover, $z_B = c_B x_B = \sum c_{B_i} x_{B_i}$ is the value of objective function corresponding to the BFS x_B.
▶ As $\beta_1, \cdots, \beta_m$ are linearly independent, so $a_j = \sum_{i=1}^{m} \beta_i y_{ij} = By_j \Rightarrow y_j = B^{-1}a_j$.
▶ <u>Net evaluation</u> is $z_j - c_j = c_B y_j - c_j = \sum c_{B_i} y_{ij} - c_j = c_B B^{-1} a_j - c_j$.

[Do It Yourself] 1.24. *Show that the given LPP has degenarate solution*

$$\begin{array}{c} Maximize,\ z = x_1 - 3x_2 + 2x_3 \\ Subject\ to, \\ 3x_1 - x_2 + 2x_3 \leq 7 \\ -2x_1 + 4x_2 \leq 12 \\ -x_1 + 3x_2 + 8x_3 \leq 10 \\ x_1, x_2, x_3 \geq 0. \end{array}$$

Theorem 1.8. *The minimum value of the objective function z can be expressed as the negative of maximum of $(-z)$ i.e. $Min(z) = -Max(-z)$ with the same solution set.*

1.2.10 Charne's M-technique (Big-M Method)

Example 1.10.

$$\begin{array}{c} Solve\ the\ LPP.\ using\ Big\text{-}M\ method \\ Minimize,\ z = 4x_1 + 3x_2 \\ Subject\ to, \\ x_1 + 2x_2 \geq 8 \\ 3x_1 + 2x_2 \geq 12 \\ x_1, x_2 \geq 0. \end{array}$$

⇒ *The given LPP is a minimization problem with $b_i \geq 0$, $\forall i$ (i.e. $8, 12 \geq 0$) and the constraints has ' $\geq$ ' sign. Introducing the surplus variables x_3, x_4 one in each constraint, we get the following converted equations.*

$$\begin{aligned} x_1 + 2x_2 - x_3 \qquad &= 8 \\ 3x_1 + 2x_2 \qquad - x_4 &= 12 \end{aligned}$$

Here the coefficient matrix does not contain a unit basis matrix, so we will add two artificial variables x_5 and x_6 to the above system.

$$\begin{aligned} x_1 + 2x_2 - x_3 \qquad + x_5 \qquad &= 8 \\ 3x_1 + 2x_2 \qquad - x_4 \qquad + x_6 &= 12 \end{aligned}$$

Now the coefficient matrix contain a unit basis. The adjusted objective function z' is given by: Maximize, $z' = -4x_1 - 3x_2 + 0.x_3 + 0.x_4 - Mx_5 - Mx_6$.

		c_j	-4	-3	0	0	$-M$	$-M$	
Basis	*B*	b	a_1	a_2	a_3	a_4	a_5	a_6	*Min Ratio*
a_5	$-M$	8	1	(2)	-1	0	1	0	$8/2 = 4 \leftarrow$
a_6	$-M$	12	3	2	0	-1	0	1	$12/2 = 6$
$z_j - c_j$		$-20M$	$-4M + 4$	$-4M+3$	M	M	0	0	
a_2	-3	4	$1/2$	1	$-1/2$	0	$1/2$	0	$4/(1/2) = 8$
a_5	$-M$	4	(2)	0	1	-1	-1	1	$4/2 = 2 \leftarrow$
$z_j - c_j$		$-4M - 12$	$-2M+5/2$	0	$-M + 3/2$	M	$2M - 3/2$	0	
a_2	-3	3	0	1	$-3/4$	$1/4$	$3/4$	$-1/4$	
a_1	-4	2	1	0	$1/2$	$-1/2$	$-1/2$	$1/2$	
$z_j - c_j$		-17	0	0	$1/4$	$5/4$	$M - 9/4$	$M - 2$	

As all $z_j - c_j > 0$, $\forall j$, the obtained result is an optimal solution. Further, since no artificial vector is present in the final basis. Hence the optimal solution is a BFS and the maximum value of z' is -17 at $x_1 = 2, x_2 = 3$. Therefore, $min(z) = -max(z') = 17$ at $x_1 = 2, x_2 = 3$.

[Do It Yourself] 1.25. *Show that the given LPP has unbounded solution*

$$\begin{aligned} &\text{Maximize, } z = 3x_1 - x_2 \\ &\text{Subject to,} \\ &-x_1 + x_2 \geq 2 \\ &5x_1 - 2x_2 \geq 2 \\ &x_1, x_2 \geq 0. \end{aligned}$$

▶ A solution is said to be unbounded if for the lowest $z_j - c_j$, a_j's are negative.

[Do It Yourself] 1.26. *Show that the given LPP has unbounded solution*

$$\begin{aligned} &\text{Maximize, } z = 2x_1 + x_2 + 3x_3 \\ &\text{Subject to,} \\ &x_1 + x_2 + 2x_3 \le 5 \\ &2x_1 + 3x_2 + 4x_3 = 12 \\ &x_1, x_2, x_3 \ge 0. \end{aligned}$$

<u>*Hint*</u> : $z = x_1 + 2x_2 + x_3 + 0.x_4 - M.x_5$, $x_1 + x_2 + 2x_3 + x_4 + 0.x_5 = 10$, $2x_1 + 3x_2 + 4x_3 + 0.x_4 + x_5 = 12$. *[Ans: 8, (3,2,0)]*

[Do It Yourself] 1.27. *Show that the given LPP has more than one optimal solution*

$$\begin{aligned} &\text{Maximize, } z = x_1 + x_2 + 3x_3 \\ &\text{Subject to,} \\ &x_1 + x_2 + 3x_3 \le 10 \\ &x_1 + x_2 \le 5 \\ &x_1 \le 1 \\ &x_1, x_2, x_3 \ge 0. \end{aligned}$$

▶ An LPP has infinitely many solutions if $z_j - c_j \ge 0$ but $z_j - c_j = 0$ for the basic variables. [Ans: 10, (0,0,10/3), (5,0,5/3), their any convex combination]

Theorem 1.9. *If there is a tie on finding Min Ratio, then the solution is degenerate.*

□ To break the tie in selecting min ratio, we will use Charne's Perturbation Method.

▶ Suppose 3 ties with a hypothetical table below:

		c_j	5	2	2	0	0	0	0	
Basis	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	Min Ratio
a_4	0	30	3	2	−2	1	0	0	0	$30/3 = 10 \leftarrow$
a_5	0	20	2	3	1	0	1	0	0	$20/2 = 10 \leftarrow$
a_6	0	40	4	2	−1	2	0	1	0	$40/4 = 10 \leftarrow$
a_7	0	20	1	5	2	0	0	0	1	$20/1 = 20$
$z_j - c_j$		0	$\boxed{-5}$	−2	−2	0	0	0	0	

Now $\min\{\frac{old\ a_{ij}}{a_{1j}}\} = \min\{\frac{a_{41}}{3}, \frac{a_{42}}{2}, \frac{a_{43}}{4}\} = \min\{\frac{1}{3}, \frac{0}{2}, \frac{2}{4}\} = 0$ i.e., we will replace a_5.

▶ If we choose any one of them then it is possible to get the solution but number of iterations will be more.

▶ Big-M usually not used in computer programming as we do not know the exact value of M. For that reason, we use two-phase method.

1.3 Duality in LPP

□ The term 'dual' means double. In linear programming (LP), duality infer that each LPP can be analyzed in two different ways with equivalent solutions. Any LPP can be stated in another equivalent form based on the same data. The new LPP is called dual linear programming problem or, dual.

► Computational cost mainly depends upon the number of constraints rather than the number of variables. The idea of converting a primal problem to its dual version is to reduce the computational cost.

Theorem 1.10. *The optimal solution of a dual LPP is same as the optimal solution of the primal LPP.*

Theorem 1.11. *Dual of the dual is the primal LPP.*

1.3.1 Conversion of a Primal to it's Dual

Primal (Max, ≤)	*Dual*
Maximize, $z = c_1x_1 + \cdots + c_nx_n$	*Minimize*, $w = b_1y_1 + \cdots + b_my_m$
Subject to,	*Subject to,*
$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$	$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_n \geq c_1$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$	$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_n \geq c_2$
$\vdots \quad \vdots \quad \cdots \quad \vdots$	$\vdots \quad \vdots \quad \cdots \quad \vdots$
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$	$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n$
With $x_j \geq 0,\ j = 1(1)n$.	With $y_i \geq 0,\ i = 1(1)m$.

► If the primal is not in the standard form, we will convert it into the standard form. Also if there is an '=' in the primal constraints, we write it into two parts '≤' and '≥'.

► If the primal has feasible solution but the dual has not feasible solution then primal objective function is unbounded.

► If the dual has feasible solution but the primal has not feasible solution then dual objective function is unbounded.

► If both the primal and dual has not a feasible solution then the solution does not exists.

Example 1.11. *Write down the dual of the LPP. Also verify that dual of dual is primal.*

$$
\begin{aligned}
&\textit{Minimize},\ z = x_1 + 3x_3 - 2x_4\\
&\textit{Subject to,}\\
&x_1 + 7x_2 + 3x_3 \leq 10\\
&x_1 - 3x_4 \geq 6\\
&x_1 - x_2 + x_3 + 5x_4 = 5\\
&x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$

⇒ *Since the LPP is not in the standard form, we can write it into the standard form below*

$$\begin{array}{c} \text{Maximize, } z* = -x_1 - 3x_3 + 2x_4 \\ \text{Subject to,} \\ x_1 + 7x_2 + 3x_3 \leq 10 \\ -x_1 + 3x_4 \leq -6 \\ x_1 - x_2 + x_3 + 5x_4 \leq 5 \\ -x_1 + x_2 - x_3 - 5x_4 \leq -5 \\ x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

Therefore, the corresponding dual problem is

$$\begin{array}{c} \text{Minimize, } w = 10w_1 - 6w_2 + 5w_3 - 5w_4 \\ \text{Subject to,} \\ w_1 - w_2 + w_3 - w_4 \geq -1 \\ 7w_1 - w_3 + w_4 \geq 0 \\ 3w_1 + w_3 - w_4 \geq -3 \\ 3w_2 + 5w_3 - 5w_4 \geq 2 \\ w_1, w_2, w_3, w_4 \geq 0. \end{array}$$

Again, to find the dual of the above problem (i.e. dual of dual), we need to write it into the standard form i.e.

$$\begin{array}{c} \text{Maximize, } w* = -10w_1 + 6w_2 - 5w_3 + 5w_4 \\ \text{Subject to,} \\ -w_1 + w_2 - w_3 + w_4 \leq 1 \\ -7w_1 + w_3 - w_4 \leq 0 \\ -3w_1 - w_3 + w_4 \leq 3 \\ -3w_2 - 5w_3 + 5w_4 \leq -2 \\ w_1, w_2, w_3, w_4 \geq 0. \end{array}$$

Therefore, the corresponding dual problem is

$$\begin{array}{c} \text{Minimize, } v = v_1 + 3v_3 - 2v_4 \\ \text{Subject to,} \\ -v_1 - 7v_2 - 3v_3 \geq -10 \\ v_1 - 3v_4 \geq 6 \\ -v_1 + v_2 - v_3 - 5v_4 \geq -5 \\ v_1 - v_2 + v_3 + 5v_4 \geq 5 \\ w_1, w_2, w_3, w_4 \geq 0. \end{array}$$

$$\begin{array}{c} \text{After rewriting,} \\ \text{Minimize, } v = v_1 + 3v_3 - 2v_4 \\ \text{Subject to,} \\ v_1 + 7v_2 + 3v_3 \leq 10 \\ v_1 - 3v_4 \geq 6 \\ v_1 - v_2 + v_3 + 5v_4 = 5 \\ w_1, w_2, w_3, w_4 \geq 0. \end{array}$$

[Do It Yourself] 1.28. *Write down the dual of the LPP. Also verify that dual of dual is primal.*

$$\begin{array}{c} \text{Minimize, } z = x_1 + 3x_3 - 2x_4 \\ \text{Subject to,} \\ x_1 + 7x_2 + 3x_3 \leq 10 \\ x_1 - 3x_4 \geq 6 \\ x_1 - x_2 + x_3 + 5x_4 = 5 \\ x_1, x_2, x_3 \geq 0, \ x_4 \text{ is unrestricted in sign} \end{array}$$

[*Hint* : $x_4 = x_4' - x_4''$, where, $x_4', x_4'' \geq 0$]

Theorem 1.12. *Fundamental Duality Theorem: If either the primal or, the dual has a finite optimal solution, then the other problem will also have a finite optimal solution.*

Theorem 1.13. *If either the primal or, the dual has unbounded solution, then the other problem will have no feasible solution..*

Theorem 1.14. *If either the primal problem has feasible solutions and the dual has no feasible solotion then the primal problem is said to have unbounded solution and vice versa.*

Theorem 1.15. *If the k^{th} constraint of a primal be an equation, the k^{th} dual variable will be unrestricted in sign.*

Theorem 1.16. *If any variable of the primal problem be unrestricted in sign, the corresponding dual constraint will be strictly an equation.*

Question 1.1. *Write down the advantages of duality.*

★ *The are various advantages of Duality:*

1. *It's easy to solve the dual with a less number of constraints compare to its primal. Since the number of constraints usually equals the number of iterations required to solve the problem.*
2. *This avoids the necessity for adding surplus or artificial variables and solves the problem quickly (i.e. primal-dual method).*
3. *The dual variables provide an important economic interpretation of the final solution of an LP problem.*
4. *It is quite useful when investigating changes in the parameters of an LPP (i.e. sensitivity or, post-optimality analysis).*
5. *Duality is used to solve an LPP by the simplex method in which the initial solution is infeasible (i.e. dual simplex method).*

subject to,

$$\begin{array}{rcl}
a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m & \geq & c_1 \\
a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m & \geq & c_2 \\
\vdots \ + \ \vdots \ + \cdots + \ \vdots & \leq & \vdots \\
a_{1k}w_1 + a_{2k}w_2 + \cdots + a_{mk}w_m & \geq & c_k \\
-a_{1k}w_1 - a_{2k}w_2 - \cdots - a_{mk}w_m & \geq & -c_k \\
\vdots \ + \ \vdots \ + \cdots + \ \vdots & \leq & \vdots \\
a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m & \geq & c_n
\end{array}$$

with $w_1, w_2, \cdots, w_m \geq 0$.
The two constraints with right hand side $c_k, -c_k$ are equivalent to an equation $a_{1k}w_1 + a_{2k}w_2 + \cdots + a_{mk}w_m = c_k$. Hence the k^{th} dual constraint is an equation. □

Question 1.1. *Write down the advantages of duality.*

★ *The are various advantages of Duality:*

1. *It's easy to solve the dual with a less number of constraints compare to its primal. Since the number of constraints usually equals the number of iterations required to solve the problem.*
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4. *It is quite useful when investigating changes in the parameters of an LPP (i.e. sensitivity or, post-optimality analysis).*
5. *Duality is used to solve an LPP by the simplex method in which the initial solution is infeasible (i.e. dual simplex method).*

1.3.2 Dual Simplex Method

◘ Here the dual problem is solved instead of the primal as sometimes solving dual is easier (due to its omission of the artificial variables) than primal. The name Dual Simplex Method as we use simplex method on dual problems.
► It's not always possible to solve an LPP by using dual simplex method.
◘ In simplex method we start from IBFS (i.e. non-optimal but feasible $[b_j > 0]$ solution) and move towards the optimal finite BFS i.e. for maximization we will iterate the process till we get all $z_j - c_j \geq 0$.

◘ In dual simplex method we start from optimal (i.e. all $z_j - c_j \geq 0$) but infeasible (i.e. $b_j < 0$) solution and move towards the feasible solution (i.e. $b_j > 0$) with maintained the optimality (i.e. all $z_j - c_j \geq 0$) in each iteration.

▶ It will be a maximization problem.

▶ Dual simplex method can't be applicable if the criteria $z_j - c_j \geq 0$ for all j doesn't hold initially.

Example 1.12. *Solve the dual problem by using dual simplex method*

$$\begin{aligned} &\text{Minimize, } z = 4x_1 + 3x_2 \\ &\text{Subject to,} \\ &x_1 + 2x_2 \geq 8 \\ &3x_1 + 2x_2 \geq 12 \\ &x_1, x_2 \geq 0. \end{aligned}$$

⇒ *The given dual is a minimization problem, so we will convert it into canonical form as:*

$$\begin{aligned} &\text{Maximize, } z^* = -4x_1 - 3x_2 \\ &\text{Subject to,} \\ &-x_1 - 2x_2 + x_3 = -8 \\ &-3x_1 - 2x_2 + x_4 = -12 \\ &x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Here the coefficient matrix contain a unit basis. The adjusted objective function z is given by: Maximize, $z^* = -4x_1 - 3x_2 + 0.x_3 + 0.x_4$.
Now we will construct the simplex table to solve the given LPP.

		c_j	-4	-3	0	0	
Basis	*B*	b	a_1	a_2	a_3	a_4	*Min Ratio*
a_3	0	-8	-1	-2	1	0	*NA*
a_4	0	-12	-3	-2	0	1	*NA*
$z_j - c_j$		0	4	3	0	0	

Here, $z_j - c_j \geq 0, \forall j$, *but* $b_j < 0$, *so the optimal solution is infeasible. Here min ratio will not work and we will find 'Min* $\{b_j : b_j < 0\}$*' to select departing vector.*

		c_j	-4	-3	0	0	
Basis	*B*	b	a_1	a_2	a_3	a_4	*Min* $\{b_j : b_j < 0\}$
a_3	0	-8	-1	-2	1	0	-8
a_4	0	-12	-3	-2	0	1	$-12 \leftarrow$
$z_j - c_j$		0	4	3	0	0	

To find the entering vector with respect to −12, we will find $\max\{\frac{z_j-c_j}{y_{2j}} : y_{2j} < 0\}$ *i.e.* $\max\{\frac{4}{-3}, \frac{3}{-2}\} = -\frac{4}{3}$ *i.e. the entering element is 4.*

		c_j	−4	−3	0	0	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	*Min* $\{b_j : b_j < 0\}$
a_3	0	−8	−1	−2	1	0	−8
a_4	0	−12	(−3)	−2	0	1	−12 ←
$z_j - c_j$		0	[4]	3	0	0	

Therefore the simplex table is

		c_j	−4	−3	0	0	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	*Min* $\{b_j : b_j < 0\}$
a_3	0	−8	−1	−2	1	0	−8
a_4	0	−12	(−3)	−2	0	1	−12 ←
$z_j - c_j$		0	[4]	3	0	0	
a_3	0	−4	0	$(-\frac{4}{3})$	1	$-\frac{1}{3}$	←
a_1	−4	4	1	$\frac{2}{3}$	0	$-\frac{1}{3}$	×
$z_j - c_j$		−16	0	[1/3]	0	$-\frac{4}{3}$	$\max\{\frac{1/3}{-4/3}, \frac{4/3}{-1/3}\} = -\frac{1}{4}$
a_2	−3	3	0	1	$-\frac{3}{4}$	$-\frac{1}{4}$	
a_1	−4	2	1	0	$\frac{1}{2}$	$-\frac{1}{6}$	
$z_j - c_j$		−17	0	0	$\frac{1}{4}$	$\frac{17}{12}$	

As all $z_j - c_j > 0$, $\forall j$ *and* $b_j \geq 0$, *the obtained result is an optimal solution and it is a BFS. Therefore,* $max(z^*) = -17$ *at* $x_1 = 2, x_2 = 3$ *i.e.* $min(z) = 17$ *at* $x_1 = 2, x_2 = 3$.

[Do It Yourself] 1.29. *Solve the LPP by using dual simplex method*

$$\begin{aligned} &Minimize,\ z = 5x_1 + 6x_2 \\ &Subject\ to, \\ &\quad x_1 + x_2 \geq 2 \\ &\quad 4x_1 + x_2 \geq 4 \\ &\quad x_1, x_2 \geq 0. \end{aligned}$$

Example 1.13. *Solve the dual problem by using dual simplex method*

$$\begin{aligned}
&\textit{Minimize}, \; z = 4x_1 + 2x_2 \\
&\textit{Subject to,} \\
&3x_1 + x_2 \geq 27 \\
&x_1 + x_2 \geq 21 \\
&x_1 + 2x_2 \geq 30 \\
&x_1, x_2 \geq 0.
\end{aligned}$$

⇒ *The given dual is a minimization problem, so we will convert it into canonical form as:*

$$\begin{aligned}
&\textit{Maximize}, \; z^* = -4x_1 - 2x_2 \\
&\textit{Subject to,} \\
&-3x_1 - x_2 + x_3 = -27 \\
&-x_1 - x_2 + x_4 = -21 \\
&-x_1 - 2x_2 + x_5 = -30 \\
&x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}$$

Here the coefficient matrix contain a unit basis. The adjusted objective function z is given by: Maximize, $z^* = -4x_1 - 2x_2 + 0.x_3 + 0.x_4$.
Now we will construct the simplex table to solve the given LPP.

		c_j	-4	-2	0	0	0	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	a_5	*Min Ratio*
a_3	0	-27	-3	-1	1	0	0	*NA*
a_4	0	-21	-1	-1	0	1	0	*NA*
a_5	0	-30	-1	-2	0	0	1	*NA*
$z_j - c_j$		0	4	2	0	0	0	

Here, $z_j - c_j \geq 0$, $\forall j$, *but* $b_j < 0$, *so the optimal solution is infeasible. Here min ratio will not work and we will find 'Min* $\{b_j : b_j < 0\}$*' to select departing vector.*

Therefore the simplex table is

		c_j	-4	-2	0	0	0	
Basis	B	b	a_1	a_2	a_3	a_4	a_5	$Min\ \{b_j :\ b_j < 0\}$
a_3	0	-27	-3	-1	1	0	0	-27
a_4	0	-21	-1	-1	0	1	0	-21
a_5	0	-30	-1	(-2)	0	0	1	$-30 \leftarrow$
$z_j - c_j$		0	4	$\boxed{2}$	0	0	0	$max\{\frac{4}{-1}, \frac{2}{-2}\} = -1$
a_3	0	-12	$(-5/2)$	0	1	0	$-1/2$	$\leftarrow$
a_4	0	-6	$-1/2$	0	0	1	$-1/2$	
a_2	-2	15	$1/2$	1	0	0	$-1/2$	
$z_j - c_j$		-30	$\boxed{3}$	0	0	0	1	$max\{\frac{3}{-5/2}, \frac{1}{-1/2}\} = -6/5$
a_1	-4	$24/5$	1	0	$-2/5$	0	$1/5$	
a_4	0	$-18/5$	0	0	$-1/5$	1	$(-2/5)$	$\leftarrow$
a_2	-2	$63/5$	0	1	$1/5$	0	$-3/5$	
$z_j - c_j$		$-222/5$	0	0	$6/5$	0	$\boxed{2/5}$	$max\{\frac{6/5}{-1/5}, \frac{2/5}{-2/5}\} = -1$
a_1	-4	3	1	0	$-1/2$	$1/2$	0	
a_5	0	9	0	0	$1/2$	$-5/2$	1	
a_2	-2	18	0	1	$1/2$	$-3/2$	0	
$z_j - c_j$		-48	0	0	1	1	0	

As all $z_j - c_j > 0$, $\forall j$ and $b_j \geq 0$, the obtained result is an optimal solution and it is a BFS. Therefore, $max(z^*) = -48$ at $x_1 = 3, x_2 = 18$ i.e. $min(z) = 48$ at $x_1 = 3, x_2 = 18$.

[Do It Yourself] 1.30. *Solve the LPP (if possible) with dual simplex method [CU 80]*

$$\begin{aligned} &Maximize,\ z = -2x_1 - 3x_2 - x_3 \\ &\quad Subject\ to, \\ &\qquad 2x_1 + x_2 + 2x_3 \geq 3 \\ &\qquad 3x_1 + 2x_2 + x_3 \geq 4 \\ &\qquad x_1, x_2, x_3 \geq 0. \end{aligned}$$

[Do It Yourself] 1.31. *Solve the LPP (if possible) with dual simplex method [KU 86]*

$$\begin{array}{l} \textit{Maximize, } z = -3x_1 - 2x_2 \\ \textit{Subject to,} \\ \quad x_1 + x_2 \geq 1 \\ \quad x_1 + x_2 \leq 7 \\ \quad x_1 + x_2 \geq 10 \\ \quad x_2 \leq 3 \\ \quad x_1, x_2 \geq 0. \end{array}$$

[Do It Yourself] 1.32. *Solve the LPP (if possible) with dual simplex method [MU 85]*

$$\begin{array}{l} \textit{Maximize, } z = 6x_1 + 7x_2 + 3x_3 + 5x_4 \\ \textit{Subject to,} \\ \quad 5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12 \\ \quad x_2 + 5x_3 - 6x_4 \geq 10 \\ \quad 2x_1 + 5x_2 + x_3 + x_4 \geq 8 \\ \quad x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

Question 1.2. *What are the differences between simplex, dual simplex and revised simplex methods.*

★ *Simplex*: *Simplex method starts with a non-optimal but feasible solution. It maintains the feasibility during successive iterations. It follows the shortest route to reach the optimal solution from the starting point. It is less efficient and accurate as compared to revised simplex.*

Dual Simplex: *Dual simplex method starts with an optimal but infeasible solution. It maintains the optimality during successive iterations. It starts with optimal basis and finds feasible basis. It can handle bounds efficiently.*

Revised Simplex: *Revised simplex is an improvement over simplex method. It is computationally more efficient and accurate. It clearly comprehends in case of large LPP. Instead of maintaining a tableau unlike Simplex method, it maintains a representation of a basis of the matrix representing the constraints. It is more efficient and accurate as compared to simplex.*

1.3.3 Artificial Constraints Method

□ Dual simplex method is applicable if $z_j - c_j \geq 0$. However, if $z_j - c_j < 0$ *for at least one* j, we can't use dual simplex method, so we will use artificial constraints method. Here we will introduce a new constraint: $\sum_j x_j \leq M$, where $M(> 0)$ is large. Introducing a slack

variable x_M, we have $\sum_j x_j + x_M = M$.
► Suppose we find: $max\ |z_j - c_j|$ at $j = \alpha$, so replace x_α as $x_\alpha = M - (x_M + \sum_{j \neq \alpha} x_j)$.

Example 1.14.

Solve the LPP by dual simplex method

$$\text{Maximize, } z = 3x_1 + 2x_2$$
Subject to,
$$2x_1 + x_2 \leq 5$$
$$x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

⇒ *The given LPP is a maximization problem, so we will convert it into canonical form as:*

$$\text{Maximize, } z = 3x_1 + 2x_2$$
Subject to,
$$2x_1 + x_2 + x_3 = 5$$
$$x_1 + x_2 + x_4 = 3$$

Here the coefficient matrix contain a unit basis. The adjusted objective function z is given by: Maximize, $z = 3x_1 + 2x_2 + 0.x_3 + 0.x_4$.
Now we will construct the simplex table to solve the given LPP.

		c_j	3	2	0	0	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	*Min Ratio*
a_2	0	5	2	1	1	0	*NA*
a_3	0	3	1	1	0	1	*NA*
$z_j - c_j$		0	−3	−2	0	0	$z_j = B_j a_{ij}$

As all $z_j - c_j < 0$, $j = 1, 2$, *so we can't apply dual simplex method. Therefore, we will use artificial constraints method. Here we will introduce a new constraint:* $x_1 + x_2 \leq M$, *where* $M(> 0)$ *is large. Introducing a slack variable* x_M, *we have* $x_1 + x_2 + x_M = M$. *Now,* $\max\{|z_1 - c_1|, |z_2 - c_2|\} = 3$ *corresponding to* a_1. *So, we replace* x_1 *by* $x_1 = M - x_2 - x_M$.

Now, the above canonical reduces as:

$$\text{Maximize, } z = 3M - x_2 - 3x_M$$
Subject to,
$$-x_2 + x_3 - 2x_M = 5 - 2M$$
$$x_4 - x_M = 3 - M$$
$$x_1 + x_2 + x_M = M$$

		c_j	0	−1	0	0	−3	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	a_M	*Min* $\{b_j : b_j < 0\}$
a_3	0	$5-2M$	0	−1	1	0	−2	←
a_4	0	$3-M$	0	0	0	1	−1	
a_1	0	M	1	1	0	0	1	
z_j-c_j		0	0	1	0	0	3	

To find the entering vector with respect to $5-2M$, *we will find* $\max\{\frac{z_j-c_j}{y_{1j}} : y_{1j} < 0\}$ *i.e.* $\max\{\frac{1}{-1}, \frac{3}{-2}\} = -1$ *i.e. the entering element is* -1.

		c_j	0	−1	0	0	−3	
Basis	*B*	*b*	a_1	a_2	a_3	a_4	a_M	*Min* $\{b_j : b_j < 0\}$
a_3	0	$5-2M$	0	(−1)	1	0	−2	←
a_4	0	$3-M$	0	0	0	1	−1	
a_1	0	M	1	1	0	0	1	
z_j-c_j		0	0	$\boxed{1}$	0	0	3	

Therefore the simplex table is

		c_j	0	−1	0	0	−3	
Basis	c_B	*b*	a_1	a_2	a_3	a_4	a_M	*Min* $\{b_j : b_j < 0\}$
a_3	0	$5-2M$	0	(−1)	1	0	−2	←
a_4	0	$3-M$	0	0	0	1	−1	
a_1	0	M	1	1	0	0	1	
z_j-c_j		0	0	$\boxed{1}$	0	0	3	
a_2	−1	$2M-5$	0	1	−1	0	2	
a_4	0	$-M+3$	0	0	0	1	(−1)	←
a_1	0	$-M+5$	1	0	1	0	1	
z_j-c_j		$-2M+5$	0	0	1	0	$\boxed{1}$	
a_2	−1	1	0	1	−1	2	0	
a_M	−3	$M-3$	0	0	0	−1	1	←
a_1	0	2	1	0	1	−1	0	
z_j-c_j		$-3M+8$	0	0	1	1	0	

As all $z_j - c_j \geq 0, \ \forall j$ *and* $b_j \geq 0, \ \forall j$, *so the obtained result is an optimal solution and it is a BFS. Therefore,* $max(z) = 8$ *at* $x_1 = 2, x_2 = 1$.

[Do It Yourself] 1.36. *Using dual simplex with artificial constraints methods to solve the following LPP.*

$$\begin{array}{c} Maximize,\ z = 4x_1 - 2x_2 + 2x_3 \\ Subject\ to, \\ 2x_1 + 3x_2 - 5x_3 \geq 4 \\ -x_1 + 9x_2 - 2x_3 \geq 2 \\ 4x_1 + 6x_2 + 2x_3 \geq 7 \\ x_1, x_2, x_3 \geq 0. \end{array}$$

[Do It Yourself] 1.37. *Using dual simplex with artificial constraints methods to solve the following LPP.*

$$\begin{array}{c} Maximize,\ z = 4x_3 \\ Subject\ to, \\ -x_1 + 2x_2 - 2x_3 \geq 16 \\ -x_1 + x_2 + x_3 \leq 8 \\ 2x_1 - 1x_2 + 4x_3 \leq 20 \\ x_1, x_2, x_3 \geq 0. \end{array}$$

1.4 Post Optimality Analysis

□ The idea of post optimality analysis (or, sensitivity analysis) is to infer about the solution (without solving it) of the LPP if there is a slight change in the parameter.

► If slight changes are made in the parameters or the structure of a given LPP after its optimum solution has been attained. An analysis of such post-optimal problems is known as post-optimality analysis.

► Suppose we have already obtained an optimal solution of an LPP and later we have found that there was some mistake in the input paramters. So we can correct this into two ways: i) Solve the LPP again with correction, ii) Use some method in a way to save the previously done computational effort.

► Usually we will choose the second one. So the aim of post optimality analysis is to perform some additional computation on the previous problem and obtain the new optimal solution.

□ The mathematical form of an LPP is

$$\begin{array}{c} Optimize,\ z = \underset{\sim}{c}\underset{\sim}{x} \\ Subject\ to,\quad A\underset{\sim}{x}(\leq = \geq)\underset{\sim}{b},\ \ \underset{\sim}{x} \geq \underset{\sim}{0} \end{array}$$

► Now there may be changes associated to the parameters: $\underset{\sim}{c}, A, \underset{\sim}{b}$. Furthermore, there may be changes related to the number of variables, number of constraints.

► After change in the given LPP, there may be possiblity of the following: optimal solution will remain unchanged (i.e. objective function and variable values), Basic variables in the final table may be same but with different values. the overall solution changes.

1.4.1 Change in the Cost Vector 'c'

The mathematical form of an LPP is

$$\boxed{\begin{array}{c} Maximize,\ z = \underset{\sim}{c}\underset{\sim}{x} \\ Subject\ to,\quad A\underset{\sim}{x} = \underset{\sim}{b},\ \ \underset{\sim}{x} \geq \underset{\sim}{0} \end{array}}$$

Suppose we change a component (say c_k) of cost vector $\underset{\sim}{c} = (c_1, \cdots, c_n)$ as $c_k^* = c_k + \Delta c_k$.
We know that, the basic solution is $x_B = B^{-1}\underset{\sim}{b}$, where B is a nonsingular basis matrix. Therefore, it implies that the current solution x_B will remain basic feasible.
Again, the value of $z_j - c_j = c_B y_j - c_j$, $j = 1(1)n$.
► If $\boxed{c_k \notin c_B}$, then we have $z_j^* - c_j^* = z_j - c_j$, $\forall j (\neq k) = 1(1)n$ and $z_k^* - c_k^* = z_k - c_k - \Delta c_k$. Then the current solution x_B remains optimum if $z_k^* - c_k^* \geq 0 \Rightarrow \Delta c_k \leq z_k - c_k$.

► If $\boxed{c_k \in c_B}$ then $z_j^* - c_j^* = c_B y_j - c_j\ (\forall j \neq k) = \sum_{i(\neq k)=1}^{m} c_{B_i} y_{ij} + (c_{B_k} + \Delta c_{B_k}) y_{kj} - c_j = \sum_{i=1}^{m} c_{B_i} y_{ij} + \Delta c_{B_k} y_{kj} - c_j = z_j - c_j + \Delta c_k y_{kj}$.
Also, $z_k^* - c_k^* = z_k + \Delta c_k y_{kk} - c_k - \Delta c_k = z_k + \Delta c_k - c_k - \Delta c_k$ ($y_{kk} = 1$ as basis)$= z_k - c_k$.
Then the current solution x_B remains optimum if $z_j^* - c_j^* \geq 0 \Rightarrow \Delta c_k \geq -\frac{z_j - c_j}{y_{kj} > 0}$ and $\Delta c_k \leq -\frac{z_j - c_j}{y_{kj} < 0}$.
Using the above two relations, we get the value of Δc_k lies between $\left[\max_j\{-\frac{z_j - c_j}{y_{kj} > 0}\}, \min_j\{-\frac{z_j - c_j}{y_{kj} < 0}\}\right]$ or, not. If it lies within the mentioned range, then the new BFS x_B^* remain optimum and the value of optimal cost will be changed by an amount $\Delta c_k x_k$.

1.4.2 Change in the Requirement Vector 'b'

The mathematical form of an LPP is

$$\boxed{\begin{array}{c} Maximize,\ z = \underset{\sim}{c}\underset{\sim}{x} \\ Subject\ to,\quad A\underset{\sim}{x} = \underset{\sim}{b},\ \ \underset{\sim}{x} \geq \underset{\sim}{0} \end{array}}$$

Suppose we change a component (say b_k) of requirement vector $\underset{\sim}{b} = (b_1, \cdots, b_m)'$ as $b_k^* = b_k + \Delta b_k$. So that, $\underset{\sim}{b}^* = (b_1, \cdots, b_k + \Delta b_k, \cdots, b_m)'$
We know that, the basic solution is $x_B = B^{-1}\underset{\sim}{b}$, where B is a nonsingular basis matrix. Therefore, it implies that the optimal solution will change and the new solution is $x_B^* = B^{-1}\underset{\sim}{b}^* = B^{-1}\underset{\sim}{b} + B^{-1}(0, \cdots, \Delta b_k, \cdots, 0)' = x_B + \Delta b_k \underset{\sim}{\beta}_k \Rightarrow x_{B_i}^* = x_{B_i} + \Delta b_k\, b_{ik}$.

Here $\underset{\sim}{\beta}_k$ is the k^{th} column of B^{-1} and b_{ik} is the element of B^{-1} corresponding to i^{th} row and k^{th} column.
Now, the new basic solution x_B^* remains feasible if $x_B^* \geq 0 \Rightarrow x_{B_i}^* \geq 0$, $i = 1(1)m$.
It implies, $x_{B_i} + \Delta b_k\ b_{ik} \geq 0 \Rightarrow \Delta b_k \geq -\frac{x_{B_i}}{b_{ik}>0}$ and $\Delta b_k \leq -\frac{x_{B_i}}{b_{ik}<0}$.
Using the above two relations, we get the value of Δb_k lies between $\left[\max_i\{-\frac{x_{B_i}}{b_{ik}>0}\}, \min_i\{-\frac{x_{B_i}}{b_{ik}<0}\}\right]$ or, not. If it lies within the mentioned range, then the new BFS x_B^* is feasible and again its optimum as $z_j - c_j = c_B y_j - c_j$ does not depend on $\underset{\sim}{b}$. Furthermore, the value of optimal cost will be changed by an amount $\sum_{i=1}^{m} c_{B_i}(\Delta b_k)\ b_{ik}$.

1.4.3 Change in the Coefficient Matrix 'A'

The mathematical form of an LPP is

$$\boxed{\begin{array}{c} Maximize,\ z = \underset{\sim}{c}\underset{\sim}{x} \\ Subject\ to,\quad A_{m\times n}\ \underset{\sim}{x} = \underset{\sim}{b},\ \ \underset{\sim}{x} \geq \underset{\sim}{0} \end{array}}$$

Here, $Rank(A) = m$, now suppose we change an element corresponding to j^{th} row and k^{th} column (say a_{jk}) as $a_{jk}^* = a_{jk} + \Delta a_{jk}$. However, for the sake of simplicity, we will assume all the changes lie in the same k^{th} column and new column is a_k^* instead of the old column a_k i.e. $a_k^* = a_k + (0, \cdots, \Delta a_{jk}, \cdots, 0)'_{m\times 1}$.

► If $\boxed{a_k \notin B}$, we know that, the basic solution is $x_B = B^{-1}\underset{\sim}{b}$, where $B_{m\times m}$ is a nonsingular basis matrix. Therefore, the current solution x_B will remain basic feasible.
Now the current solution x_B remains optimum if $z_k^* - c_k \geq 0 \Rightarrow c_B y_k^* - c_k \geq 0 \Rightarrow c_B B^{-1} a_k^* - c_k \geq 0 \Rightarrow c_B B^{-1} a_k + c_B \underset{\sim}{\beta}_j \Delta a_{jk} - c_k \geq 0 \Rightarrow z_k - c_k + c_B \underset{\sim}{\beta}_j \Delta a_{jk} \geq 0$.
Here $\underset{\sim}{\beta}_j$ is the j^{th} column of B^{-1}.
Now, the above relation implies, $\Delta a_{jk} \geq -\frac{z_k - c_k}{c_B \underset{\sim}{\beta}_j(>0)}$ and $\Delta a_{jk} \leq -\frac{z_k - c_k}{c_B \underset{\sim}{\beta}_j(<0)}$.
Using the above two relations, we get the value of Δa_{jk} lies between $\left[-\frac{z_k - c_k}{c_B \underset{\sim}{\beta}_j(>0)}, -\frac{z_k - c_k}{c_B \underset{\sim}{\beta}_j(<0)}\right]$ (also $[-\infty, \infty]$ if $c_B \underset{\sim}{\beta}_j = 0$) or, not. If it lies within the mentioned range, then the new BFS x_B^* is feasible and optimum.

► If $\boxed{a_k \in B}$, we know that, the basic solution is $x_B = B^{-1}\underset{\sim}{b}$, where $B_{m\times m}$ is a nonsingular basis matrix. So the feasibility of the current solution may be changed.
Let $B = (\alpha_{ij}) = (\alpha_1, \cdots, \alpha_k, \cdots, \alpha_m)$ and $B^{-1} = (\beta_{ij}) = (\beta_1, \cdots, \beta_k, \cdots, \beta_m)$.
Here, α_i, β_i are the column vectors of B and B^{-1} respectively.
As $a_k \in B$, so any change in a_{jk} will affect only k^{th} column of B i.e. α_k.
Now, $B^* = (\alpha_1, \cdots, \alpha_k^*, \cdots, \alpha_m)$, with $\alpha_k^* = (\alpha_{1k}, \cdots, \alpha_{jk} + \Delta a_{jk}, \cdots, \alpha_{mk})'$.
Again, $B^{-1}B = I \Rightarrow B^{-1}(\alpha_1, \cdots, \alpha_k, \cdots, \alpha_m) = I \Rightarrow B^{-1}\alpha_i = e_i$, $i = 1(1)m$.
So, $B^{-1}B^* = B^{-1}(\alpha_1, \cdots, \alpha_k^*, \cdots, \alpha_m) \Rightarrow B^{-1}\alpha_k^* = B^{-1}[\alpha_k + (0, \cdots, \Delta a_{jk}, \cdots, 0)'] = e_k + \beta_j \Delta a_{jk} = \mu$ (say) $\Rightarrow B^{-1}B^* = (e_1, \cdots, e_{k-1}, \mu, e_{k+1}, \cdots, e_m)$.

Chapter 2

Transportation & Assignment problem

2.1 Transportation Problem

□ The transportation problem (TP) is a special type of linear programming problem where the objective is to minimise the cost of distributing a product from a number of sources or origins to a number of destinations. Because of its special structure the usual simplex method is not suitable for solving transportation problems. These problems require a special method of solution.

▶ The origin of a TP is the location from which shipments are despatched. The destination of a transportation problem is the location to which shipments are transported. The unit transportation cost is the cost of transporting one unit of the consignment from an origin to a destination.

▶ The objective is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply and demand limits.

▶ The application of the transportation model can be extended to other areas of operation, including inventory control, employment scheduling, and personnel assignment.

2.1.1 Assumptions

The assumptions of transportation problems are:

1. Total quantity of the item available at different sources = the total requirement at different destinations.
2. Item can be transported convinently from all sources to destinations.
3. The unit transportation cost of the item from all sources to destinations is certainly and precisely known.
4. The transportation cost on a given route is directly proportional to the number of

units shipped on that route.

5. The objective is to minimize the total transportation cost for the organsation as a whole and not for individual supply and distribution centres.

2.1.2 Mathematical Formulation

A product is available in known quantities at each of m origins. It is required that given quantities of the product be shipped to each of n destinations.
Let a_i, b_j are the quantities of the available product at origin i and requirement at destination j respectively. The cost of shipping one unit from origin i to destination j is c_{ij}.
Now, there will be two cases:
i) Balanced TP i.e. $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_j$ and ii) Unbalanced TP i.e. $\sum_{i=1}^{m} a_i \neq \sum_{i=1}^{n} b_j$.

□ Note that, for BFS of a TP can be obtained for the balanced problem only. We usually convert an unbalanced TP to a balanced TP for BFS.

□ The mathematical form of transportation problem is as follows:

	D_1	$\cdots$	D_j	$\cdots$	D_n	a_i
O_1	x_{11}	$\cdots$	x_{1j}	$\cdots$	x_{1n}	a_1
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
O_i	x_{i1}	$\cdots$	x_{ij}	$\cdots$	x_{in}	a_i
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
O_m	x_{m1}	$\cdots$	x_{mj}	$\cdots$	x_{mn}	a_m
b_j	b_1	$\cdots$	b_j	$\cdots$	b_n	

Now x_{11} is the quantity unit and c_{11} is the transportation cost per unit from origin 1 i.e. O_1 to destination 1 i.e. D_1. In general, Now x_{ij} is the quantity unit and c_{ij} is the transportation cost per unit from origin i i.e. O_i to destination j i.e. D_j. Here $i = 1(1)m$, $j = 1(1)n$.
We have the restriction $\sum_{i=1}^{m} x_{ij} = b_j$, $j = 1(1)n$ and $\sum_{j=1}^{n} x_{ij} = a_i$, $i = 1(1)m$.
Also for balanced TP: $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.
In TP, we need to find the quantity x_{ij} such that the transportation cost is minimum.
So the TP can be viewed as:

$$\begin{array}{c} Min\ z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}, \\ \text{Subject to,} \\ \sum_{i=1}^{m} x_{ij} = b_j,\ j = 1(1)n, \\ \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1(1)m, \\ \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j. \\ x_{ij} \geq 0. \end{array}$$

► Here the number of linearly independent constraints are $m + n - 1$. The number of variables are mn.
► Usually the cost c_{ij} are given in the TP matrix.

Theorem 2.1. *We can always find a FS:* $\boxed{x_{ij} = \frac{a_i b_j}{L},\ i = 1(1)m;\ j = 1(1)n}$ *for each balanced TP with m origins and n destinations. Here* $L = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Proof. The balanced TP is

$$\begin{array}{c} Min\ z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}, \\ \text{Subject to,} \\ \sum_{i=1}^{m} x_{ij} = b_j,\ j = 1(1)n, \\ \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1(1)m, \\ \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j. \\ x_{ij} \geq 0. \end{array}$$

As all a_i, b_j are non-negative $\Rightarrow x_{ij} \geq 0,\ \forall i, j$.
Now, $x_{ij} = \frac{a_i b_j}{L} \Rightarrow \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{L} = b_j \sum_{i=1}^{m} \frac{a_i}{L} = b_j,\ j = 1(1)n$.
Again, $x_{ij} = \frac{a_i b_j}{L} \Rightarrow \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{a_i b_j}{L} = a_i \sum_{j=1}^{n} \frac{b_j}{L} = a_i,\ i = 1(1)m$.
It satisfies the balanced TP i.e. $x_{ij} = \frac{a_i b_j}{L},\ i = 1(1)m;\ j = 1(1)n$ is a FS. □

Theorem 2.2. *Show that the exact number of basic variables are $m+n-1$ for a balanced TP with m origins and n destinations.*

Proof. The balanced TP is

$$\begin{array}{c} Min\ z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}, \\ \text{Subject to,} \\ \sum_{i=1}^{m} x_{ij} = b_j,\ j = 1(1)n, \\ \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1(1)m, \\ \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j. \\ x_{ij} \geq 0. \end{array}$$

There are $m+n$ linear constraints with mn variables and $mn > m+n-1$ $(m, n \geq 2)$.
Now, $\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \ \cdots\ (1)$

Again, $\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \cdots (2)$
Therefore by (1)-(2) we have, $\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} - \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = b_n \Rightarrow \sum_{i=1}^{m} x_{in} = b_n$

which is the last or, n^{th} constraint of the balanced TP. Therefore, there are only $(m+n-1)$ linearly independent equations with mn variables $(mn > m+n-1)$. Thus from the definition of the basic solution, we can say that the number of basic variables is exactly $(m+n-1)$. □

2.1.3 North West Corner Rule

Example 2.1. *Find the IBFS and associated cost of following balanced TP by North-West Corner rule*

	D_1	D_2	D_3	D_4	a_i
O_1	16	25	42	12	8
O_2	62	20	45	48	12
O_3	25	18	50	12	16
b_j	7	6	9	14	

⇒ *The method starts at the north west-corner cell i.e.* $(1,1)$ *of the table and we allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount. Here we can allocate* $\min\{7,8\} = 7$ *in the* $(1,1)$ *cell and freeze/ cross out the column* D_1 *(see red blocks). It implies that no further assignments can be made in that column. Now the value of* a_1 *corresponding to the row* O_1 *changes to* 1.
Again we apply north west-corner rule to modified table i.e. in the cell $(1,2)$, *allocate* $\min\{6,1\} = 1$ *in the* $(1,2)$ *cell and freeze the row* O_1 *(see grey blocks). It implies that no further assignments can be made in that row. Now the value of* b_1 *corresponding to the column* D_2 *changes to* 5.
Again we apply north west-corner rule to modified table i.e. in the cell $(2,2)$, *allocate* $\min\{5,12\} = 5$ *in the* $(2,2)$ *cell and freeze the column* D_2 *(see blue blocks). It implies that no further assignments can be made in that column. Now the value of* a_2 *corresponding to the row* O_2 *changes to* 7.
Again we apply north west-corner rule to modified table i.e. in the cell $(2,3)$, *allocate* $\min\{9,7\} = 7$ *in the* $(2,3)$ *cell and freeze the row* O_2 *(see orange blocks). It implies that no further assignments can be made in that row. Now the value of* b_3 *corresponding to the column* D_3 *changes to* 2.
Now only two cells $(3,3),(3,4)$ *left with obvious allocation* 2 *and* 14 *respectively and we have the allocated table below.*

	D_1	D_2	D_3	D_4	a_i	
O_1	7 / 16	1 / 25	42	12	8	1
O_2	62	5 / 20	7 / 45	48	12	7
O_3	25	18	2 / 50	14 / 12	16	
b_j	7	6	9	14		
		5	2			

The solution is a basic feasible solution as:
The number of allocated cells = 6 = 3+4-1 = No. of rows + No. of columns - 1.
Also the allocated cells does not form a loop.
Therefore the IBFS is $x_{11} = 7$, $x_{12} = 1$, $x_{22} = 5, x_{23} = 7$, $x_{33} = 2$, $x_{34} = 14$. *Also the associated cost is=* $7.16 + 1.25 + 5.20 + 7.45 + 2.50 + 14.12 = 820$ *unit.*

2.1.4 Matrix Minima Method

Example 2.2. *Find the IBFS and associated cost of following balanced TP by Matrix Minima Method*

	D_1	D_2	D_3	D_4	a_i
O_1	16	25	42	12	8
O_2	62	20	45	48	12
O_3	25	18	50	12	16
b_j	7	6	9	14	

⇒ *The method starts at the cell with minimum cost i.e.* $(1,4)$ *or,* $(3,4)$ *of the table (we choose* $(3,4)$*) and we allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount. Here we can allocate* $\min\{14,16\} = 14$ *to the* $(3,4)$ *cell and freeze/ cross out the column* D_4 *(see red blocks). It implies that no further assignments can be made in that column. Now the value of* a_3 *corresponding to the row* O_3 *changes to* 2.
Again we apply matrix minima rule to modified table i.e. in the cell $(1,1)$*, allocate* $\min\{7,8\} = 7$ *in the* $(1,1)$ *cell and freeze the column* D_1 *(see blue blocks). It implies that no further assignments can be made in that column. Now the value of* a_1 *corresponding to the row* O_1 *changes to* 1.
Again we apply matrix minima rule to modified table i.e. in the cell $(3,2)$*, allocate* $\min\{6,2\} = 2$ *in the* $(3,2)$ *cell and freeze the row* O_3 *(see grey blocks). It implies that no*

further assignments can be made in that row. Now the value of b_2 corresponding to the column D_2 changes to 4.
Again we apply matrix minima rule to modified table i.e. in the cell (2,2), allocate $\min\{4,8\} = 4$ *in the (2,2) cell and freeze the column D_2 (see orange blocks). It implies that no further assignments can be made in that column. Now the value of a_2 corresponding to the row O_2 changes to 8.*
Now only two cells (1,3), (2,3) left with obvious allocations 1 and 8 respectively and we have the allocated table below.

	D_1	D_2	D_3	D_4	a_i	
O_1	7		1		8	1
	16	25	42	12		
O_2		4	8		12	8
	62	20	45	48		
O_3		2		14	16	2
	25	18	50	12		
b_j	7	6	9	14		
		4	8			

The solution is a basic feasible solution as:
The number of allocated cells = 6 = 3+4-1 = No. of rows + No. of columns - 1.
Also the allocated cells does not form a loop.
Therefore the IBFS is $x_{11} = 7$, $x_{13} = 1$, $x_{22} = 4, x_{23} = 8$, $x_{32} = 2$, $x_{34} = 14$. *Also the associated cost is*$= 7.16 + 1.42 + 4.20 + 8.45 + 2.18 + 14.12 = 798$ *unit.*

2.1.5 Vogel's Approximation Method (VAM)

Example 2.3. *Find an IBFS and associated cost of following balanced TP by Vogel's Approximation Method*

	D_1	D_2	D_3	D_4	a_i
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	14	

⇒ *The method starts by determining the penalty measure by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column). We will determine penalty terms for each row and column. Here, the*

penalty corresponding to b_1 *is* $(40 - 19 = 21)$, a_1 *is* $(19 - 10 = 9)$ *and so on.*
Next, we will identify the row or, column with the largest penalty i.e. 22 *corresponding to* b_2 *(also break the ties arbitrarily if any). Allocate as much as possible to the variable with the least unit cost in the selected row or column i.e. we will chose cell* $(3, 2)$ *and allocate* $\min\{8, 18\} = 8$ *in that cell. Then we adjust the supply and demand, and cross out the satisfied row or column i.e. we freeze* D_2 *and change* 18 *in* a_3 *as* $18 - 8 = 10$. *Note that, if a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).*
Again we will identify the row or, column with the largest penalty i.e. 21 *corresponding to* b_1. *Allocate as much as possible to the variable with the least unit cost in the selected row or column i.e. we will chose cell* $(1, 1)$ *and allocate* $\min\{5, 7\} = 5$ *in that cell. Then we adjust the supply and demand, and cross out the satisfied column i.e. we freeze* D_1 *and change* 7 *in* a_1 *as* $7 - 5 = 2$.
Again we will identify the row or, column with the largest penalty i.e. 50 *corresponding to* a_3. *Allocate as much as possible to the variable with the least unit cost in the selected row i.e. we will chose cell* $(3, 4)$ *and allocate* $\min\{10, 14\} = 10$ *in that cell. Then we adjust the supply and demand, and cross out the satisfied row i.e. we freeze* O_3 *and change* 14 *in* b_4 *as* $14 - 10 = 4$.
Again we will identify the row or, column with the largest penalty i.e. 50 *corresponding to* b_4. *Allocate as much as possible to the variable with the least unit cost in the selected column i.e. we will chose cell* $(1, 4)$ *and allocate* $\min\{4, 2\} = 2$ *in that cell. Then we adjust the supply and demand, and cross out the satisfied row i.e. we freeze* O_1 *and change* 4 *in* b_4 *as* $4 - 2 = 2$.

	D_1	D_2	D_3	D_4	a_i	
O_1	19	30	50	10	7	7(9)
O_2	70	30	40	60	9	9(10)
O_3	40	8 8	70	20	18	18(12)
b_j	5	8	7	14		
	5(21)	*8(22)*	7(10)	14(10)		

	D_1	D_2	D_3	D_4	a_i	
O_1	(5) 19	30	50	10	7	7(9)
O_2	70	30	40	60	9	9(20)
O_3	40	(8) 8	70	20	18	10(20)
b_j	5	8	7	14		
	5 (21)	×	7(10)	14(10)		

	D_1	D_2	D_3	D_4	a_i	
O_1	(5) 19	30	50	10	7	2(40)
O_2	70	30	40	60	9	9(20)
O_3	40	(8) 8	70	(10) 20	18	*10 (50)*
b_j	5	8	7	14		
	×	×	7(10)	14(10)		

	D_1	D_2	D_3	D_4	a_i	
O_1	(5) 19	30	50	(2) 10	7	2(40)
O_2	70	30	40	60	9	9(20)
O_3	40	(8) 8	70	(10) 20	18	×
b_j	5	8	7	14		
	×	×	7(10)	*4(50)*		

Now only two cells $(2,3),(2,4)$ *left with obvious allocations* 7 *and* 2 *respectively and* *we have the allocated table below.*

	D_1	D_2	D_3	D_4	a_i	
O_1	5, 19	30	50	2, 10	7	×
O_2	70	30	7, 40	2, 60	9	9
O_3	40	8, 8	70	10, 20	18	×
b_j	5	8	7	14		
	×	×	7	2		

However, in a single table, we can write this as follows:

	D_1	D_2	D_3	D_4	a_i					
O_1	5, 19	30	50	2, 10	7	7(9)	7(9)	2(40)	2(40)	
O_2	70	30	7, 40	2, 60	9	9(10)	9(20)	9(20)	9(20)	9(20)
O_3	40	8, 8	70	10, 20	18	18(12)	10(20)	10(50)		
b_j	5	8	7	14						
	5(21)	8(22)	7(10)	14(10)						
	5(21)		7(10)	14(10)						
			7(10)	14(10)						
			7(10)	4(50)						
			7	2						

Therefore, the final allocation is in the table below:

	D_1	D_2	D_3	D_4	a_i
O_1	[5] 19	30	50	[2] 10	7
O_2	70	30	[7] 40	[2] 60	9
O_3	40	[8] 8	70	[10] 20	18
b_j	5	8	7	14	

The solution is a basic feasible solution as:
The number of allocated cells = 6 = 3+4-1 = No. of rows + No. of columns - 1.
Also the allocated cells does not form a loop.
Therefore the IBFS is $x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$. *Also the associated cost is*= $5.19 + 2.10 + 7.40 + 2.60 + 8.8 + 10.20 = 779$ *unit.*

[Do It Yourself] 2.1. *Obtain the IBFS to the following transportation problem by i) North west corner, ii) Matrix (cost) minima and iii) VAM. Also determine which one gives better result.*

	D_1	D_2	D_3	D_4	a_i
O_1	21	16	25	13	11
O_2	17	18	15	23	13
O_3	32	27	18	41	19
b_j	6	10	12	15	

2.1.6 Modified Distribution (MODI) Method

Example 2.4. *Obtain the IBFS to the following transportation problem by matrix (cost) minima method then find out an optimal solution and corresponding cost of the transportation.*

	D_1	D_2	D_3	D_4	a_i
O_1	5	4	6	14	15
O_2	2	9	8	6	4
O_3	6	11	7	13	8
b_j	9	7	5	6	

we have the allocated table below.

	D_1	D_2	D_3	D_4	a_i	
O_1	5 / 19	30	50	2 / 10	7	×
O_2	70	30	7 / 40	2 / 60	9	9
O_3	40	8 / 8	70	10 / 20	18	×
b_j	5	8	7	14		
	×	×	7	2		

However, in a single table, we can write this as follows:

	D_1	D_2	D_3	D_4	a_i					
O_1	5 / 19	30	50	2 / 10	7	7(9)	7(9)	2(40)	2(40)	
O_2	70	30	7 / 40	2 / 60	9	9(10)	9(20)	9(20)	9(20)	9(20)
O_3	40	8 / 8	70	10 / 20	18	18(12)	10(20)	10(50)		
b_j	5	8	7	14						
	5(21)	8(22)	7(10)	14(10)						
	5(21)		7(10)	14(10)						
			7(10)	14(10)						
			7(10)	4(50)						
			7	2						

Therefore, the final allocation is in the table below:

	D_1	D_2	D_3	D_4	a_i
O_1	[5] 19	30	50	[2] 10	7
O_2	70	30	[7] 40	[2] 60	9
O_3	40	[8] 8	70	[10] 20	18
b_j	5	8	7	14	

The solution is a basic feasible solution as:
The number of allocated cells = 6 = 3+4-1 = No. of rows + No. of columns - 1.
Also the allocated cells does not form a loop.
Therefore the IBFS is $x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$. *Also the associated cost is*= 5.19 + 2.10 + 7.40 + 2.60 + 8.8 + 10.20 = 779 *unit.*

[Do It Yourself] 2.1. *Obtain the IBFS to the following transportation problem by i) North west corner, ii) Matrix (cost) minima and iii) VAM. Also determine which one gives better result.*

	D_1	D_2	D_3	D_4	a_i
O_1	21	16	25	13	11
O_2	17	18	15	23	13
O_3	32	27	18	41	19
b_j	6	10	12	15	

2.1.6 Modified Distribution (MODI) Method

Example 2.4. *Obtain the IBFS to the following transportation problem by matrix (cost) minima method then find out an optimal solution and corresponding cost of the transportation.*

	D_1	D_2	D_3	D_4	a_i
O_1	5	4	6	14	15
O_2	2	9	8	6	4
O_3	6	11	7	13	8
b_j	9	7	5	6	

⇒ *The matrix-minima method starts at the cell with minimum cost i.e.* (2, 1) *of the table and we allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount. Here we can allocate* $\min\{9, 4\} = 4$ *to the* (2, 1) *cell and freeze/ cross out the row* O_2 *(see red blocks). It implies that no further assignments can be made in that column. Now the value of* b_1 *corresponding to the column* D_1 *changes to* 5.

Again we apply matrix minima rule to modified table i.e. in the cell (1, 2), *allocate* $\min\{7, 15\} = 7$ *in the* (1, 2) *cell and freeze the column* D_2 *(see blue blocks). It implies that no further assignments can be made in that column. Now the value of* a_1 *corresponding to the row* O_1 *changes to* 8.

Again we apply matrix minima rule to modified table i.e. in the cell (1, 1), *allocate* $\min\{5, 8\} = 5$ *in the* (1, 1) *cell and freeze the column* D_1 *(see grey blocks). It implies that no further assignments can be made in that column. Now the value of* a_1 *corresponding to the row* O_1 *changes to* 3.

Again we apply matrix minima rule to modified table i.e. in the cell (1, 3), *allocate* $\min\{3, 5\} = 3$ *in the* (1, 3) *cell and freeze the row* O_1 *(see orange blocks). It implies that no further assignments can be made in that row. Now the value of* b_3 *corresponding to the column* D_3 *changes to* 2.

Now only two cells (3, 3), (3, 4) *left with obvious allocations* 2 *and* 6 *respectively and we have the allocated table below.*

	D_1	D_2	D_3	D_4	a_i			
O_1	5 / 5	7 / 4	3 / 6	14	15	15	8	3
O_2	4 / [illegible]	[illegible]	[illegible]	[illegible]	4			
O_3	6	11	2 / 7	6 / 13	8	8	8	8
b_j	9	7	5	6				
	5	7	5	6				
	5		5	6				
			5	6				

So, the allocation by matrix-minima method is

	D_1	D_2	D_3	D_4	a_i
O_1	[5] 5	[7] 4	[3] 6	14	15
O_2	[4] 2	9	8	6	4
O_3	6	11	[2] 7	[6] 13	8
b_j	9	7	5	6	

Now it is a balanced transportation problem with $m = 3$ origins and $n = 4$ destinations. Here the exact number of allocations = exact number of basic variable = $m + n - 1 = 6$.
Now to check for the optimum allocation, we have to compute the net-evaluation corresponding to the non basic cells. If all the net-evaluations are negative, then we reach the optimum allocation else we need to form a loop and readjust the basic variables in a way that after adjustment all the net-evaluations are negative.

To compute net evaluations, we need to calculate u_i and v_j from the basic cells that $u_i + v_j = c_{ij}$. Therefore, we have the equations: $u_1 + v_1 = 5$, $u_1 + v_2 = 4$, $u_1 + v_3 = 6$, $u_2 + v_1 = 2$, $u_3 + v_3 = 7$, $u_3 + v_4 = 13$.
Take $u_1 = 0$, we have $v_1 = 5, v_2 = 4, v_3 = 6, u_2 = -3, u_3 = 1, v_4 = 12$.

	D_1	D_2	D_3	D_4	u_i
O_1	[5] 5	[7] 4	[3] 6	14	0
O_2	[4] 2	9	8	6	−3
O_3	6	11	[2] 7	[6] 13	1
v_j	5	4	6	12	

Now, for the non-basic cells the net evaluations $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$.
So for the cells $(1,4), (2,2), (2,3), (2,4), (3,1), (3,2)$ the net evaluations are $-2, -8, -5, 3, 0, -6$ respectively.

	D_1	D_2	D_3	D_4	u_i
O_1	[5] 5	[7] 4	[3] 6	[−2] 14	0
O_2	[4] 2	[−8] 9	[−5] 8	[3] 6	−3
O_3	[0] 6	[−6] 11	[2] 7	[6] 13	1
v_j	5	4	6	12	

Here, the net evaluation corresponding to the cell (2,4) is positive, it implies the solution is not optimum. Now, we try to form a loop starting from (2,4) i.e. the red cell which covers some (or all) of the basic cells. The loop is constructed as follows: 3(red) → 6 → 2 → 3 → 7 → 5 → 4 → 3(red) *as below table.*

	D_1	D_2	D_3	D_4	u_i
O_1	[5] 5	[7] 4	[3] 6	14	0
O_2	[4] 2	9	8	[θ] 6	−3
O_3	6	11	[2] 7	[6] 13	1
v_j	5	4	6	12	

Now, we insert the value $\theta > 0$, for the cell (2,4) and readjust the basic variables in the cells containing the loop.

	D_1	D_2	D_3	D_4	u_i
O_1	[$5+\theta$] 5	[7] 4	[$3-\theta$] 6	14	0
O_2	[$4-\theta$] 2	9	8	[θ] 6	−3
O_3	6	11	[$2+\theta$] 7	[$6-\theta$] 13	1
v_j	5	4	6	12	

Now, put $\theta = \min\{3, 4, 6\} = 3$, *rewrite the cell allocation and calculate the net evaluations again*

	D_1	D_2	D_3	D_4	u_i
O_1	8 / 5	7 / 4	−3 / 6	−6 / 14	0
O_2	1 / 2	−8 / 9	−8 / 8	3 / 6	−3
O_3	3 / 6	−3 / 11	5 / 7	3 / 13	4
v_j	5	4	3	9	

Here, the net evaluation corresponding to the cell (3,1) is positive, it implies the solution is not optimum. Now, we try to form a loop starting from (3,1) i.e. the red cell which covers some (or all) of the basic cells. The loop is constructed as follows: $3(red) \to 3 \to 6 \to 1 \to 3(red)$ *as below table.*

	D_1	D_2	D_3	D_4	u_i
O_1	8 / 5	7 / 4	6	14	0
O_2	1 / 2	9	8	3 / 6	−3
O_3	3 / 6	11	5 / 7	3 / 13	4
v_j	5	4	3	9	

Now, we insert the value $\theta > 0$*, for the cell* $(3,1)$ *and readjust the basic variables in the cells containing the loop. Also, put* $\theta = \min\{1,3\} = 1$*, rewrite the cell allocation and calculate the net evaluations again*

	D_1	D_2	D_3	D_4	u_i
O_1	8 / 5	7 / 4	0 / 6	−2 / 14	−1
O_2	−3 / 2	−11 / 9	−8 / 8	4 / 6	−7
O_3	1 / 6	−6 / 11	5 / 7	2 / 13	0
v_j	6	5	7	13	

As all the net evaluations $z_{ij} - c_{ij} \le 0$*, it implies the solution* $x_{11} = 8$, $x_{12} =$

7, $x_{24} = 7$, $x_{31} = 4$, $x_{33} = 5$, $x_{43} = 2$ is optimal. Therefore, the minimum cost of the transportation is $\hat{z} = \sum \hat{c}_{ij}\hat{x}_{ij} = 5.8 + 4.7 + 6.4 + 6.1 + 7.5 + 13.2 = 159$ units.

■ If $z_{ij} - c_{ij} = 0$ for a non-basic cell ⇒ Multiple solutions exists.

[Do It Yourself] 2.2. *Find optimal solution and associated cost of following balanced TP [CU 86,90]*

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	4	30
O_2	3	4	7	8	15
O_3	9	6	5	8	15
b_j	10	25	18	7	

2.1.7 Degenerate and Maximization Transportation

Example 2.5. *Find an IBFS and associated cost of following balanced TP by Vogel's Approximation Method*

	D_1	D_2	D_3	D_4	a_i
O_1	24	32	38	23	6
O_2	27	32	34	48	10
O_3	42	18	57	25	12
O_4	12	18	35	41	14
b_j	9	16	10	7	

⇒ *The method starts by determining the penalty measure by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column). We will determine penalty terms for each row and column. Here, the penalty corresponding to b_1 is $(24 - 12 = 12)$, a_1 is $(24 - 23 = 1)$ and so on.*
Next, we will identify the row or, column with the largest penalty i.e. 12 corresponding to b_1 (also break the ties arbitrarily if any). Allocate as much as possible to the variable with the least unit cost in the selected row or column i.e. we will chose cell $(4, 1)$ and allocate $\min\{9, 14\} = 9$ in that cell. Then we adjust the supply and demand, and cross out the satisfied row or column i.e. we freeze D_1 and change 14 in a_4 as $14 - 9 = 5$. Note that, if a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).
Again we will identify the row or, column with the largest penalty i.e. 17 corresponding to a_4. Allocate as much as possible to the variable with the least unit cost in the selected

row or column i.e. we will chose cell $(4,2)$ *and allocate* $\min\{16,5\} = 5$ *in that cell. Then we adjust the supply and demand, and cross out the satisfied row i.e. we freeze* O_4 *and change* 16 *in* b_2 *as* $16 - 5 = 11$.
Again we will identify the row or, column with the largest penalty i.e. 14 *corresponding to* b_2. *Allocate as much as possible to the variable with the least unit cost in the selected column i.e. we will chose cell* $(3,2)$ *and allocate* $\min\{11,12\} = 11$ *in that cell. Then we adjust the supply and demand, and cross out the satisfied row i.e. we freeze* D_2 *and change* 12 *in* a_3 *as* $12 - 11 = 1$.
Again we will identify the row or, column with the largest penalty i.e. 32 *corresponding to* a_3. *Allocate as much as possible to the variable with the least unit cost in the selected column i.e. we will chose cell* $(3,4)$ *and allocate* $\min\{7,1\} = 1$ *in that cell. Then we adjust the supply and demand, and cross out the satisfied row i.e. we freeze* O_3 *and change* 7 *in* b_4 *as* $7 - 1 = 6$.
Again we will identify the row or, column with the largest penalty i.e. 25 *corresponding to* b_4. *Allocate as much as possible to the variable with the least unit cost in the selected column i.e. we will chose cell* $(1,4)$ *and allocate* $\min\{7,6\} = 6$ *in that cell. Then we adjust the supply and demand, and cross out the satisfied row i.e. we freeze* O_1 *and supply and demand automaticallay adjusted. Here we also allocate* 0 *in the cell* $(2,4)$ *and freeze* D_4. *It also implies that the solution is degenerate.*
Now only two cells $(2,3)$ *left with obvious allocation* 10 *and we have the allocated table below.*

	D_1	D_2	D_3	D_4	a_i					
O_1	24	32	38	6 / 23	6(1)	6(9)	6(9)	6(15)	6(15)	
O_2	27	32	10 / 34	0 / 48	10(5)	10(2)	10(2)	10(14)	10(14)	10
O_3	30	11 / 18	57	1 / 25	12(7)	12(7)	12(7)	1(32)		
O_4	9 / 12	5 / 18	35	41	14(6)	5(17)				
b_j	9(12)	16(0)	10(1)	7(2)						
		16(0)	10(1)	7(2)						
		11(14)	10(4)	7(2)						
			10(4)	7(2)						
			10(4)	6(25)						
			10							

Therefore, the final allocation is in the table below:

	D_1	D_2	D_3	D_4	a_i
O_1	24	32	38	[6] 23	6
O_2	27	32	[10] 34	48	10
O_3	42	[11] 18	57	[1] 25	12
O_4	[9] 12	[5] 18	35	41	14
b_j	9	16	10	7	

The solution is a degenerate basic feasible solution as:
The number of allocated cells = 5 (not counting cell with allocation 0) $\neq$ 3+4-1 = No. of rows + No. of columns - 1.
Also the allocated cells does not form a loop.
Therefore the degenerate IBFS is $x_{14} = 6$, $x_{23} = 10$, $x_{32} = 11$, $x_{34} = 1$, $x_{41} = 9$, $x_{42} = 5$.
Also the associated cost is$= 6.23 + 10.34 + 11.18 + 1.25 + 9.12 + 5.18 = 899$ *unit.*

Example 2.6. *Find an IBFS and associated cost of following balanced TP by matrix (cost) minima method then find out an optimal solution and corresponding cost of the transportation.*

	D_1	D_2	D_3	D_4	a_i
O_1	24	32	38	23	6
O_2	27	32	34	48	10
O_3	42	18	57	25	12
O_4	12	18	35	41	14
b_j	9	16	10	7	

$\Rightarrow$ *The method starts at the cell with minimum cost i.e.* $(4,1)$ *of the table and we allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount. Here we can allocate* $\min\{9,14\} = 9$ *to the* $(4,1)$ *cell and freeze/ cross out the column* D_1 *(see red blocks). It implies that no further assignments can be made in that column. Now the value of* a_4 *corresponding to the row* O_4 *changes to* $14 - 9 = 5$.
Again we apply matrix minima rule to modified table i.e. in the cell $(4,2)$, *allocate* $\min\{5,16\} = 5$ *in the* $(4,2)$ *cell and freeze the row* O_4 *(see blue blocks). It implies that*

no further assignments can be made in that row. Now the value of b_2 *corresponding to the column* D_2 *changes to* 11.
Again we apply matrix minima rule to modified table i.e. in the cell $(3,2)$*, allocate* $\min\{11,12\} = 11$ *in the* $(3,2)$ *cell and freeze the column* D_2 *(see grey blocks). It implies that no further assignments can be made in that column. Now the value of* a_3 *corresponding to the row* O_3 *changes to* 1.
Again we apply matrix minima rule to modified table i.e. in the cell $(1,4)$*, allocate* $\min\{6,7\} = 6$ *in the* $(4,1)$ *cell and freeze the row* O_1 *(see orange blocks). It implies that no further assignments can be made in that row. Now the value of* b_4 *corresponding to the column* D_4 *changes to* 1.
Again we apply matrix minima rule to modified table i.e. in the cell $(3,4)$*, allocate* $\min\{1,1\} = 1$ *in the* $(3,4)$ *cell and freeze the row* O_3*, column* D_4 *simultaneously. In that case, we will allocate* 0 *to either* $(2,4)$ *or,* $(3,3)$*. However, we allocated* 0 *on the cell* $(2,4)$*. Now only cell* $(2,3)$ *left with obvious allocations* 10 *and we have the allocated table below.*

	D_1	D_2	D_3	D_4	a_i					
O_1	21	32	38	(6) 23	6	6	6	6		
O_2	27	32	(10) 34	(0) 48	10	10	10	10	10	10
O_3	12	(11) 18	57	(1) 25	12	12	12	1	1	
O_4	(9) 12	(5) 18	36	41	14	5				
b_j	9	16	10	7						
		16	10	7						
		11	10	7						
			10	7						
			10	1						
			10							

Therefore, the final allocation is in the table below:

	D_1	D_2	D_3	D_4	a_i
O_1	24	32	38	[6] 23	6
O_2	27	32	[10] 34	[0] 48	10
O_3	42	[11] 18	57	[1] 25	12
O_4	[9] 12	[5] 18	35	41	14
b_j	9	16	10	7	

Here the solution is a degenerate IBFS. Now to check for the optimum allocation, we have to compute the net-evaluation corresponding to the non basic cells. If all the net-evaluations are negative, then we reach the optimum allocation else we need to form a loop and readjust the basic variables in a way that after adjustment all the net-evaluations are negative.

To compute net evaluations, first we put $\epsilon > 0$ on the cell $(2,4)$ and we need to calculate u_i and v_j from the basic cells that $u_i + v_j = c_{ij}$. Therefore, we have the equations: $u_1 + v_4 = 23$, $u_2 + v_3 = 34$, $u_2 + v_4 = 48$, $u_3 + v_2 = 18$, $u_3 + v_4 = 25$, $u_4 + v_1 = 12$, $u_4 + v_2 = 18$.
Take $v_4 = 0$, we have $u_1 = 23, u_2 = 48, u_3 = 25, v_3 = -14, v_2 = -7, u_4 = 25, v_1 = -13$.

	D_1	D_2	D_3	D_4	u_i
O_1	24	32	38	[6] 23	23
O_2	27	32	[10] 34	[ϵ] 48	48
O_3	42	[11] 18	57	[1] 25	25
O_4	[9] 12	[5] 18	35	41	25
v_j	−13	−7	−14	0	

Now, for the non-basic cells the net evaluations $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$.

So for the cells $(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,3),(4,3),(4,4)$ *the net evaluations are* $-14,-16,-29,8,9,-30,-46,-24,-16$ *respectively.*

	D_1	D_2	D_3	D_4	u_i
O_1	−14 / 24	−16 / 32	−29 / 38	6 / 23	23
O_2	8 / 27	9 / 32	10 / 34	ϵ / 48	48
O_3	−30 / 42	11 / 18	−46 / 57	1 / 25	25
O_4	9 / 12	5 / 18	−24 / 35	−16 / 41	25
v_j	−13	−7	−14	0	

	D_1	D_2	D_3	D_4	u_i
O_1	−14 / 24	−16 / 32	−29 / 38	6 / 23	23
O_2	8 / 27	9 / 32	10 / 34	ϵ / 48	48
O_3	−30 / 42	11 / 18	−46 / 57	1 / 25	25
O_4	9 / 12	5 / 18	−24 / 35	−16 / 41	25
v_j	−13	−7	−14	0	

Here, the net evaluation corresponding to the cell $(2,1),(2,2)$ *are positive, it implies the solution is not optimum. Now, we try to form a loop starting from* $(2,2)$ *(as it has maximum positive value 9) i.e. the red cell which covers some (or all) of the basic cells. The loop is constructed as follows:* $9(red) \to 11 \to 1 \to \epsilon \to 9(red)$ *as below table.*

	D_1	D_2	D_3	D_4	u_i
O_1	24	32	38	[6] 23	23
O_2	27	[θ] 32	[10] 34	[ϵ] 48	48
O_3	42	[11] 18	57	[1] 25	25
O_4	[9] 12	[5] 18	35	41	25
v_j	-13	-7	-14	0	

Now, we insert the value $\theta > 0$, for the cell $(2,2)$ and readjust the basic variables in the cells containing the loop.

	D_1	D_2	D_3	D_4	u_i
O_1	24	32	38	[6] 23	23
O_2	27	[θ] 32	[10] 34	[$\epsilon - \theta$] 48	48
O_3	42	[$11 - \theta$] 18	57	[$1 + \theta$] 25	25
O_4	[9] 12	[5] 18	35	41	25
v_j	-13	-7	-14	0	

Now, put $\theta = \min\{\epsilon, 11\} = \epsilon$, rewrite the cell allocation and calculate the net evaluations again

	D_1	D_2	D_3	D_4	u_i
O_1	-14 / 24	-16 / 32	-20 / 38	6 / 23	16
O_2	-1 / 27	ϵ / 32	10 / 34	-9 / 48	32
O_3	-30 / 42	$11-\epsilon$ / 18	-37 / 57	$1+\epsilon$ / 25	18
O_4	9 / 12	5 / 18	-16 / 35	-16 / 41	18
v_j	-6	0	2	7	

As all the net evaluations $z_{ij} - c_{ij} \leq 0$, it implies we obtained the optimal solution. So put $\epsilon = 0$, we have optimal solution (degenerate as at least one component is zero) as $x_{13} = 6$, $x_{22} = 0$, $x_{23} = 10$, $x_{32} = 11$, $x_{34} = 1$, $x_{41} = 9$, $x_{42} = 5$ is optimal. Therefore, the minimum cost of the transportation is $\widehat{z} = \sum \widehat{c}_{ij}\widehat{x}_{ij} = 6.23 + 0.32 + 10.34 + 11.18 + 1.25 + 9.12 + 5.18 = 899$ units.

Example 2.7. *Four products are produced in three machines and their profit margins are given by the table below with capacity a_i and requirement b_j. Find out an optimal allocation so that the profit is maximized. Also discuss about its multiple optimal solutions.*

	P_1	P_2	P_3	P_4	a_i
M_1	6	4	1	5	14
M_2	8	9	2	7	18
M_3	4	3	6	2	7
b_j	6	10	15	8	

$\Rightarrow$ *As the transportation problem is a minimization problem and the given problem is a maximization problem, so we will use the relation $Max(z) = -Min(-z)$. Now we will change the cost matrix as*

	P_1	P_2	P_3	P_4	a_i
M_1	-6	-4	-1	-5	14
M_2	-8	-9	-2	-7	18
M_3	-4	-3	-6	-2	7
b_j	6	10	15	8	

To find the IBFS, we will use VAM. The method starts by determining the penalty *measure by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column). We will determine penalty terms for each row and column. Here, the penalty corresponding to* b_1 *is* $(-6-(-8)=2)$, a_1 *is* $(-5-(-6)=1)$ *and so on.*

Next, we will identify the row or, column with the largest penalty i.e. 5 corresponding to b_2 *(also break the ties arbitrarily if any). Allocate as much as possible to the variable with the least unit cost in the selected row or column i.e. we will chose cell* $(2,2)$ *and allocate* $\min\{10,18\}=10$ *in that cell. Then we adjust* a_i *and* b_j*, and cross out the satisfied row or column i.e. we freeze (red) column* P_2 *and change 18 in* a_2 *as* $18-10=8$*.* ***Note that, if a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero allocation.***

Again we will identify the row or, column with the largest penalty i.e. 4 corresponding to b_3*. Allocate as much as possible to the variable with the least unit cost in the selected row or column i.e. we will chose cell* $(3,3)$ *and allocate* $\min\{15,7\}=7$ *in that cell. Then we adjust* a_i, b_j*, and cross out the satisfied row i.e. we freeze (blue) row* M_3 *and change 15 in* b_3 *as* $15-7=8$*.*

Again we will identify the row or, column with the largest penalty i.e. 2 corresponding to b_1 *(*b_4 *is also possible). Allocate as much as possible to the variable with the least unit cost in the selected row i.e. we will chose cell* $(2,1)$ *and allocate* $\min\{6,8\}=6$ *in that cell. Then we adjust* a_i, b_j*, and cross out the satisfied column i.e. we freeze (gray) column* P_1 *and change 8 in* a_2 *as* $8-6=2$*.*

Again we will identify the row or, column with the largest penalty i.e. 5 corresponding to a_2*. Allocate as much as possible to the variable with the least unit cost in the selected row i.e. we will chose cell* $(2,4)$ *and allocate* $\min\{2,8\}=2$ *in that cell. Then we adjust* a_i, b_j*, and cross out the satisfied row i.e. we freeze (orange) row* M_2 *and change 8 in* b_4 *as* $8-2=6$*. Now only cell* $(1,3)$, $(1,4)$ *are left with obvious allocations* $8,6$ *respectively and we have the allocated table below.*

	P_1	P_2	P_3	P_4	a_i				
M_1	−6	−4	8 / −1	6 / −5	14	14(1)	14(1)	14(1)	14(4)
M_2	6 / −8	10 / −9	−2	2 / −7	18	18(1)	8(1)	8(1)	2(5)
M_3	−4	−2	7 / −6	−2	7	7(2)	7(2)		
b_j	6	10	15	8					
	6(2)	10(5)	15(4)	8(2)					
	6(2)		15(4)	8(2)					
	6(2)		8(1)	8(2)					
			8(1)	8(2)					

So, the allocation by VAM is

	P_1	P_2	P_3	P_4	a_i
M_1	−6	−4	8 / −1	6 / −5	14
M_2	6 / −8	10 / −9	−2	2 / −7	18
M_3	−4	−3	7 / −6	−2	7
b_j	6	10	15	8	

Now it is a balanced transportation problem with $m = 3$ origins and $n = 4$ destinations. Here the exact number of allocations = exact number of basic variable = $m + n - 1 = 6$.
Now to check for the optimum allocation, we have to compute the net-evaluation corresponding to the non basic cells. If all the net-evaluations are negative, then we reach the optimum allocation else we need to form a loop and readjust the basic variables in a way that after adjustment all the net-evaluations are negative.

To compute net evaluations, we need to calculate u_i and v_j from the basic cells that $u_i + v_j = c_{ij}$. Therefore, we have the equations: $u_1 + v_3 = -1$, $u_1 + v_4 = -5$, $u_2 + v_1 = -8$, $u_2 + v_2 = -9$, $u_2 + v_4 = -7$, $u_3 + v_3 = -6$.
Take $u_2 = 0$, we have $v_1 = -8, v_2 = -9, v_4 = -7, u_1 = 2, v_3 = -3, u_3 = -3$.

	P_1	P_2	P_3	P_4	u_i
M_1	−6	−4	8 / −1	6 / −5	2
M_2	6 / −8	10 / −9	−2	2 / −7	0
M_3	−4	−3	7 / −6	−2	−3
v_j	−8	−9	−3	−7	

Now, for the non-basic cells the net evaluations $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$.
So for the cells $(1,1), (1,2), (2,3), (3,1), (3,2), (3,4)$ the net evaluations are $0, -3, -1, -7, -9, -8$ respectively.

	P_1	P_2	P_3	P_4	u_i
M_1	0 / −6	−3 / −4	8 / −1	6 / −5	2
M_2	6 / −8	10 / −9	−1 / −2	2 / −7	0
M_3	−7 / −4	−9 / −3	7 / −6	−8 / −2	−3
v_j	−8	−9	−3	−7	

As all the net evaluations $z_{ij} - c_{ij} \leq 0$, it implies the solution $x_{13} = 8$, $x_{14} = 6$, $x_{21} = 6$, $x_{22} = 10$, $x_{24} = 2$, $x_{33} = 7$ is optimal. Therefore, the minimum cost of the modified transportation problem is $Min(-z) = \sum \hat{c}_{ij}\hat{x}_{ij} = 8.(-1) + 6.(-5) + 6.(-8) + 10.(-9) + 2.(-7) + 7.(-6) = -232$. So, the maximum profit of the original transportation problem i.e. $Max(z) = -Min(-z) = 232$ units.

□ *As at least one $z_j - c_j = 0$, it implies there exists multiple optimal soluions. Now to find an another solution, we observe the third step of VAM where we can have the option to choose between $b_1 = 6(2)$ and $b_4 = 8(2)$, we used b_1 in our problem. Now, if we choose b_4, then we have an alternative solution.*

[Do It Yourself] 2.3. *Find optimal solution and associated cost of following balanced TP. Also show that the solution is degenerate and there exists multiple solutions.*

	D_1	D_2	D_3	D_4	a_i
O_1	5	2	4	6	9
O_2	2	4	1	5	11
O_3	4	2	3	1	8
b_j	6	7	7	8	

[Do It Yourself] 2.4. *Messrs Hindustan Construction Company Limited require 3, 3, 4 and 5 million cubic feet of fill at four earthen dam sites D_1, D_2, D_3 and D_4 in the district of Birbhum, West Bengal. The company can transfer the fill from thre mounds M_1, M_2 and M_3 where 2, 6 and 7 million cubic feet of fill are available respectively. The cost of transporting one million cubic feet of fill from the mounds to the dam sites (expressed in Rs. Lakh) are shown in the cost matrix given below:*

	D_1	D_2	D_3	D_4
M_1	15	10	17	18
M_2	16	13	12	13
M_3	12	17	20	11

a) Formulate the above problem as a LPP. b) Find the optimal solution. [CU 85]

2.1.8 Unbalanced Transportation

Example 2.8. *Obtain the IBFS to the following unbalanced transportation problem by matrix (cost) minima method then find out an optimal solution and corresponding cost of the transportation.*

	D_1	D_2	D_3	D_4	a_i
O_1	5	4	6	14	12
O_2	2	9	8	6	4
O_3	6	11	7	13	8
b_j	9	7	5	6	

⇒ *The given problem is an unbalanced TP as* $\sum a_i = 24$ *and* $\sum b_j = 27$. *As* $\sum b_j > \sum a_i$, *so we use a fake origin* O_4 *with* $a_4 = 27 - 24 = 3$. *We also assign the zero cost for all the components of* O_4. *So we can rewrite the transportation problem as:*

	D_1	D_2	D_3	D_4	a_i
O_1	5	4	6	14	12
O_2	2	9	8	6	4
O_3	6	11	7	13	8
O_4	0	0	0	0	3
b_j	9	7	5	6	

The matrix-minima method starts at the cell with minimum cost i.e. any cell of O_4, *we choose* $(4,1)$ *of the table and we allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount. Here we can allocate* $\min\{9,3\} = 3$ *to the* $(4,1)$ *cell and freeze/ cross out the row* O_4 *(see red blocks). It implies that no further assignments can be made in that column. Now the value of* b_1 *corresponding to the column* D_1 *changes to* $9 - 3 = 6$.
Again we apply matrix minima rule to modified table i.e. in the cell $(2,1)$, *allocate*

$\min\{4,6\} = 4$ *in the* $(2,1)$ *cell and freeze the row* O_2 *(see blue blocks). It implies that no further assignments can be made in that column. Now the value of* b_1 *corresponding to the column* D_1 *changes to* $6-4=2$.
Again we apply matrix minima rule to modified table i.e. in the cell $(1,2)$, *allocate* $\min\{7,12\} = 7$ *in the* $(1,2)$ *cell and freeze the column* D_2 *(see grey blocks). It implies that no further assignments can be made in that column. Now the value of* a_1 *corresponding to the row* O_1 *changes to* $12-7=5$.
Again we apply matrix minima rule to modified table i.e. in the cell $(1,1)$, *allocate* $\min\{2,5\} = 2$ *in the* $(1,1)$ *cell and freeze the column* D_1 *(see orange blocks). It implies that no further assignments can be made in that column. Now the value of* a_1 *corresponding to the row* O_3 *changes to* $5-2=3$.
Again we apply matrix minima rule to modified table i.e. in the cell $(1,3)$, *allocate* $\min\{3,5\} = 3$ *in the* $(1,3)$ *cell and freeze the row* O_1 *(see purple blocks). It implies that no further assignments can be made in that row. Now the value of* b_3 *corresponding to the column* D_3 *changes to* $5-3=2$.
Now only two cells $(3,3),(3,4)$ *left with obvious allocations* 2 *and* 6 *respectively and we have the allocated table below.*

	D_1	D_2	D_3	D_4	a_i					
O_1	2 (5)	7 (4)	3 (6)	(11)	12	12	12	5	3	
O_2	4 (2)	(9)	(8)	(6)	4	4				
O_3	(6)	(11)	2 (7)	6 (13)	8	8	8	8	8	8
O_4	3 (0)	(0)	(0)	(0)	3					
b_j	9	7	5	6						
	6	7	5	6						
	2	7	5	6						
	2		5	6						
			5	6						
			2	6						

So, the allocation by matrix-minima method is

	D_1	D_2	D_3	D_4	a_i
O_1	[2] 5	[7] 4	[3] 6	14	12
O_2	[4] 2	9	8	6	4
O_3	6	11	[2] 7	[6] 13	8
O_4	[3] 0	0	0	0	3
b_j	9	7	5	6	

Now it is a balanced transportation problem with $m = 3$ origins and $n = 4$ destinations. Here the exact number of allocations = exact number of basic variable = $m + n - 1 = 6$.
Now to check for the optimum allocation, we have to compute the net-evaluation corresponding to the non basic cells. If all the net-evaluations are negative, then we reach the optimum allocation else we need to form a loop and readjust the basic variables in a way that after adjustment all the net-evaluations are negative.

To compute net evaluations, we need to calculate u_i and v_j from the basic cells that $u_i + v_j = c_{ij}$. Therefore, we have the equations: $u_1 + v_1 = 5$, $u_1 + v_2 = 4$, $u_1 + v_3 = 6$, $u_2 + v_1 = 2$, $u_3 + v_3 = 7$, $u_3 + v_4 = 13$, $u_4 + v_1 = 0$.
Take $u_1 = 0$, we have $v_1 = 5, v_2 = 4, v_3 = 6, u_2 = -3, u_3 = 1, v_4 = 12, u_4 = -5$.

	D_1	D_2	D_3	D_4	u_i
O_1	[2] 5	[7] 4	[3] 6	14	0
O_2	[4] 2	9	8	6	−3
O_3	6	11	[2] 7	[6] 13	1
O_4	[3] 0	0	0	0	−5
v_j	5	4	6	12	

Now, for the non-basic cells the net evaluations $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$.
So for the cells $(1,4), (2,2), (2,3), (2,4), (3,1), (3,2), (4,2), (4,3), (4,4)$ the net evaluations

are $-2, -8, -5, 3, 0, -6, -1, 1, 7$ *respectively.*

	D_1	D_2	D_3	D_4	u_i
O_1	[5] 5	[7] 4	[3] 6	[−2] 14	0
O_2	[4] 2	[−8] 9	[−5] 8	[3] 6	−3
O_3	[0] 6	[−6] 11	[2] 7	[6] 13	1
O_4	[3] 0	[−1] 0	[1] 0	[7] 0	−5
v_j	5	4	6	12	

Here, the net evaluation corresponding to the cell $(2,3), (4,3), (4,4)$ *are positive, it implies the solution is not optimum. Now, we try to form a loop starting from the maximum positive cell* $(4,4)$ *i.e. the red cell which covers some (or all) of the basic cells. The loop is constructed as follows:* $7(red) \to 6 \to 2 \to 3 \to 5 \to 3 \to 7(red)$ *as below table.*

	D_1	D_2	D_3	D_4	u_i
O_1	[2] 5	[7] 4	[3] 6	14	0
O_2	[4] 2	9	8	6	−3
O_3	6	11	[2] 7	[6] 13	1
O_4	[3] 0	0	0	[7] 0	−5
v_j	5	4	6	12	

Now, we insert the value $\theta > 0$, *for the cell* $(4,4)$ *and readjust the basic variables in the cells containing the loop.*

	D_1	D_2	D_3	D_4	u_i
O_1	[$2+\theta$] 5	[7] 4	[$3-\theta$] 6	14	0
O_2	[4] 2	9	8	6	-3
O_3	6	11	[$2+\theta$] 7	[$6-\theta$] 13	1
O_4	[$3-\theta$] 0	0	0	[θ] 0	-5
v_j	5	4	6	12	

Now, put $\theta = \min\{3, 6\} = 3$, rewrite the cell allocation and calculate the net evaluations again

	D_1	D_2	D_3	D_4	u_i
O_1	[5] 5	[7] 4	[0] 6	14	0
O_2	[4] 2	9	8	6	-3
O_3	6	11	[5] 7	[3] 13	1
O_4	[0] 0	0	0	[3] 0	-5
v_j	5	4	6	12	

It is to be noted that, due to reallocation there are two zeros on the cell $(1, 3)$ and $(4, 1)$. Here, we will choose one zero in such a way that it does not form a loop. Here, we choose $(1, 3)$ and calculate the net evaluations as:

	D_1	D_2	D_3	D_4	u_i
O_1	5 / 5	7 / 4	0 / 6	−2 / 14	0
O_2	4 / 2	−8 / 9	−5 / 8	3 / 6	−3
O_3	0 / 6	−6 / 11	5 / 7	3 / 13	1
O_4	−7 / 0	−8 / 0	−6 / 0	3 / 0	−12
v_j	5	4	6	12	

Here, the net evaluation corresponding to the cell (2,4) is positive, it implies the solution is not optimum. Now, we try to form a loop starting from (2,4) i.e. the red cell which covers some (or all) of the basic cells. The loop is constructed as follows: $3(red) \to 3 \to 5 \to 0 \to 5 \to 4 \to 3(red)$ *as below table.*

	D_1	D_2	D_3	D_4	u_i
O_1	5 / 5	7 / 4	ϵ / 6	14	0
O_2	4 / 2	9	8	3 / 6	−3
O_3	6	11	5 / 7	3 / 13	1
O_4	0	0	0	3 / 0	−5
v_j	5	4	6	12	

Now, we insert the value $\theta > 0$, for the cell (2,4) and readjust the basic variables in the cells containing the loop. Also, put $\theta = \min\{3, \epsilon, 4\} = \epsilon$, rewrite the cell allocation and calculate the net evaluations again

	D_1	D_2	D_3	D_4	u_i
O_1	$5+\epsilon$ / 5	7 / 4	-3 / 6	-5 / 14	9
O_2	$4-\epsilon$ / 2	-8 / 9	-8 / 8	ϵ / 6	6
O_3	3 / 6	-3 / 11	$5+\epsilon$ / 7	$3-\epsilon$ / 13	13
O_4	-4 / 0	-5 / 0	-6 / 0	3 / 0	0
v_j	-4	-5	-6	0	

Put $\epsilon = 0$ and form the loop 3(red) $\to 3 \to 0 \to 4 \to$ 3(red) we have the

	D_1	D_2	D_3	D_4	u_i
O_1	5 / 5	7 / 4	6	14	0
O_2	4 / 2	9	8	0 / 6	-3
O_3	3 / 6	11	5 / 7	3 / 13	1
O_4	0	0	0	3 / 0	-5
v_j	5	4	6	12	

Now, we insert the value $\theta > 0$, for the cell (3,1) and readjust the basic variables in the cells containing the loop. Also, put $\theta = \min\{3, 4\} = 3$, rewrite the cell allocation and calculate the net evaluations again

	D_1	D_2	D_3	D_4	u_i
O_1	5 / 5	7 / 4	0 / 6	−5 / 14	5
O_2	1 / 2	−8 / 9	−5 / 8	$\epsilon + 3$ / 6	2
O_3	3 / 6	−6 / 11	5 / 7	−3 / 13	6
O_4	−4 / 0	−5 / 0	−3 / 0	3 / 0	−4
v_j	0	−1	1	4	

As all the net evaluations $z_{ij} - c_{ij} \leq 0$, we put $\epsilon = 0$ and the solution $x_{11} = 5$, $x_{12} = 7$, $x_{21} = 1$, $x_{24} = 3$, $x_{31} = 3$, $x_{33} = 5$, $x_{44} = 3$ is optimal. Therefore, the minimum cost of the transportation is $\widehat{z} = \sum \widehat{c}_{ij}\widehat{x}_{ij} = 5.5 + 4.7 + 2.1 + 6.3 + 6.3 + 7.5 + 0.3 = 126$ units.

■ Note that, the fake amount of 3 units are not supplied to the destination D_4 whereas the demand requires 3 units more. So, D_4 will be demand short of 3 units and $x_{44} = 3$ is the idle capacity of D_4.

[Do It Yourself] 2.5. *The cost matrix of a TP given below. There is a precondition that if all demands of a particular destination does not fulfill, one unit of penalty is to be charged for each unit of goods not supplied. Find the minimum cost of TP and mention in which destination(s), demand will not be met.*

	D_1	D_2	D_3	D_4	a_i
O_1	12	9	13	7	14
O_2	9	7	11	7	10
O_3	12	10	8	9	11
b_j	9	12	13	6	

[Hint: Add a row O_4 with $a_4 = 5$ and each $c_{4j} = 1$]

2.2 Assignment Problem

■ Assignment Problem (AP) is a special case of Transportation problem with equal number of source and destination. One source is assigned for one destination in a way such that the assignment cost is optimum.

► It can be a maximization (profit matrix) or, a minimization (cost matrix) problem.

The mathematical form of assignment problem is as follows:

	D_1	$\cdots$	D_j	$\cdots$	D_n
O_1	x_{11}	$\cdots$	x_{1j}	$\cdots$	x_{1n}
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
O_i	x_{i1}	$\cdots$	x_{ij}	$\cdots$	x_{in}
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
O_n	x_{n1}	$\cdots$	x_{nj}	$\cdots$	x_{nn}

Now x_{11} is the quantity unit and c_{11} is the transportation cost per unit from origin 1 i.e. O_1 to destination 1 i.e. D_1. In general, Now x_{ij} is the quantity unit and c_{ij} is the transportation cost per unit from origin i i.e. O_i to destination j i.e. D_j. Here $i = 1(1)n$, $j = 1(1)n$.

We have the restriction that each row and column has only assignment i.e. for example of $n = 3$, if $O_1 \to D_2$ then there will be no further assignment on O_1 row and D_2 column. Further if $O_2 \to D_3$ then then there will be no further assignment on O_2 row and D_3 column. It means $O_3 \to D_1$ is obvious.

So the variable $x_{ij} = 1$ if i^{th} origin assigned to j^{th} destination and $x_{ij} = 0$ otherwise.

If we compare to balanced TP, for AP we have: $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j = n$.

In AP, we need to find the quantity x_{ij} such that the assignment cost is optimum. In other words, we can say finding the allocation i.e. (i, j) pair for which $x_{ij} = 1$.

So the AP can be viewed as:

$$\text{Optimize } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

Subject to,

$$\sum_{i=1}^{n} x_{ij} = b_j = 1, \; j = 1(1)n,$$
$$\sum_{j=1}^{n} x_{ij} = a_i = 1, \; i = 1(1)n,$$
$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j = n.$$
$$x_{ij} = 1 \text{ or}, \; 0.$$

- Here the number of allocation i.e. n is less than number of linearly independent constraints i.e. $2n - 1$. Therefore, the solution obtained is a degenerate solution (comparing to no. of basic variables i.e. $m + n - 1$ for TP).
- Usually the cost c_{ij} are given in the AP square matrix.

Theorem 2.3. *Suppose (c_{ij}) is a cost matrix of order n. Now if α_i, β_j are arbitrary constants, then the matrix $(c^*_{ij}) = (c_{ij} - \alpha_i - \beta_j)$ and (c_{ij}) have identical solutions.*

Proof. Let x_{ij} be the quantity unit with c_{ij} be the transportation cost per unit from origin i i.e. O_i to destination j i.e. D_j for $i = 1(1)n$, $j = 1(1)n$. So the assignment problem is

$$\begin{aligned} &Optimize\ z = \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}, \\ &\text{Subject to,} \\ &\textstyle\sum_{i=1}^{n} x_{ij} = 1,\ j = 1(1)n, \\ &\textstyle\sum_{j=1}^{n} x_{ij} = 1,\ i = 1(1)n \end{aligned}$$

Now if we change cost matrix (c_{ij}) to the matrix $(c^*_{ij}) = (c_{ij} - \alpha_i - b_j)$, then the objective function z changes to $z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} c^*_{ij}x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} (c^*_{ij} - \alpha_i - \beta_j)x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} c^*_{ij}x_{ij} - \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} \beta_j \sum_{i=1}^{n} x_{ij} = z - \sum_{i=1}^{n} \alpha_i - \sum_{j=1}^{n} \beta_j$.
As α_i, β_j are arbitrary constants, so z^* and z are only differ by a constant i.e., $z^* = z - constant$, it implies $Optimize\ z^* = Optimize\ z - constant$.
Therefore, the matrix (c^*_{ij}) and (c_{ij}) have identical solutions. □

□ The optimum assignment remain same if we subtract an arbitrary element from each row or, column of a cost matrix (See Theorem 2.3).

Example 2.9. *Solve the following assignment problem and find the optimal (minimum) assignment cost from the cost matrix below.*

	A	B	C	D
1	11	8	8	7
2	10	9	7	8
3	10	8	7	9
4	11	11	10	10

⇒ *Since the cost matrix is given, its a minimization problem. So subtract the lowest element 7 from all the element we have,*

	A	B	C	D
1	4	1	1	0
2	3	2	0	1
3	3	1	0	2
4	4	4	3	3

Select the lowest element from each row and subtract it from rest of the elements of that row we have,

	A	B	C	D
1	4	1	1	0
2	3	2	0	1
3	3	1	0	2
4	1	1	0	0

Select the lowest element from each column and subtract it from rest of the elements of that column we have,

	A	B	C	D
1	3	0	1	0
2	2	1	0	1
3	2	0	0	2
4	0	0	0	0

Now we will connect the zeros by minimum number of vertical and horizontal blocks.

	A	B	C	D
1	3	0	1	0
2	2	1	0	1
3	2	0	0	2
4	0	0	0	0

Now we will select 4 zeros in such a way that each row and column has exactly one zero. In left panel $(A,4)$, $(C,2)$ has to selected as A column has only one zero and 2 row has only one zero. In the right panel, remaining $(D,1)$ and $(B,3)$ has to be chosen.

	A	B	C	D
1	3	0	1	0
2	2	1	0	1
3	2	0	0	2
4	0	0	0	0

	A	B	C	D
1	3	0	1	0
2	2	1	0	1
3	2	0	0	2
4	0	0	0	0

Here, the optimal assignment is unique and the assignment is $1 \to D$, $2 \to C$, $3 \to B$, $4 \to A$. Hence the optimal cost will be the sum of the given cost matrix at the assign-

ment cells i.e. $7 + 7 + 8 + 11 = 33$ *units.*

[Do It Yourself] 2.6. *Solve the following assignment problem and find the optimal (minimum) assignment cost from the cost matrix below.*

	A	B	C	D	E
1	11	10	9	8	6
2	7	9	7	8	10
3	10	9	8	5	7
4	10	7	6	11	5
5	8	9	8	10	7

2.2.1 Hungarian Method

☐ It's a method for solving Assignment Problems. It has the following steps:

1. If the problem is 'Maximization' then convert it to 'Minimization' by $Max(z) = -Min(-z)$. Now for minimization problem, go to Step 2.

2. Suppose the cost matrix has size n, then subtract the minimum element from all the elements. Now select the lowest element from each row and subtract it from rest of the elements of that row. Then select the lowest element from each column and subtract it from rest of the elements of that column.

3. Connect all the zeros by minimum number of horizontal (rows) and vertical (columns) lines. If the total number of lines (say L) $< n$ then follow Step 4 else go to Step 5.

4. If $L < n$, then the allocation is not optimum. We select those elements not covered by L lines, select the minimum element and subtract it from the rest. Simultaneously, we also add that selected element to the intersection square of L lines. Now for the new matrix, follow Step 2, 3.

5. If $L = n$, then the allocation is optimum. Here it is possible to find n zeros in a way such that each row and column has a single zero i.e. for each origin (O_i) there is a destination (D_j). It forms n pair of assignment $(O_i \to D_j)$. These n pair of assignment may or, may not be unique and the minimum cost will be $\sum c_{ij}$.

Example 2.10. *Solve the following assignment problem and find the optimal assignment profit from the profit matrix below.*

	A	B	C	D	E
1	5	7	6	8	7
2	10	7	9	11	7
3	5	12	11	13	7
4	11	9	15	10	7
5	5	11	8	11	11

⇒ *Since profit matrix is given, it's a maximization problem. Now first we will convert it into a minimization problem i.e.* $Max(z) = -Min(-z) = -Min(z^*)$ *with* $z^* = -z$. *Therefore, we change the sign of each element of the profit matrix as below*

	A	B	C	D	E
1	−5	−7	−6	−8	−7
2	−10	−7	−9	−11	−7
3	−5	−12	−11	−13	−7
4	−11	−9	−15	−10	−7
5	−5	−11	−8	−11	−11

Subtract the lowest element -15 *from all the element we have,*

	A	B	C	D	E
1	10	8	9	7	8
2	5	8	6	4	8
3	10	3	4	2	8
4	4	6	0	5	8
5	10	4	7	4	4

Select the lowest element from each row and subtract it from rest of the elements of that row we have,

	A	B	C	D	E
1	3	1	2	0	1
2	1	4	2	0	4
3	8	1	2	0	6
4	4	6	0	5	8
5	6	0	3	0	0

Select the lowest element from each column and subtract it from rest of the elements of that column we have,

	A	B	C	D	E
1	2	1	2	0	1
2	0	4	2	0	4
3	7	1	2	0	6
4	3	6	0	5	8
5	5	0	3	0	0

Now we will connect the zeros by minimum number of vertical and horizontal blocks (lines).

	A	B	C	D	E
1	2	1	2	0	1
2	0	4	2	0	4
3	7	1	2	0	6
4	3	6	0	5	8
5	5	0	3	0	0

Since the number of lines are 4 and it < 5, the allocation is not optimum. We select those elements not covered by 4 lines, select the minimum element i.e. 1 and subtract it from the rest. Simultaneously, we also add that selected element to the intersection square of 4 lines and we get

	A	B	C	D	E
1	1	0	1	0	0
2	0	4	2	1	4
3	6	0	1	1	5
4	3	6	0	6	8
5	5	0	3	1	0

Now we will connect the zeros by minimum number of vertical and horizontal blocks (lines).

	A	B	C	D	E
1	1	0	1	0	0
2	0	4	2	1	4
3	6	0	1	1	5
4	3	6	0	6	8
5	5	0	3	1	0

Now we will select 5 zeros in such a way that each row and column has exactly one zero. In left panel $(2, A)$, $(4, C)$ has to be selected as 2 row, A column has only one zero and 4 row, C column has only one zero. Also $(3, B)$ is fixed as row 3 has only one zero. In the right panel, remaining $(5, E)$ and $(1, D)$ has to be chosen.

	A	B	C	D	E
1	1	0	1	0	0
2	0	4	2	1	4
3	6	0	1	1	5
4	3	6	0	6	8
5	5	0	3	1	0

	A	B	C	D	E
1	1	0	1	0	0
2	0	4	2	1	4
3	6	0	1	1	5
4	3	6	0	6	8
5	5	0	3	1	0

Here, the optimal assignment is unique (note that in 5^{th} matrix it is possible to choose A column instead 2 Row, then (D,2) zero won't vanish in 6^{th} matrix, may leads to multiple solutions) and the assignment is $1 \to D$, $2 \to A$, $3 \to B$, $4 \to C$, $5 \to E$. Hence the optimal cost will be the sum of the given profit matrix at the assignment cells i.e. $-Min(z^*) = -(-8-10-12-15-11) = 56$ units.

[Do It Yourself] 2.7. *There are five pumps available for developing five wells. The* *efficiency of each pump in producing the maximum yield at each well is shown in the following table. In what way the pumps be assigned to the wells so as to maximize the overall efficiency? [CU 96]*

	1	2	3	4	5
A	45	40	65	25	55
B	50	30	25	60	30
C	25	20	10	20	40
D	35	25	30	25	20
E	80	60	50	70	50

[Do It Yourself] 2.8. *Find the optimal assignment for the assignment problem with the following cost matrix: [CU 99]*

	M_1	M_2	M_3	M_4	M_5
J_1	3	8	2	10	3
J_2	8	7	2	9	7
J_3	6	4	2	7	5
J_4	8	4	2	3	5
J_5	9	10	6	9	10

[Do It Yourself] 2.9. *A car hire company has one car in each of the five depots a, b, c, d and e. A customer in each of the five towns A, B, C, D and E requires a car. The distance (in km) between the depots (origins) and the towns (destinations) where the customers are, given by the following distance matrix: [CU 77]*

	a	b	c	d	e
A	20	40	30	50	40
B	70	40	60	80	40
C	20	90	80	100	40
D	80	60	120	70	40
E	20	80	50	80	80

[Do It Yourself] 2.10. *A car paint company has 5 types of cars $(C_1, C_2, C_3, C_4, C_5)$ and 5 paint machines $(M_1, M_2, M_3, M_4, M_5)$. Any car can be painted through any machines and the time (hr) taken by the machines are given below:*

	C_1	C_2	C_3	C_4	C_5
M_1	16	11	19	21	22
M_2	12	18	27	12	15
M_3	15	12	7	7	7
M_4	9	15	13	11	16
M_5	18	20	21	15	23

Select machines for cars such that the time minimizes.

[Do It Yourself] 2.11. *A car paint company has 5 types of cars* $(C_1, C_2, C_3, C_4, C_5)$ *and 5 paint machines* $(M_1, M_2, M_3, M_4, M_5)$*. Any car can be painted through any machines and the selling profit (Rs) based on paint by the machines are given below:*

	C_1	C_2	C_3	C_4	C_5
M_1	13	19	18	20	12
M_2	10	12	14	11	13
M_3	11	9	13	22	14
M_4	14	22	12	19	15
M_5	15	17	16	20	13

Select machines for cars such that the profit maximizes.

2.2.2 Special Cases of Assignment Problem

□ Restricted Assignment Problem: For an assignment problem, it is sometimes possible that a particular machine (or, a person) fails to perform a specific job.

▶ For example, among four persons (P_1, P_2, P_3, P_4) one person (P_2) can't repair heavy vehicle and they are suppose to repair bike (V_1), auto (V_2), light motor vehicle (V_3) and heavy vehicle (V_4). Let, the matrix represent the hour for repairing then there will be no element in the entry corresponding to P_2V_4.

Example 2.11. *Solve the following assignment problem and find the optimal assignment from the cost matrix below.*

Chapter 4

Inventory Management

4.1 Introduction

▪ An Inventory consists of usable but idle (not used immediately) resources such as men, machines, materials or money. Inventory is very important for a company as it directly connected to the revenew. If the resource involved is a material, the inventory is also called 'stock'. There are mainly three types of inventory: Raw materials, Work in progress and finished goods.

▸ Raw Materials : Items use to make your finished goods.

▸ Work in Progress : Items in the process of making finished goods for sales.

▸ Finished Goods : The products you sell to your customers.

▪ Suppose for the wooden furniture company Godrej Interio: The wood bought from the supplier to make furniture are raw materials. All unfinished parts before the assembly of the finished product are work in progress inventory. It including labor, shape cutting, assemble with gum, design, polish, etc. The finished goods are ready for sale product as we see in the showroom.

▸ We will discuss a detailed classification of inventories later.

4.1.1 Requirement of Inventory

▪ Although inventory is an idle resouces but most of the companies retain that for effective and smooth performance. Lack of inventory causes various trouble for an organisation and ultimately hinder the profit.

▸ Suppose a company has no inventory, then after receiving an order they have to purchase and recieve raw materials, then start production. It's a time consuming process makes the customers wait for a long delivery time of the products, may causes to lose the customer.

► Depending on the type, most of companies allocate a certain proportion of its budget for the inventory and its management.

■ Inventory management is the process of tracking and controlling the business inventory as it is bought, manufactured, stored and used. It manages the complete progress of inventories from purchasing to sale. It also determines that based on the need, the right quantities of the right item in the right location at the right time are always available.
► At a basic level, inventory management works by tracking products, components and ingredients across suppliers, stock on hand, production and sales to ensure that stock is used as efficiently and effectively as possible. This management process will get complex as we consider various dimensions in it. Also most of the companies try to improve their existing inventory management process.
► The application of operations research techniques in this area is providing a powerful tool for managing inventories and it's known as scientific inventory management.

Question 4.1. *Why it is essential for mantaining an inventory for almost all the companies?*

★ *There are several reasons behind the need of inventory management:*

1. *It helps the company to flow in a smooth and efficient way.*
2. *Reduces the waiting time for a customer which leads to customer satisfaction and future growth.*
3. *Maintaining inventory requires bulk purchasing which is cost effective. Also it takes the advantage of market price i.e. if there is no inventory then company has to purchase the inventory after the order received and it may possible that product price is high on that time.*
4. *Reduces product cost due to uninterrupted flow. Inventories also plays a critical role due to product scarcity (or, slow raw material received) at the market.*
5. *For long distance customers, inventories saved a significant amount of time. For example customer transite time is six days and if there is an inventory then product making requires small amount of time and the customer received the order quickly.*
6. *Suppose a customer transite time is four days and they just created the order before journey. So if the company has inventory, then it may possbile to create the required product within four days.*

■ Note that, efficient inventory management is required else it will impact heavily on the company profit.

4.1.2 Classification of Inventories

Inventories are mainly classified into two categories: Direct and Indirect.

Direct Inventories: Inventories directly used for production and **are a part of the finished goods**. It has several classifications such as: i) production inventories (includes raw materials, components, etc.); ii) work in progress inventories; iii) finished goods inventories; iv) MRO inventories (stands for maintenance, repair and operating. This is the inventory you use to support the manufacturing process); v) Miscellaneous inventories (scrap, stationary items etc.)

Indirect Inventories: Inventories are required for production and **are not a part of the finished goods**. Some example includes fuel, maintenance, oil, spare parts, coolants, tools, etc). It has several classifications such as: i) Transport Inventories (items currently under transportation, for example coal [or, petrolium oil] being transported from coal fields [or, pertolium oil fields] to a thermal plant [or, petrolium industry]); ii) Buffer Inventories (balancing the fluctuations on supply and demand, if there is a requirement of above average product then buffer inventories play a crucial role, also if there is a delay in supply then buffer inventories help to run the process smoothly); iii) Decouple Inventories (here inventory managers reserve a portion of the stock for each node of production, for example, the computer manufacturer would set aside a portion of the parts needed at each stage of building a laptop as a buffer against inventory interruptions in the operation's nodes. The decoupling inventories act as shock absorbers in case of varying work-rates, machine breakdowns or failures, etc.); iv) Seasonal Inventories (inventories with seasonal demands e.g. demand of woollen textiles in winter, air conditioners in summer, etc. Seasonal inventories has to be maintained to smooth production under high seasonal demand); v) Anticipation/ Provision Inventories (used to meet the anticipated demand e.g. purchasing of crackers well before Diwali, cricket match. Storing of raw materials for some expected strike, etc.); vi) Lot Size Inventories (used to take advantages of discounts for purchase of large quantities. It includes: price discounts, transportation costs discount, handling costs discounts.)

Question 4.2. *Discuss various types of costs associated with inventory control models.*
★ *Inventory costs are mainly four types: Purchasing Cost, Holding Cost, Setup Cost and Shortage Cost. We will briefly discuss these costs as follows:*

1. *Purchase Costs: Purchase cost is the price that is paid for purchasing an item. It may be constant per unit or may vary with the quantity purchased. If the cost is constant, it does not affect the inventory control decision. However, the purchase cost is definitely considered when it varies as in quantity discount situations. At times the item is offered at a discount if the order size exceeds a certain amount, which is a factor in deciding how much to order.*

2. *Holding Costs: It consists of Interest costs on capital (e.g. an interest has to be paid to the bank against the capital amount), Storage costs (e.g. rent for storage and its ideal conditions i.e. heat, cold, pressure, light, etc. Note that, for the company's own warehouse the rental cost is an opportunity cost i.e. there is a possibility to use the warehouse for other purposes is compromised), Depreciation costs (e.g. this cost mainly associated with seasonal, fashion, chemical, fragile, etc. items),*

Pilferage costs (i.e. small theft performed repeatedly over a long period of time e.g. an employee stealing small amounts of office supplies from their workplace every few days), Obsolescence costs (i.e. inventory has not been sold or used for a long period of time and is not expected to be sold in the future. So the inventory at the end of its product life cycle. For example, clothing and apparel retailers have the most difficulty with obsolescence. As every season and every year clothing styles change. So the retailers can't afford to used outdated fashion due to decrease in sales. It causes season sales as every reatiler want to get rid of the inventory before it becomes obsolete and worthless to them), Handling costs (i.e. costs associated with movement of stock e.g. labour cost, fork lifter cost, cranes cost, battery truck, etc.), Record Keeping costs (i.e. costs associated with a systematic process that enables the accountability about the whole stock and its very important) and Tax-insurance costs (i.e. tax liabiliies and costs associated with insurance cover against possible loss from damage, theft, fire, business interruption etc.)

3. *Setup Costs: These include the fixed cost associated with placing of an order or setting up a machinery before starting production. It includes various clerical and administrative costs (e.g. costs of purchase, requisition, follow up, receiving the goods, quality control, cost of communications, salaries of persons for accounting and auditing, etc. i.e. Ordering costs). Setup costs are independent of the quantity ordered although it directly proportional to the number of orders placed.*

   ▷ *There is an inverse relationship between setup and holding cost. Higher the order quantity reduces the cost due to setup (w.r.t. a given demand) but it will block capital and related holding costs increases. Consequently, lower the order quantity decreases the block capital and related holding costs but it increases the number of ordering and the consequent setup costs. In inventory management, we usually optimize these two costs.*

4. *Shortage Costs: These costs are the damage due to lack of supply (i.e. backlog) for a particular demand and the cost is usually proportional to the deficit items and delay time. Shortage cost also includes the inadequacy to supply the demand at all (i.e. demand lost). It includes potential loss of income, cancelled orders, lost sales, profit and the more subjective cost of loss in customer's goodwill.*

Moreover, if the purchase cost is constant and independent of the quantity purchased, it is not considered in the model else it is considered in the model i.e., usually
Inventory Cost = Purchase cost + Holding cost + Setup cost + Shortage cost. □

4.1.3 Inventory Control Problems

■ Inventory modeling deals with detennining the level of a commodity that a business must maintain to ensure smooth operation.

► The basis for the decision is a model that balances the cost of capital resulting from

holding too much inventory against the penalty cost resulting from inventory shortage.
► The principal factor affecting the solution is the nature of the demand: deterministic or probabilistic. In real life, demand is usually probabilistic, but in some cases the simpler deterministic approximation may be acceptable.
► The complexity of the inventory problem does not allow the development of a general model that covers all possible situations. So we will use different model to represent various situations.

■ Usually an inventory model seeks three basic results; i) When to order ii) How much to order and iii) How much safety stock.
► i) **When to order**: This is related to the lead time i.e. the time interval between the placement of an order for an item and its receipt in stock. There should be enough stock for each item so that customers' orders can be reasonably met from this stock until refill and this stock level is known as reorder level.
► ii) **How much to order**: This is related to both the holding and setup costs. An inventory cost model balances these two costs.
► iii) **How much safety stock**: This is related to both the over-stock and out-stock costs. An inventory cost model balances these two costs.

■ **Importance of Demand**: The inventory control policy of an organisation depends upon the demand characteristics. The demand for an item may be deterministic or probabilistic. Also demand may or, may not vary over the time. The forecasting of demand is very important and usually we assume four types of demand in inventory modelling:

1. Deterministic and constant (static) with time.
2. Deterministic and variable (dynamic) with time.
3. Probabilistic and stationary over time (i.e. statistical properties are constant over time).
4. Probabilistic and nonstationary over time (i.e. statistical properties changes over time).

Now to develop an inventory model the demand type 1 is the simplest after that the complexity increases respectively for type 2, 3 and 4. However, in practical situation the order reverses i.e. 4^{th} is most likely and 1^{st} is less likely. So in modelling process one aim for a balance between model simplicity (not too simple to explain a practical situation inefficiently) and model accuracy (not too complex to handle analytically).
► To choose the demand type for inventory modelling we follow certain rules. Suppose we have a monthly consumption data for a product for 10 years (say $2010 - 2020$) i.e. 120 data points. Now to check the demand for a particular month we will find mean i.e. m and coefficient of variation i.e. $C.O.V. = \frac{SD}{Mean} \times 100\%$. So there will be 12 m and C.O.V. values w.r.t. each month by using 10 year data.
▹ We will use Type 1 demand if m is approximately constant and C.O.V. is less than 20%.
▹ We will use Type 2 demand if m is not approximately constant and C.O.V. is less than 20%.
▹ We will use Type 3 demand if m and C.O.V. are approximately constant and C.O.V

greater than 20%.
▹ We will use Type 4 demand if m and C.O.V. are not approximately constant and C.O.V greater than 20%.

[Do It Yourself] 4.1. *Suppose a hospital ordered first aid kit from a manufacturer during each month for 8 years as follows: If the manufacturer want to forecast the future demand*

Monthly Number of First Aid Kit (in thousands) Consumption for 8 Years												
Year	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
2015	210	165	106	130	350	205	237	391	374	331	362	254
2016	281	161	109	166	320	246	240	355	382	355	354	264
2017	275	148	116	172	287	231	222	353	383	356	386	300
2018	290	178	137	157	310	234	221	352	392	332	383	287
2019	200	159	114	109	301	237	244	389	384	352	381	273
2020	178	169	138	156	345	231	230	346	410	349	367	309
2021	276	146	107	174	289	201	234	353	389	335	375	247
2022	266	155	128	184	298	202	232	323	407	347	381	276

Table 4.1: Data for Demand Type

using inventory modelling, which type of demand he will consider and why?

Example 4.1. *Briefly write down various components of inventory models.*
★ *Inventory models can be identified with lots of components. Here we are briefly discussing its components below:*

► *A)* *Demand Rate*: *It has mainly four types*

1. *Deterministic and constant (static) with time.*
2. *Deterministic and variable (dynamic) with time.*
3. *Probabilistic and stationary over time (i.e. statistical properties are constant over time).*
4. *Probabilistic and nonstationary over time (i.e. statistical properties changes over time).*

► *B)* *Replenishment Type*: *It has mainly two types*

1. *Instant replenishment (rate infinite) i.e. lead time zero.*
2. *Non-instant replenishment (rate finite) i.e. lead time non-zero. So stock gradually increases.*

► *C)* *Shortage Type*: *It has mainly two types*

1. *Shortage not allowed i.e. inventory level never goes below zero.*
2. *Shortage allowed i.e. inventory level may goes below zero.*

► *D)* *Quantity Discount*: *It has mainly two types*

1. *Quantity discount allowed i.e. inventory unit cost vary with respect to quantity.*
2. *Quantity discount not allowed i.e. inventory unit cost does not vary with respect to quantity.*

► *E)* *Cost Related*: *Based on the above components mainly four types of cost used in inventory models:*

1. *Purchase cost.*
2. *Setup cost.*
3. *Holding cost.*
4. *Shortage cost.* □

4.2 Deterministic Inventory Models

◘ Since a general inventory model is very complex and may not be possible to solve analytically. So we will use different models to study various scenerio.

► For Deterministic Inventory Models two broad categories are there: i) Demand rate is constant and ii) Demand rate is variable.

► The most common inventory situation faced by manufacturers, retailers, and wholesalers is that stock levels are depleted over time and then are replenished by the arrival of a batch of new units. A simple model representing this situation is the economic order quantity (EOQ) model.

◘ Some Notations: D = Demand rate (units/time), I = No. of inventories, Q = Order quantity (units) at each ordering cycle.

► Cost Notations: C_1 = Purchase cost per unit , C_2 = Ordering cost (per order), C_3 = Holding cost per inventory per time, C_4 = Shortage cost.

► Time Notations: t = Ordering cycle length (time), T = Total time to hold the inventory.

◘ Some Terms: Lead Time = Time between the placement of an order and its reciepence, Reorder Point = Inventory level at which the order is placed.

4.2.1 EOQ Model - I [A] (Shortage Not Allowed Type)

♦ [A]: Demand Rate Uniform, Instant Replenishment.

An organisation must control its inventory for smooth performance. A basic inventory situation focouses on stock levels that are decreasing with time and stock refill with new units. A simple represenation of this situtaion can be explained through economic order quantity (EOQ) model. The EOQ models are based on some elementary assumptions.

◘ Assumptions: The main assumptions of classical EOQ models are

1. A known constant demand rate (D).
2. Instantaneous order (quantity Q) replenishment (restoration of a stock) i.e. replenish rate infinite. In other words, lead time is 0 i.e. instantaneous order placed and received.
3. No shortages are allowed i.e. Inventory cost = Purchase + Setup + Holding. □

The objective of EOQ model is to determine the frequency of order i.e. when and by how much (i.e. number of unit) we should order the inventory so that it minimizes the sum of these costs per unit time.

■ Objective: The main target of classical EOQ models are to determine

1. The frequency of order i.e. optimal ordering cycle length (t).
2. The number of unit in each order i.e. optimum order quantity units (Q) at each ordering cycle. □

Suppose units of the product under consideration are assumed to be withdrawn from inventory continuously at a known constant rate D per unit time (Assumption 1). It is further assumed that inventory is replenished when needed by ordering a batch of fixed size (Q units), where all Q units arrive simultaneously (Assumption 2) at the desired time (Assumption 3). The time between consecutive replenishments of inventory is referred to as a cycle (t). For the basic EOQ model, the only costs to be considered are: C_1 = Purchase cost per unit, C_2 = Ordering cost (per order), C_3 = Holding cost per inventory per time. For the fixed demand rate, shortages can be avoided by replenishing inventory each time the inventory level drops to zero (See Figure 4.1), and this also will minimize the holding cost. We assume the cost of the items remains constant over time i.e. no quantity discounts.

From the schematic diagram (Figure 4.1) it is observed that, x-axis represents time and y-axis represents inventory level. At the begining, we start fixed Q units (see the line OB) of inventory and as time goes, the inventory decreases with a constant demand rate D i.e. at an instance t_1 the $(Q - D \times t_1)$ units of inventory present in the store. Also the inventory units goes to zero at the end of the cycle (i.e. after time t) it implies $Q - D \times t = 0 \Rightarrow Q = Dt$. So the total holding inventory units during the time t i.e. the cycle length OA is = the area of the triangle OAB = $\frac{Qt}{2}$.

At the point A i.e. after the first cycle having length t, the inventory goes to zero and instant restoration the stock with Q units (see the line AB_1). The process continues upto n orders.

So, the various costs for the first cycle (i.e. per cycle) are:
Purchase cost = Item units × Cost per item = QC_1, Setup Cost = C_2, Holding Cost = Inventory units × Cost per item = $\frac{QtC_3}{2}$.
So the total cost per cycle = $QC_1 + C_2 + \frac{QtC_3}{2} = DC_1t + C_2 + \frac{DC_1C_3t^2}{2}$.
Hence, the total cost per unit time is $C = DC_1 + \frac{C_2}{t} + \frac{DC_1C_3t}{2}$.

Now we will minimize C with respect to t, $\frac{dC}{dt} = 0 \Rightarrow -\frac{C_2}{t^2} + \frac{DC_1C_3}{2} = 0 \Rightarrow \boxed{t = \sqrt{\frac{2C_2}{DC_1C_3}}}$.

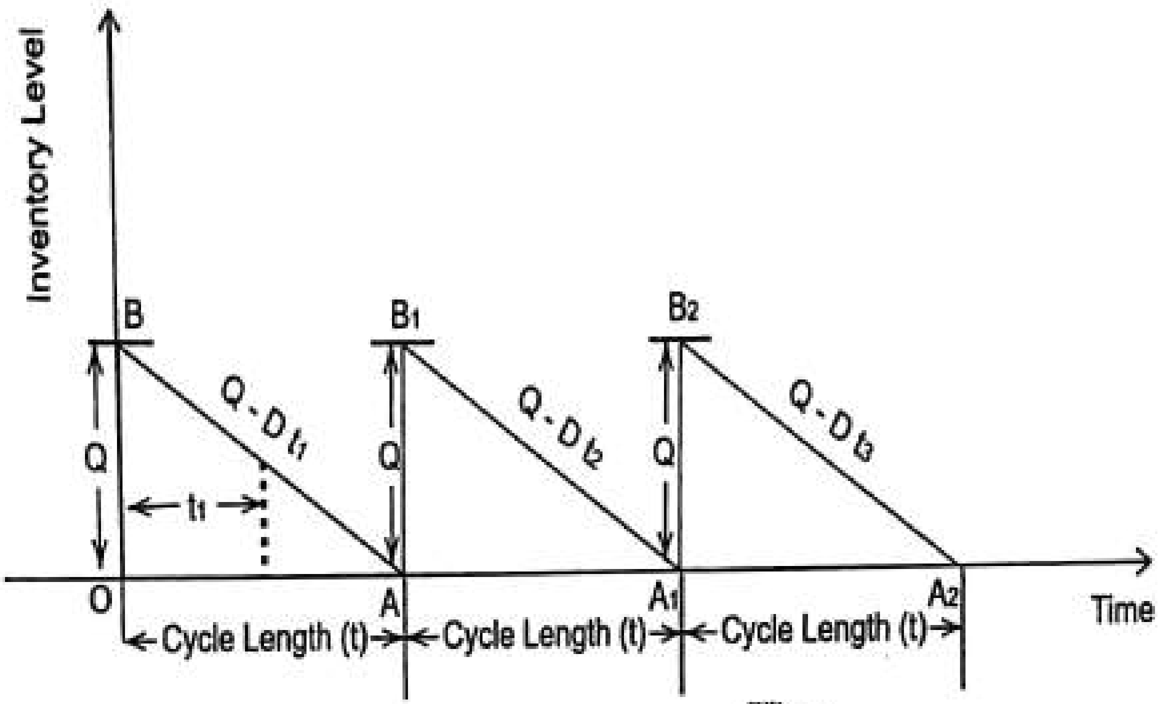


Figure 4.1: Inventory Situation with uniform demand rate and instantaneous order replenishment.

Further, $\frac{d^2C}{dt^2} > 0$ at $t = \sqrt{\frac{2C_3}{DC_1C_3}}$.

The optimal time interval of ordering or, cycle length (say t_{opt}) is $\boxed{t_{opt} = \sqrt{\frac{2C_3}{DC_1C_3}}}$.

The optimal order quantity (or, lot size) (say Q_{opt}) is $\boxed{Q_{opt} = Dt_{opt} = \sqrt{\frac{2C_3D}{C_1C_3}}}$.

The above expression is known as economic order quantity formula or, square root formula.

Example 4.2: *Write down some limitations of EOQ model.*

4.2.2 EOQ Model - I [B] (Shortage Not Allowed Type)

♦ [B]: Demand Rate Non-Uniform, Instant Replenishment.

An organisation must control its inventory for smooth performance. A basic inventory situation focouses on stock levels that are decreasing with time and stock refill with new units. A simple represenation of this situtaion can be explained through economic order quantity (EOQ) model. The EOQ models are based on some elementary assumptions.

■ Assumptions: The main assumptions of the above EOQ models are

1. A non-uniform demand rate at each cycle with total demand D_T.
2. Instantaneous order (quantity Q) replenishment (restoration of a stock) i.e. replenish rate infinite. In other words, lead time is 0 i.e. instantaneous order placed and received.
3. No shortages are allowed i.e. Inventory cost = Purchase + Setup + Holding. □

The objective of EOQ model is to determine the frequency of order i.e. when and by how much (i.e. number of unit) we should order the inventory so that it minimizes the total costs per unit time.

■ Objective: The main target of the EOQ model (I[B]) are to determine

1. The frequency of order i.e. optimal ordering cycle length (t). However, it will not applicable here as cycle lengths are different due to non-uniform demand.
2. The number of unit in each order i.e. optimum order quantity (fixed) units (Q) at each ordering cycle. □

Suppose units of the product under consideration are assumed to be withdrawn from inventory continuously at non-uniform rate $D_1, D_2, \cdots, D_n$ corresponding to the cycle of length $t_1, t_2, \cdots, t_n$ respectively per unit time (Assumption 1). The total time is $T = t_1 + \cdots + t_n$ and $nQ = D_T \Rightarrow n = \frac{D_T}{Q}$. It is further assumed that inventory is replenished when needed by ordering a batch of fixed size (Q units), where all Q units arrive simultaneously (Assumption 2) at the desired time (Assumption 3). The time between consecutive replenishments of inventory is referred to as a cycle and we assume n cycles are there i.e. there will be n orders. For the above EOQ model, the only costs to be considered are: C_1 = Purchase cost per unit, C_2 = Ordering cost (per order), C_3 = Holding cost per inventory per time. For the non-uniform demand rate, shortages can be avoided by replenishing inventory each time the inventory level drops to zero (See Figure 4.2), and this also will minimize the holding cost. We assume the cost of the items remains constant over time i.e. no quantity discounts.

From the schematic diagram (Figure 4.2) it is observed that, x-axis represents time and y-axis represents inventory level. At the begining, we start fixed Q units (see the line OB) of inventory and as time goes, the inventory decreases with a constant demand rate D_1 i.e. at an instance t the $(Q - D_1 \times t)$ units of inventory present in the store. Also the inventory units goes to zero at the end of the cycle (i.e. after time t_1) it implies $Q - D_1 \times t_1 = 0 \Rightarrow Q = D_1 t_1$. So the total holding inventory units during the time t_1 i.e. the cycle length OA is = the area of the triangle OAB = $\frac{Qt_1}{2}$.

At the point A i.e. after the first cycle having length t_1, the inventory goes to zero and instant restoration the stock with Q units (see the line AB_1). The process continues upto n orders.

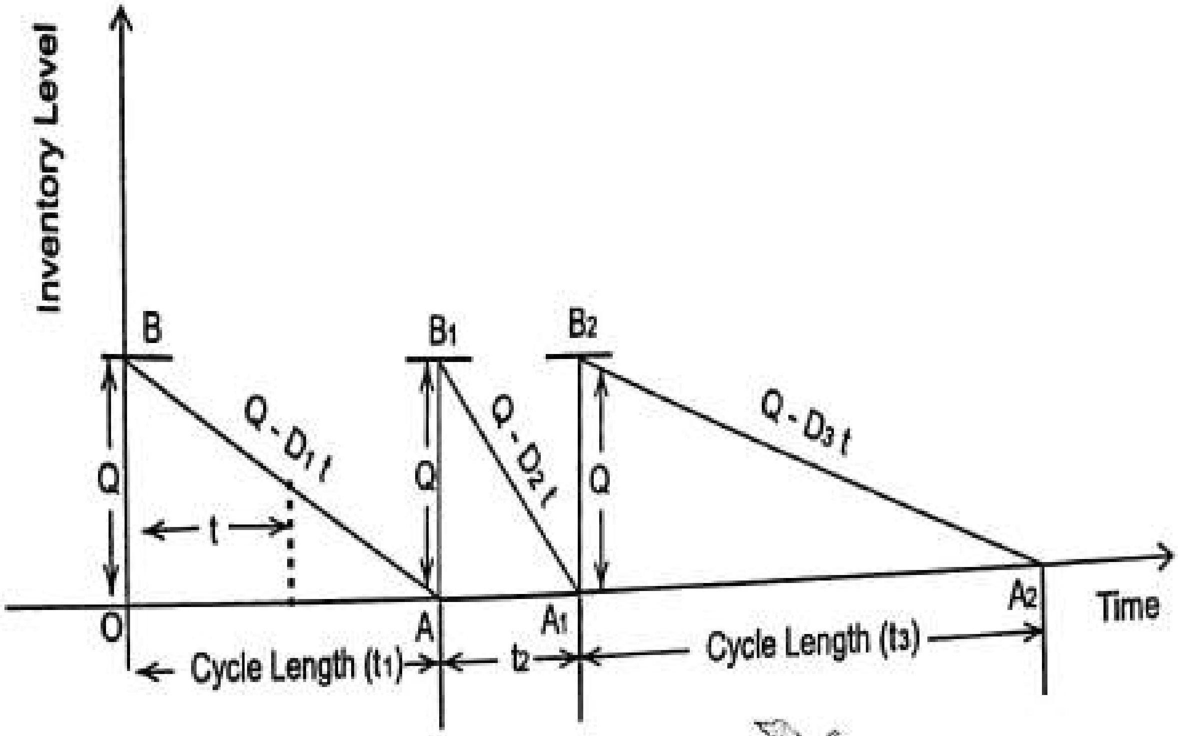


Figure 4.2: Inventory Situation with non-uniform demand rate and instantaneous order replenishment.

So, the various costs for the first cycle (i.e. per cycle) are:
Purchase cost = Item units × Cost per item = QC_1, Setup Cost = C_2, Holding Cost = Inventory units × Cost per item = $\frac{Qt_1C_3}{2}$.
So the total cost per cycle = $QC_1 + C_2 + \frac{Qt_1C_3}{2}$.
Hence, the total cost for n cycles is $C = nQC_1 + nC_2 + \frac{QC_3}{2}(t_1 + \cdots + t_n) = D_TC_1 + \frac{D_TC_2}{Q} + \frac{QC_3T}{2}$.
Now we will minimize C with respect to Q, $\frac{dC}{dQ} = 0 \Rightarrow -\frac{D_TC_2}{Q^2} + \frac{C_3T}{2} = 0 \Rightarrow \boxed{Q = \sqrt{\frac{2D_TC_2}{C_3T}}}$.
Further, $\frac{d^2C}{dQ^2} > 0$ at $Q = \sqrt{\frac{2D_TC_2}{C_3T}}$.
The optimal order quantity (or, lot size) (say Q_{opt}) is $\boxed{Q_{opt} = \sqrt{\frac{2D_TC_2}{C_3T}}}$.
The optimal cost (say C_{opt}) is $\boxed{C_{opt} = D_TC_1 + \frac{D_TC_2}{Q_{opt}} + \frac{Q_{opt}C_3T}{2} = D_TC_1 + \sqrt{2D_TC_2C_3T}}$.

4.2.3 EOQ Model - I [C] (Shortage Not Allowed Type)

♦ [C]: Demand Rate Uniform, Non Instant Replenishment.

An organisation must control its inventory for smooth performance. A basic inventory situation focouses on stock levels that are decreasing with time and stock refill with new units. A simple represenation of this situtaion can be explained through economic order quantity (EOQ) model. The EOQ models are based on some elementary assumptions.

■ **Assumptions**: The main assumptions of the above EOQ models are

1. A uniform demand rate at each cycle with demand rate D.
2. Non Instantaneous order replenishment (restoration of a stock) i.e. replenish rate finite. Here each cycle has two parts: First part starts with zero inventory and inventory increases with a specific rate (I) upto certain time (say t_1) (while demand is simultaneously acting). Second part there will be no supply of inventory, demand continues and after a certain time (say t_2) the inventory goes to zero. So each cycle has length $t = t_1 + t_2$. Same continues in all the cycle.
3. No shortages are allowed i.e. Inventory cost = Purchase + Setup + Holding. □

The objective of EOQ model is to determine the frequency of order i.e. when and by how much (i.e. number of unit) we should order the inventory so that it minimizes the total costs per unit time.

■ **Objective**: The main target of the EOQ model (I[C]) are to determine

1. The optimal cycle length (t).
2. The number of unit in each order i.e. optimum order quantity units (Q) at each cycle. □

Suppose units of the product under consideration are assumed to be withdrawn from inventory continuously at an uniform rate D at each cycle of length t per unit time (Assumption 1). The total item produced is $Q = D \times t$. It is further assumed that in each cycle lenth t, inventory is replenished with a rate I per time ($> D$ as no shortage allowed by Assumption 3) for time t_1, and the remaining inventory units (say Q_r) goes to zero at time t_2 with demand rate D such that $t = t_1 + t_2$. The time between consecutive replenishments of inventory is referred to as a cycle and we assume n cycles are there. For the above EOQ model, the only costs to be considered are: C_1 = Purchase cost per unit, C_2 = Ordering cost (per order), C_3 = Holding cost per inventory per time. For the uniform demand rate, shortages can be avoided by replenishing inventory each time the inventory level drops to zero (See Figure 4.3), and this also will minimize the holding cost. We assume the cost of the items remains constant over time i.e. no quantity discounts.

From the schematic diagram (Figure 4.3) it is observed that, x-axis represents time and y-axis represents inventory level. At the begining, we start from zero units (see the point O) of inventory with replenishment rate I and demand rate D per unit time ($I > D$) upto time t_1. At the point E, the inventory present is $BE = Q_r$ units. After t_1 unit of time, there will be no inventory replenishment, only demand with rate D, an instance t'' the $(Q_r - D \times t'')$ units of inventory present in the store. Also the inventory units goes to zero at the end of the cycle (i.e. after time t_2) it implies $Q_r - D \times t_2 = 0 \Rightarrow Q_r = Dt_2$. Again, at point B, $(I - D)t_1 = Q_r$ i.e. $\frac{I-D}{D} = \frac{t_2}{t_1} \Rightarrow \frac{I}{D} = \frac{t}{t_1} \Rightarrow t_1 = \frac{Dt}{I} \Rightarrow \boxed{Q_r = \frac{(I-D)Dt}{I}}$. So the total holding inventory units during the time $t = t_1 + t_2$ i.e. the cycle length OA

is = the area of the triangle OAB = $\frac{Q_r t}{2}$.

At the point A i.e. after the first cycle having length t, the inventory goes to zero and inventory replenishment with rate I upto AE_1 (demand acts simultaneously) and then inventory decreases due to demand upto E_1A_1. The process continues upto n orders.

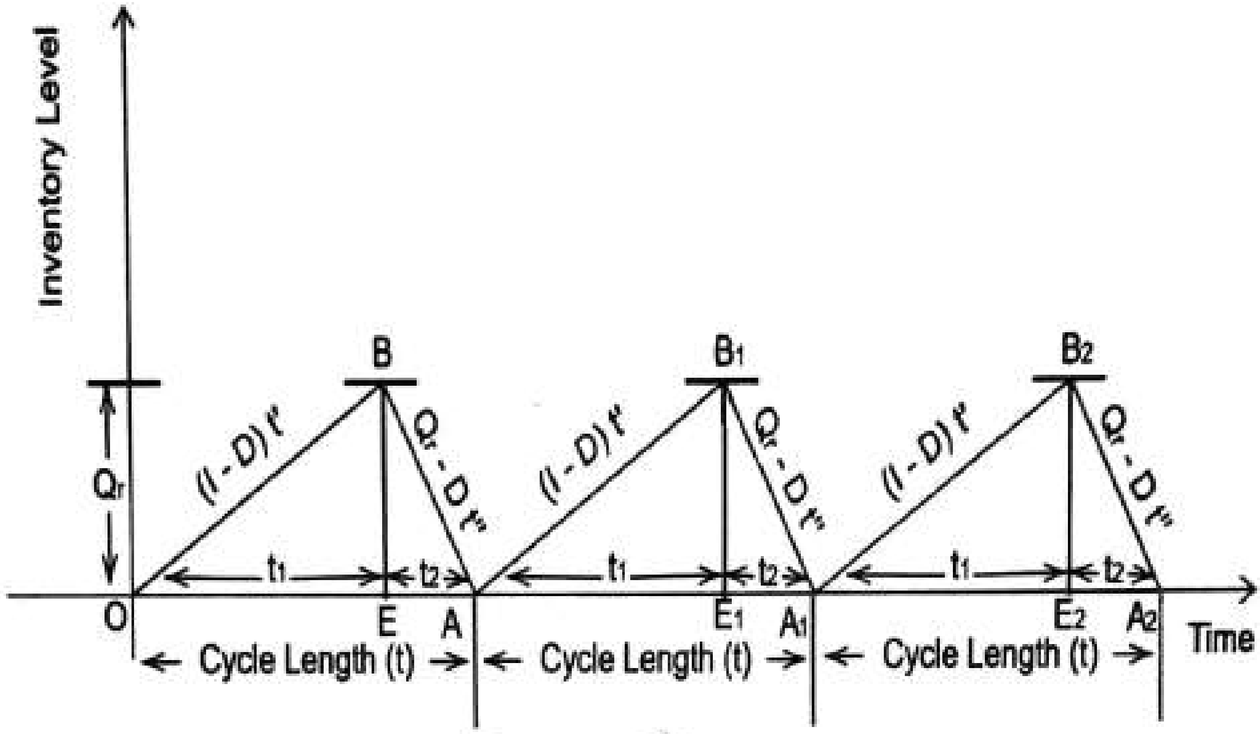


Figure 4.3: Inventory Situation with uniform demand rate and non-instantaneous order replenishment.

So, the various costs for the first cycle (i.e. per cycle) are:
Purchase cost = Item units × Cost per item = DtC_1, Setup Cost = C_2, Holding Cost = Inventory units × Cost per item = $\frac{Q_r t C_3}{2}$.
So the total cost per cycle = $DtC_1 + C_2 + \frac{Q_r t C_3}{2} = DtC_1 + C_2 + \frac{(I-D)Dt^2C_3}{2I}$.
Hence, the total cost per cycle per unit time is $C = DC_1 + \frac{C_2}{t} + \frac{(I-D)DtC_3}{2I}$.
Now we will minimize C with respect to t, $\frac{dC}{dt} = 0 \Rightarrow -\frac{C_2}{t^2} + \frac{(I-D)DC_3}{2I} = 0 \Rightarrow \boxed{t = \sqrt{\frac{2IC_2}{(I-D)DC_3}}}$.
Further, $\frac{d^2C}{dt^2} > 0$ at $t = \sqrt{\frac{2IC_2}{(I-D)DC_3}}$.

The optimal cycle length (say t_{opt}) is $\boxed{t_{opt} = \sqrt{\frac{2IC_2}{(I-D)DC_3}}}$.

The optimal order quantity (or, lot size) (say Q_{opt}) is $\boxed{Q_{opt} = Dt_{opt} = \sqrt{\frac{2IDC_2}{(I-D)C_3}}}$.

The optimal cost per cycle (say C_{opt}) is $\boxed{C_{opt} = Dt_{opt}C_1 + C_2 + \frac{(I-D)Dt_{opt}^2C_3}{2I}}$.

4.2.4 EOQ Model - II [A] (Shortage Allowed Type)

♦ [A]: Demand Rate Uniform, Instant Replenishment, Shortage Allowed.

An organisation must control its inventory for smooth performance. A basic inventory situation focouses on stock levels that are decreasing with time and stock refill with new units. A simple represenation of this situtaion can be explained through economic order quantity (EOQ) model. The EOQ models are based on some elementary assumptions.

◘ Assumptions: The main assumptions of the above EOQ models are

1. A uniform demand rate at each cycle with demand rate D.
2. Instantaneous order (quantity Q) replenishment (restoration of a stock) i.e. replenish rate infinite. In other words, lead time is 0 i.e. instantaneous order placed and received.
3. Shortages are allowed i.e. Inventory cost = Purchase + Setup + Holding + Shortage. Now, the total cycle lenth t can be split into two parts: t_1 i.e. after which inventory remains zero and t_2 i.e. upto that time there is shortage of inventory. In that t_2 time period, the demand accumulated and after t_2 time Q units of replenishment inventory instantly arrives. So $t = t_1 + t_2$ in t_1 part there will be holding cost and no shortage cost, whereas in t_2 part there will be shortage cost and no holding cost. □

The objective of EOQ model is to determine the frequency of order i.e. when and by how much (i.e. number of unit) we should order the inventory so that it minimizes the total costs per unit time.

◘ Objective: The main target of the EOQ model (II[A]) are to determine

1. The optimal cycle length (t).
2. The number of unit in each order i.e. optimum order quantity units (Q) at each cycle.
3. Also the optimum shortage unit in each cycle. □

Suppose units of the product under consideration are assumed to be withdrawn from inventory continuously at an uniform rate D at each cycle of length t per unit time (Assumption 1). The total item ordered is $Q = D \times t$. It is further assumed that in each cycle lenth t, inventory is replenished instantly with Q units (Assumption 2). With a demand rate D per time for time t_1 the inventory goes to zero, and for time t_2 with demand rate D there is a shortage of inventory (as shortage allowed by Assumption 3) such that $t = t_1 + t_2$. The time between consecutive replenishments of inventory is referred to as a cycle and we assume n cycles are there. For the above EOQ model, the only costs to be considered are: C_1 = Purchase cost per unit, C_2 = Ordering cost (per order), C_3 = Holding cost per inventory per time, C_4 = Shortage cost per inventory per time. For the uniform demand rate with shortages usually inventory replenishing each time after there is some shortage upto a certain time (See Figure 4.4), and this also will lower the holding cost. We assume the cost of the items remains constant over time i.e. no quantity discounts.

From the schematic diagram (Figure 4.4) it is observed that, x-axis represents time and y-axis represents inventory level. At the begining, we start from Q' units (see the line OB) with demand rate D per unit time, at time t_1 the Q' unit reduces to zero (at point A). So at the point A, $Q' = Dt_1$. After t_1 unit of time, there will be no inventory, only demand with rate D, so a shortage continue for the time t_2. At the point F, Q unit of inventory (i.e. B_1E) instantly added in the system. So, at point F, $Q - Dt_2 = Q' \Rightarrow Q = Dt$. So the total holding inventory units during the time t_1 is the area of triangle $OAB = \frac{Q't_1}{2}$, inventory shortage during time t_2 is the area of the triangle $AEF = \frac{(Q-Q')t_2}{2}$ and the cycle length OF is $= t = t_1 + t_2$.

▶ Here consider similar triangles EB_1E_1 and FB_1A_1, then we have:

$\frac{B_1F}{B_1E} = \frac{FA_1}{EE_1} \Rightarrow \frac{Q'}{Q} = \frac{t_1}{t} \Rightarrow \boxed{t_1 = \frac{Q't}{Q} = \frac{Q'}{D}}$,

Again $\frac{A_1F_1}{FF_1} = \frac{F_1E_1}{B_1E} \Rightarrow \frac{t_2}{t} = \frac{Q-Q'}{Q} \Rightarrow \boxed{t_2 = \frac{(Q-Q')t}{Q}}$.

At the point F i.e. after the first cycle having length t, the inventory goes to negative value (i.e. deficit EF) and inventory replenishment instantly Q units (i.e. EB_1). The process continues upto n orders.

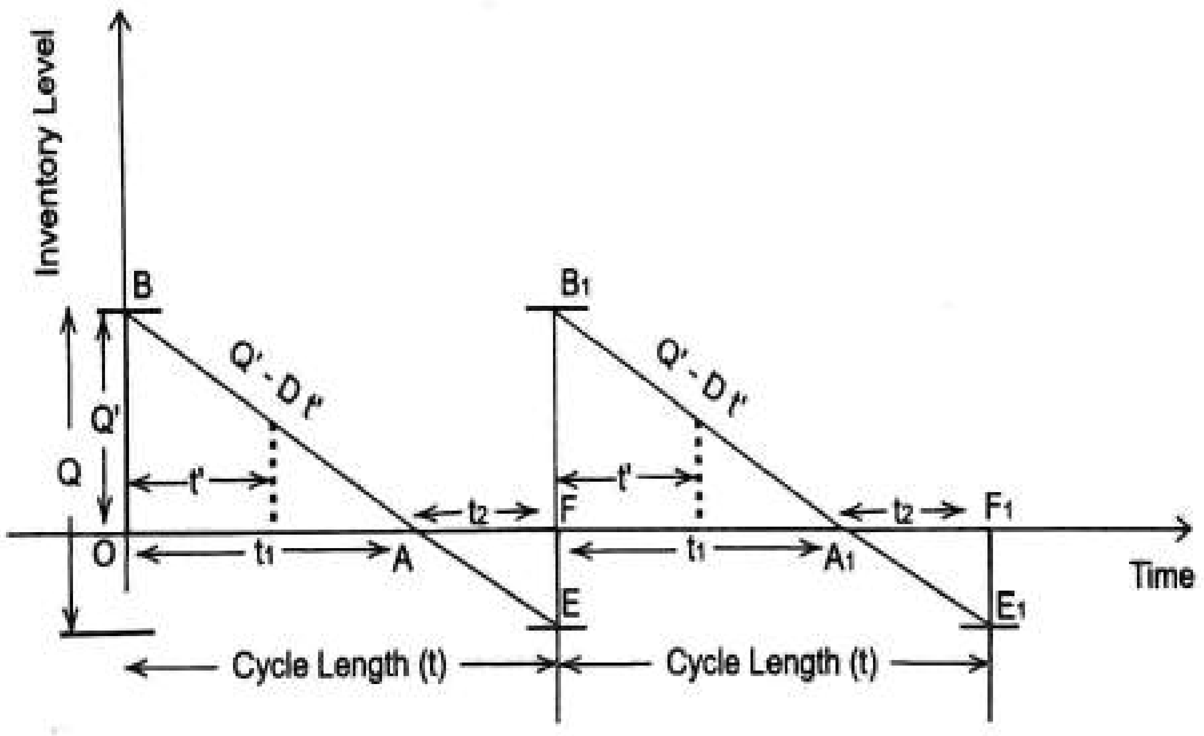


Figure 4.4: Inventory situation with uniform demand rate, instantaneous order replenishment and shortage allowed.

So, the various costs for the first cycle is:
Purchase cost = Item units × Cost per item = QC_1, Setup Cost = C_2, Holding Cost =

Inventory units × Cost per item = $\frac{Q' t_1 C_3}{2}$, Shortage Cost = Inventory shortage units × Shortage cost per item = $\frac{(Q-Q')t_2 C_4}{2}$.

So the total cost per cycle = $QC_1 + C_2 + \frac{Q' t_1 C_3}{2} + \frac{(Q-Q')t_2 C_4}{2} = DtC_1 + C_2 + \frac{Q'^2 t C_3}{2Q} + \frac{(Q-Q')^2 t C_4}{2Q}$.

Hence, the total cost per cycle per unit time is $C = DC_1 + \frac{C_2}{t} + \frac{Q'^2 C_3}{2Q} + \frac{(Q-Q')^2 C_4}{2Q} = DC_1 + \frac{C_2 D}{Q} + \frac{Q'^2 C_3}{2Q} + \frac{(Q-Q')^2 C_4}{2Q}$.

Now we will minimize C with respect to Q and Q', $\frac{\partial C}{\partial Q} = 0$ and $\frac{\partial C}{\partial Q'} = 0$.

Now, $\frac{\partial C}{\partial Q} = 0 \Rightarrow -\frac{C_2 D}{Q^2} - \frac{Q'^2 C_3}{2Q^2} + \frac{C_4}{2} \times \frac{Q.2(Q-Q') - (Q-Q')^2}{Q^2} = 0 \Rightarrow 2C_2 D + Q'^2 C_3 + C_4(Q'^2 - Q^2) = 0 \quad \cdots \quad (1)$.

$\frac{\partial C}{\partial Q'} = 0 \Rightarrow \frac{2Q' C_3}{2Q} - \frac{C_4}{2} \times \frac{2(Q-Q')}{Q} = 0 \Rightarrow Q' C_3 = (Q-Q')C_4 \Rightarrow QC_4 = Q'(C_3+C_4) \quad \cdots \quad (2)$

Solving (1) and (2), we have $(Q^2 - Q'^2)C_4 - Q'^2 C_3 = 2C_2 D \Rightarrow Q'^2\{[\frac{(C_3+C_4)^2}{C_4^2} - 1]C_4 - C_3\} = 2C_2 D \Rightarrow Q'^2\{[\frac{C_3^2 + 2C_3 C_4}{C_4}] - C_3\} = 2C_2 D \Rightarrow Q'^2[\frac{C_3^2 + C_3 C_4}{C_4}] = 2C_2 D \Rightarrow \boxed{Q' = \sqrt{\frac{2C_2 D}{C_3}}\sqrt{\frac{C_4}{C_3+C_4}}}$

So by (2), $\boxed{Q = \sqrt{\frac{2C_2 D}{C_3}}\sqrt{\frac{C_3+C_4}{C_4}}}$.

Further, $\frac{\partial^2 C}{\partial Q^2}\frac{\partial^2 C}{\partial Q'^2} - \frac{\partial^2 C}{\partial Q \partial Q'} > 0$ at $Q = \sqrt{\frac{2C_2 D}{C_3}}\sqrt{\frac{C_3+C_4}{C_4}}$, $Q' = \sqrt{\frac{2C_2 D}{C_3}}\sqrt{\frac{C_4}{C_3+C_4}}$.

The optimal order quantity (or, lot size) (say Q_{opt}) is $\boxed{Q_{opt} = \sqrt{\frac{2C_2 D}{C_3}}\sqrt{\frac{C_3+C_4}{C_4}}}$.

The optimal shortage (say $S_{opt} = Q_{opt} - Q'_{opt}$) is $\boxed{S_{opt} = \sqrt{\frac{2C_2 D}{C_3}}\sqrt{\frac{C_3^2}{C_4(C_3+C_4)}} = \sqrt{\frac{2C_2 D}{C_4}}\sqrt{\frac{C_3}{C_3+C_4}}}$.

The optimal cycle length (say t_{opt}) is $\boxed{t_{opt} = \frac{Q_{opt}}{D} = \sqrt{\frac{2C_2}{C_3 D}}\sqrt{\frac{C_3+C_4}{C_4}}}$.

The optimal cost per cycle (say C_{opt}) is $\boxed{C_{opt} = Dt_{opt}C_1 + C_2 + \frac{Q'^2_{opt} t_{opt} C_3}{2Q_{opt}} + \frac{S^2_{opt} t_{opt} C_4}{2Q_{opt}}}$.

4.2.5 EOQ Model - II [B] (Shortage Allowed Type)

♦ [B]: Demand Rate Uniform, Instant Replenishment, Shortage Allowed, Fixed cycle length.

An organisation must control its inventory for smooth performance. A basic inventory situation focouses on stock levels that are decreasing with time and stock refill with new units. A simple represenation of this situtaion can be explained through economic order quantity (EOQ) model. The EOQ models are based on some elementary assumptions.

□ Assumptions : The main assumptions of the above EOQ models are

1. A uniform demand rate at each cycle with demand rate D.
2. Instantaneous order (quantity Q) replenishment (restoration of a stock) i.e. replen-

ish rate infinite. In other words, lead time is 0 i.e. instantaneous order placed and received.

3. Shortages are allowed i.e. Inventory cost = Purchase + Setup + Holding + Shortage. Now, the total cycle lenth t can be split into two parts: t_1 i.e. after which inventory remains zero and t_2 i.e. upto that time there is shortage of inventory. In that t_2 time period, the demand accumulated and after t_2 time Q units of replenishment inventory instantly arrives. So $t = t_1 + t_2$ in t_1 part there will be holding cost and no shortage cost, whereas in t_2 part there will be shortage cost and no holding cost.
4. Fixed cycle length i.e. t is fixed $\Rightarrow Q$ is fixed as $Q = D \times t$. □

The objective of EOQ model is to determine the frequency of order i.e. when and by how much (i.e. number of unit) we should order the inventory so that it minimizes the total costs per unit time.

☒ Objective: The main target of the EOQ model (II[B]) are to determine

1. The optimal cycle length (t). However, it is not applicable as t is fixed.
2. The number of unit in each order i.e. optimum order quantity units (Q) at each cycle. However, it is not applicable as Q is fixed.
3. Also the optimum shortage unit in each cycle. □

Suppose units of the product under consideration are assumed to be withdrawn from inventory continuously at an uniform rate D at each cycle of length t per unit time (Assumption 1). The total item ordered is $\boxed{Q = D \times t}$. It is further assumed that in each fixed cycle lenth t (Assumption 4), inventory is replenished instantly with Q units (Assumption 2). With a demand rate D per time for time t_1 the inventory goes to zero, and for time t_2 with demand rate D there is a shortage of inventory (as shortage allowed by Assumption 3) such that $t = t_1 + t_2$. The time between consecutive replenishments of inventory is referred to as a cycle and we assume n cycles are there. For the above EOQ model, the only costs to be considered are: C_1 = Purchase cost per unit, C_2 = Ordering cost (per order), C_3 = Holding cost per inventory per time, C_4 = Shortage cost per inventory per time. For the uniform demand rate with shortages usually inventory replenishing each time after there is some shortage upto a certain time (See Figure 4.5), and this also will lower the holding cost. We assume the cost of the items remains constant over time i.e. no quantity discounts.

From the schematic diagram (Figure 4.5) it is observed that, x-axis represents time and y-axis represents inventory level. At the begining, we start from Q' units (see the line OB) with demand rate D per unit time, at time t_1 the Q' unit reduces to zero (at point A). So at the point A, $Q' = Dt_1$. After t_1 unit of time, there will be no inventory, only demand with rate D, so a shortage continue for the time t_2. At the point F, Q unit of inventory (i.e. B_1E) instantly added in the system. So, at point F, $Q - Dt_2 = Q' \Rightarrow Q = Dt$. So the total holding inventory units during the time t_1 is the area of triangle $OAB = \frac{Q't_1}{2}$, inventory shortage during time t_2 is the area of the triangle $AEF = \frac{(Q-Q')t_2}{2}$ and the cycle length OF is $= t = t_1 + t_2$ is fixed.

▶ Here consider similar triangles EB_1E_1 and FB_1A_1, then we have:

$\frac{B_1F}{B_1E} = \frac{FA_1}{EE_1} \Rightarrow \frac{Q'}{Q} = \frac{t_1}{t} \Rightarrow \boxed{t_1 = \frac{Q't}{Q} = \frac{Q'}{D}}$,

Again $\frac{A_1F_1}{FF_1} = \frac{F_1E_1}{B_1E} \Rightarrow \frac{t_2}{t} = \frac{Q-Q'}{Q} \Rightarrow \boxed{t_2 = \frac{(Q-Q')t}{Q}}$.

At the point F i.e. after the first cycle having length t, the inventory goes to negative value (i.e. deficit EF) and inventory replenishment instantly Q units (i.e. EB_1). The process continues upto n orders.

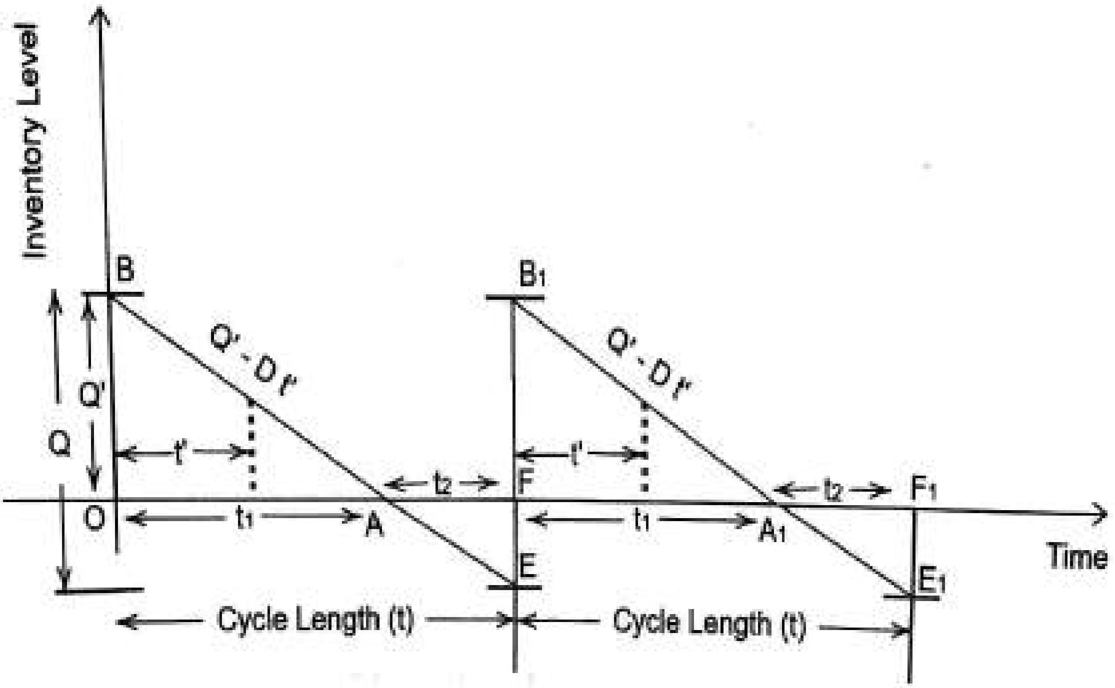


Figure 4.5: Inventory situation with uniform demand rate, instantaneous order replenishment and shortage allowed. Fixed cycle length.

So, the various costs for the first cycle is:
Purchase cost = Item units × Cost per item = QC_1, Setup Cost = C_2, Holding Cost = Inventory units × Cost per item = $\frac{Q't_1C_3}{2}$, Shortage Cost = Inventory shortage units × Shortage cost per item = $\frac{(Q-Q')t_2C_4}{2}$.
So the total cost per cycle = $QC_1 + C_2 + \frac{Q't_1C_3}{2} + \frac{(Q-Q')t_2C_4}{2} = DtC_1 + C_2 + \frac{Q'^2tC_3}{2Q} + \frac{(Q-Q')^2tC_4}{2Q}$,
Hence, the total cost per cycle per unit time is $C = DC_1 + \frac{C_2}{t} + \frac{Q'^2C_3}{2Q} + \frac{(Q-Q')^2C_4}{2Q} = DC_1 + \frac{C_2D}{Q} + \frac{Q'^2C_3}{2Q} + \frac{(Q-Q')^2C_4}{2Q}$.
Since, t is fixed $\Rightarrow Q$ is fixed. So the first two terms of C are constant.
Now we will minimize C with respect to Q', i.e. $\frac{dC}{dQ'} = 0$.
Now, $\frac{dC}{dQ'} = 0 \Rightarrow \frac{2Q'C_3}{2Q} - \frac{C_4}{2} \times \frac{2(Q-Q')}{Q} = 0 \Rightarrow Q'C_3 = (Q-Q')C_4 \Rightarrow QC_4 = Q'(C_3+C_4) \Rightarrow$

$\boxed{Q' = \frac{QC_4}{C_3+C_4}}$.

Further, $\frac{d^2C}{dQ'^2} > 0$ at $Q' = \frac{QC_4}{C_3+C_4}$.

The optimal shortage (say $S_{opt} = Q - Q'_{opt}$) is $\boxed{S_{opt} = Q - \frac{QC_4}{C_3+C_4} = \frac{QC_3}{C_3+C_4} = \frac{DtC_3}{C_3+C_4}}$.

The optimal cost per cycle (say C_{opt}) is $\boxed{C_{opt} = DtC_1 + C_2 + \frac{Q'^2_{opt}tC_3}{2Dt} + \frac{S^2_{opt}tC_4}{2Dt}}$.

4.2.6 EOQ Model - II [C] (Shortage Allowed Type)

♦ [C]: Demand Rate Uniform, Non Instant Replenishment, Shortage Allowed.

An organisation must control its inventory for smooth performance. A basic inventory situation focouses on stock levels that are decreasing with time and stock refill with new units. A simple represenation of this situtaion can be explained through economic order quantity (EOQ) model. The EOQ models are based on some elementary assumptions.

☐ Assumptions: The main assumptions of the above EOQ models are

1. A uniform demand rate at each cycle with demand rate D.
2. Non Instantaneous order replenishment (restoration of a stock) i.e. replenish rate finite. Here each cycle has two parts: First part starts with zero inventory and inventory increases with a specific rate (I) upto certain time (say t_1) (while demand is simultaneously acting). Second part there will be no supply of inventory, demand continues and after a certain time (say t_2) the inventory goes to zero.
3. Shortages are allowed i.e. Inventory cost = Purchase + Setup + Holding + Shortage. Now, the total cycle lenth t can be split into two parts: $t_1 + t_2$ i.e. after which inventory remains zero and $t_3 + t_4$ i.e. upto that time there is shortage of inventory. In that $t_3 + t_4$ shortage time period, upto t_3 time there is a shortage and no supply of inventory. After t_3 the supply of inventory started and upto t_4 time the inventory again goes to zero from the shortage. So $t = t_1 + t_2 + t_3 + t_4$ in $t_1 + t_2$ part there will be holding cost and no shortage cost, whereas in $t_3 + t_4$ part there will be shortage cost and no holding cost. So each cycle has length $t = t_1 + t_2 + t_3 + t_4$. Same continues in all the cycle. ☐

The objective of EOQ model is to determine the frequency of order i.e. when and by how much (i.e. number of unit) we should order the inventory so that it minimizes the total costs per unit time.

☐ Objective: The main target of the EOQ model (II[C]) are to determine

1. The optimal cycle length (t).
2. The number of unit in each order i.e. optimum order quantity units (Q) at each cycle.
3. Also the optimum shortage unit in each cycle. ☐

Suppose units of the product under consideration are assumed to be withdrawn from inventory continuously at an uniform rate D at each cycle of length t per unit time (Assumption 1). The total item ordered is $\boxed{Q = I \times (t_1 + t_4)}$. It is further assumed that in each cycle length t, inventory is replenished with a rate I per time ($> D$ otherwise the system always runs with shortage) for time t_1 (Assumption 2), and the remaining inventory units (say Q_r) goes to zero at time t_2 with the demand rate D. For time t_3 with demand rate D there is a shortage of inventory (as shortage allowed by Assumption 3) with no replinishment rate. Lastly, for time t_4 with demand rate D there is a shortage of inventory with replinishment rate I per time such that at the end of time t_4 the inventory units goes to zero from shortage. The time between consecutive replenishments of inventory is referred to as a cycle and we assume n cycles are there. For the above EOQ model, the only costs to be considered are: C_1 = Purchase cost per unit, C_2 = Ordering cost (per order), C_3 = Holding cost per inventory per time, C_4 = Shortage cost per inventory per time. For the uniform demand rate with shortages usually inventory replenishing with respect to time after there is some shortage upto a certain time (See Figure 4.6), and this also will lower the holding cost. We assume the cost of the items remains constant over time i.e. no quantity discounts.

From the schematic diagram (Figure 4.6) it is observed that, x-axis represents time and y-axis represents inventory level. At the begining, we start from zero units (see the point O) of inventory replenishment rate I and demand rate D per unit time ($I > D$), upto time t_1. At the point E, the inventory present is $BE = Q_r$ units. After t_1 unit of time, there will be no inventory replenishment, only demand with rate D, an instance t'' the $(Q_r - D \times t'')$ units of inventory present in the store. Also the inventory units goes to zero at the end of the cycle (i.e. after time t_2) it implies $Q_r - D \times t_2 = 0 \Rightarrow$ $\boxed{Q_r = Dt_2 \quad \cdots \quad (1)}$. Again, at point B, $\boxed{(I - D)t_1 = Q_r \quad \cdots \quad (2)}$. So the Q_r unit reduces to zero at point A. After $t_1 + t_2$ unit of time (i.e. after point A), there will be no inventory, only demand with rate D, so a shortage continue for the time t_3 with a deficit $FH = S$ units. After the point H, inventory added with rate I units per time upto t_4 time and the inventory units again goes to zero (see point G) from the shortage. So at the point A, $Q_r - Dt_2 = 0$, $\boxed{S - Dt_3 = 0 \quad \cdots \quad (3)}$. Again, at point H, $Dt_3 = (I - D)t_4$. Also at G, $\boxed{S - (I - D)t_4 = 0 \quad \cdots \quad (4)}$. From (1), (3) we have $\boxed{Q_r + S = D(t_2 + t_3) \quad \cdots \quad (5)}$, and from (2), (4) we have $\boxed{Q_r + S = (I - D)(t_1 + t_4) \quad \cdots \quad (6)}$. So from (5), (6) we have $\boxed{D(t_2 + t_3) = (I - D)(t_1 + t_4) \quad \cdots \quad (7)}$. Further, from (6) we have $\boxed{Q_r + S = \frac{I-D}{I} Q \quad \cdots \quad (8)}$.

So the total holding inventory units during the time $t_1 + t_2$ is the area of triangle $OAB = \frac{Q_r(t_1+t_2)}{2}$, inventory shortage during time $t_3 + t_4$ is the area of the triangle AHG $= \frac{S(t_3+t_4)}{2}$ and cycle length OG is $= t = t_1 + t_2 + t_3 + t_4 = (t_1 + t_4)(1 + \frac{I-D}{D})$. [using (7)] It implies $t = \frac{Q}{I} \times \frac{I}{D} = \frac{Q}{D}$ [using $Q = I(t_1 + t_4)$] i.e. $\boxed{t = \frac{Q}{D} \quad \cdots \quad (9)}$.

At the point G i.e. after the first cycle having length t, the inventory goes to zero and inventory replenishment with rate I per units per time. The process continues upto n orders.

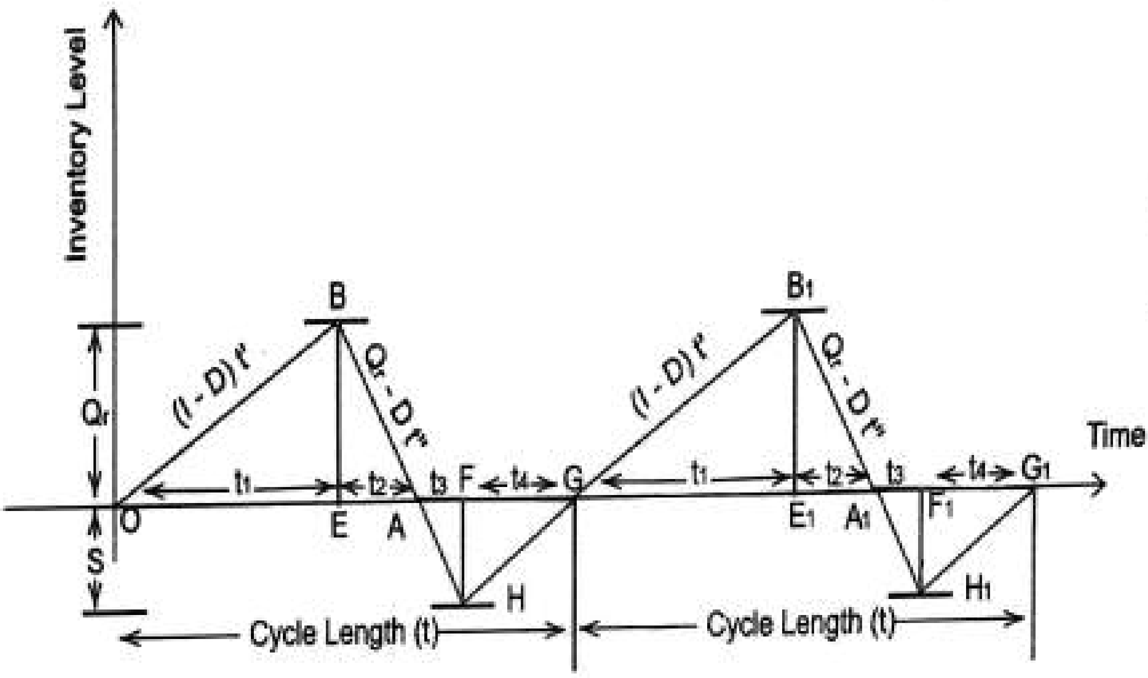


Figure 4.6: Inventory situation with uniform demand rate, non-instantaneous order replenishment and shortage allowed.

So, the various costs for the first cycle is:
Purchase cost = Item units × Cost per item = QC_1, Setup Cost = C_2, Holding Cost = Inventory units × Cost per item = $\frac{Q_r(t_1+t_2)C_3}{2}$, Shortage Cost = Inventory shortage units × Shortage cost per item = $\frac{S(t_3+t_4)C_4}{2}$.
So the total cost per cycle is $C = QC_1 + C_2 + \frac{Q_r(t_1+t_2)C_3}{2} + \frac{S(t_3+t_4)C_4}{2} = DtC_1 + C_2 + \frac{Q_rC_3}{2}(\frac{Q_r}{I-D} + \frac{Q_r}{D}) + \frac{SC_4}{2}(\frac{S}{D} + \frac{S}{I-D})$ [Using (1)-(4),(9)].
It implies, $C = DtC_1 + C_2 + \frac{Q_r^2C_3}{2}\frac{I}{D(I-D)} + \frac{S^2C_4}{2}\frac{I}{D(I-D)}$

Hence, the total cost per cycle per unit time is $C = DC_1 + \frac{C_2}{t} + \frac{Q_r^2C_3}{2t}\frac{I}{D(I-D)} + \frac{S^2C_4}{2t}\frac{I}{D(I-D)}$. Now there is relation (8) between Q and Q_r so we will replace Q_r by Q. We also replace t using (9), so we get

Therefore, $\boxed{C = DC_1 + \frac{DC_2}{Q} + \frac{[\frac{I-D}{I}Q-S]^2C_3}{2Q}\frac{I}{(I-D)} + \frac{S^2C_4}{2Q}\frac{I}{(I-D)}}$

Now C has two unknown components S and Q. So we will minimize C with respect to S and Q, i.e. $\frac{\partial C}{\partial S} = 0$ and $\frac{\partial C}{\partial S} = 0$.
Now, $\frac{\partial C}{\partial S} = 0 \Rightarrow -\frac{2[\frac{I-D}{I}Q-S]C_3}{2Q}\frac{I}{(I-D)} + \frac{2SC_4}{2Q}\frac{I}{(I-D)} = 0 \Rightarrow [\frac{I-D}{I}Q-S]C_3 = SC_4 \Rightarrow \frac{I-D}{I}Q = \frac{C_3+C_4}{C_3}S \Rightarrow \frac{S}{Q} = \frac{C_3}{C_3+C_4}\frac{I-D}{I} \quad \cdots \quad (10)$.
$\frac{\partial C}{\partial Q} = 0 \Rightarrow -\frac{DC_2}{Q^2} + [(\frac{I-D}{I})^2 - \frac{S^2}{Q^2}]\frac{C_3I}{2(I-D)} - \frac{S^2C_4}{2Q^2}\frac{I}{(I-D)} = 0 \Rightarrow \frac{DC_2}{Q^2} = C_3\frac{I-D}{2I} - \frac{I}{2(I-D)} \times$

$\frac{S^2}{Q^2}(C_3 + C_4) \Rightarrow \frac{DC_2}{Q^2} = C_3\frac{I-D}{2I} - \frac{I}{2(I-D)} \times \frac{C_3^2}{(C_3+C_4)^2}\frac{(I-D)^2}{I^2}(C_3 + C_4) \Rightarrow \frac{DC_2}{Q^2} = C_3\frac{I-D}{2I} - \frac{I-D}{2I} \times \frac{C_3^2}{C_3+C_4} \Rightarrow \frac{DC_2}{Q^2} = \frac{I-D}{2I} \times \frac{C_3C_4}{C_3+C_4} \Rightarrow \boxed{Q = \sqrt{2C_2\frac{C_3+C_4}{C_3C_4}} \times \sqrt{\frac{DI}{I-D}}}$.

So by (10), $\boxed{S = \sqrt{\frac{2C_2C_3}{C_4(C_3+C_4)}} \times \sqrt{\frac{D(I-D)}{I}}}$.

Further, $\frac{\partial^2 C}{\partial Q^2}\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 C}{\partial Q \partial S} > 0$ at $Q = \sqrt{2C_2\frac{C_3+C_4}{C_3C_4}} \times \sqrt{\frac{DI}{I-D}}$, $S = \sqrt{\frac{2C_2C_3}{C_4(C_3+C_4)}} \times \sqrt{\frac{D(I-D)}{I}}$.

The optimal order quantity (or, lot size) (say Q_{opt}) is $\boxed{Q_{opt} = \sqrt{2C_2\frac{C_3+C_4}{C_3C_4}} \times \sqrt{\frac{DI}{I-D}}}$.

The optimal cycle length (say t_{opt}) is $\boxed{t_{opt} = \frac{Q_{opt}}{D} = \sqrt{2C_2\frac{C_3+C_4}{C_3C_4}} \times \sqrt{\frac{I}{D(I-D)}}}$.

The optimal shortage (say $SH_{opt} = Dt_3 + (I-D)t_4$, now use the relation at point H) is $\boxed{SH_{opt} = 2Dt_3 = 2S_{opt} = 2\sqrt{\frac{2C_2C_3}{C_4(C_3+C_4)}} \times \sqrt{\frac{D(I-D)}{I}}}$.

The optimal cost per cycle (say C_{opt}) is $\boxed{C_{opt} = DC_1 + \frac{DC_2}{Q_{opt}} + \frac{[\frac{I-D}{I}Q_{opt} - S_{opt}]^2C_3}{2Q_{opt}}\frac{I}{(I-D)} + \frac{S_{opt}^2C_4}{2Q_{opt}}\frac{I}{(I-D)}}$.

4.3 Quantity Discount Model

❑ Here the quantity cost reduced if larger quantity ordered. This is also known as quantity discount or, price break models. For example, if we buy 5000 pens then the cost Rs. 10 per pen, for 5001 to 8000 the cost is Rs. 9.5 per pen and 8000 − 15000 the cost is Rs. 9.25 per pen.

► Now in each cycle what will be the optimal quantity unit such that maximum quantity discount avail with the minimum storage cost. Also the optimum cycle lenth is equally important.

4.3.1 Discount EOQ Model - I [A] (No Shortage)

♦ [A]: Demand Rate Uniform, Instant Replenishment.

An organisation must control its inventory for smooth performance. A basic inventory situation focouses on stock levels that are decreasing with time and stock refill with new units. A simple represenation of this situtaion can be explained through economic order quantity (EOQ) model. The EOQ models are based on some elementary assumptions.

❑ Assumptions: The main assumptions of classical EOQ models are

1. A known constant demand rate (D).
2. Instantaneous order (quantity Q) replenishment (restoration of a stock) i.e. replenish rate infinite. In other words, lead time is 0 i.e. instantaneous order placed and received.

3. No shortages are allowed i.e. Inventory cost = Purchase + Setup + Holding. □

The objective of EOQ model is to determine the frequency of order i.e. when and by how much (i.e. number of unit) we should order the inventory so that it minimizes the sum of these costs per unit time.

■ Objective: The main target of classical EOQ models are to determine

1. The frequency of order i.e. optimal ordering cycle length (t).
2. The number of unit in each order i.e. optimum order quantity units (Q) at each ordering cycle. □

Suppose units of the product under consideration are assumed to be withdrawn from inventory continuously at a known constant rate D per unit time (Assumption 1). It is further assumed that inventory is replenished when needed by ordering a batch of fixed size (Q units), where all Q units arrive simultaneously (Assumption 2) at the desired time (Assumption 3). The time between consecutive replenishments of inventory is referred to as a cycle (t). For the basic EOQ model, the only costs to be considered are: C_1 = Purchase cost per unit (later we will modify as quantity discount), C_2 = Ordering cost (per order), C_3 = Holding cost per inventory per time. For the fixed demand rate, shortages can be avoided by replenishing inventory each time the inventory level drops to zero (See Figure 4.7), and this also will minimize the holding cost. We assume the cost of the items are not constant over time i.e. there are quantity discounts.

From the schematic diagram (Figure 4.7) it is observed that, x-axis represents time and y-axis represents inventory level. At the begining, we start fixed Q units (see the line OB) of inventory and as time goes, the inventory decreases with a constant demand rate D i.e. at an instance t_1 the $(Q - D \times t_1)$ units of inventory present in the store. Also the inventory units goes to zero at the end of the cycle (i.e. after time t) it implies $Q - D \times t = 0 \Rightarrow \boxed{Q = Dt}$. So the total holding inventory units during the time t i.e. the cycle length OA is = the area of the triangle OAB = $\frac{Qt}{2}$.

At the point A i.e. after the first cycle having length t, the inventory goes to zero and instant restoration the stock with Q units (see the line AB_1). The process continues upto n orders.

So, the various costs for the first cycle (i.e. per cycle) are:
Purchase cost = Item units × Cost per item = QC_1, Setup Cost = C_2, Holding Cost = Inventory units × Cost per item = $\frac{QtC_3}{2}$.
So the total cost per cycle = $QC_1 + C_2 + \frac{QtC_3}{2} = DC_1t + C_2 + \frac{DC_3t^2}{2}$.
Hence, the total cost per unit time is $C = DC_1 + \frac{C_2}{t} + \frac{DC_3t}{2} = DC_1 + \frac{C_2D}{Q} + \frac{C_3Q}{2}$.

Now we will minimize C with respect to t, $\frac{dC}{dt} = 0 \Rightarrow -\frac{C_2}{t^2} + \frac{DC_3}{2} = 0 \Rightarrow \boxed{t = \sqrt{\frac{2C_2}{DC_3}}}$.

Further, $\frac{d^2C}{dt^2} > 0$ at $t = \sqrt{\frac{2C_2}{DC_3}}$.

The optimal time interval of ordering or, cycle length (say t_{opt}) is $\boxed{t_{opt} = \sqrt{\frac{2C_2}{DC_3}}}$.

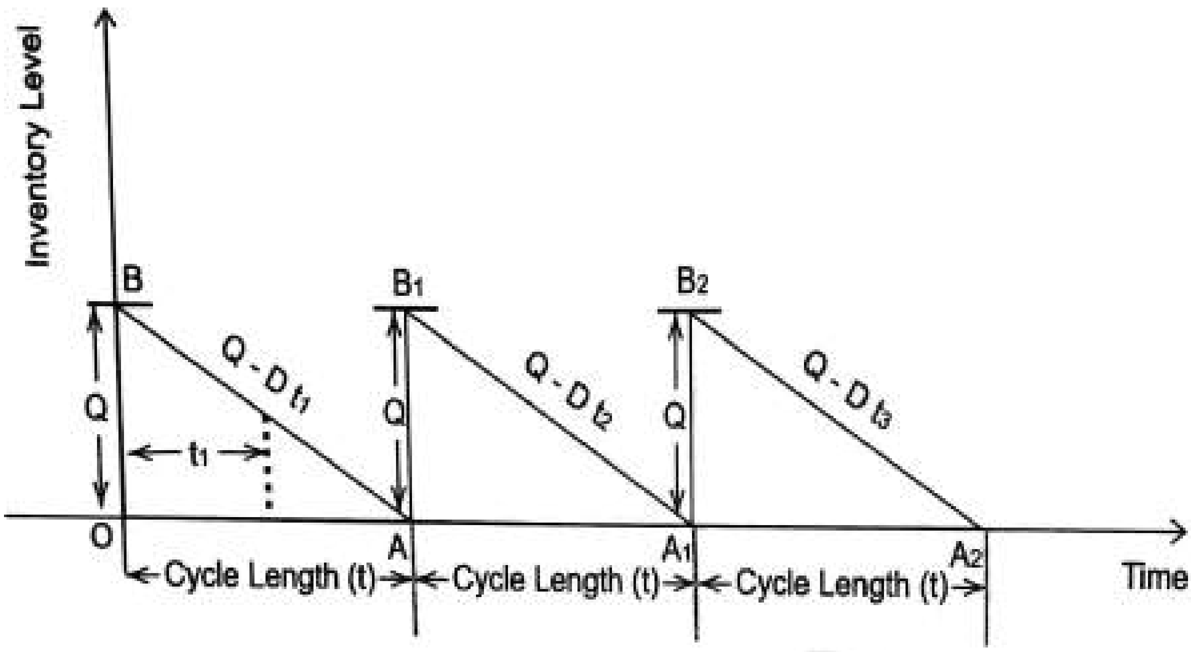


Figure 4.7: Inventory Situation with uniform demand rate and instantaneous order replenishment.

The optimal order quantity (or, lot size) (say Q_{opt}) is $\boxed{Q_{opt} = Dt_{opt} = \sqrt{\frac{2C_3D}{C_3}}}$.

☐ **Inclusion of Quantity Discount :** Suppose there is a quantity discount i.e. the purchase cost C_1 vary with respect to Q. Suppose for purchasing $[0, Q_1)$ unit the price is C_{11} per unit, $[Q_1, Q_2)$ unit the price is C_{12} per unit, $[Q_2, \infty)$ unit the price is C_{13} per unit. Here $0 < Q_1 < Q_2$ and $C_{11} > C_{12} > C_{13}$.
So we can write,

$$C_1 = \begin{cases} C_{11} & \text{if} Q \in [0, Q_1) \\ C_{12} & \text{if} Q \in [Q_1, Q_2) \\ C_{13} & \text{if} Q \in [Q_2, \infty) \end{cases}$$

We know, the total cost per unit time is $C = DC_1 + \frac{C_3D}{Q} + \frac{C_3Q}{2}$.

Hence,

$$C = \begin{cases} DC_{11} + \frac{C_3D}{Q} + \frac{C_3Q}{2} & \text{if} Q \in [0, Q_1) \\ DC_{12} + \frac{C_3D}{Q} + \frac{C_3Q}{2} & \text{if} Q \in [Q_1, Q_2) \\ DC_{13} + \frac{C_3D}{Q} + \frac{C_3Q}{2} & \text{if} Q \in [Q_2, \infty) \end{cases}$$

Now the optimal order quantity is $Q_{opt} = \sqrt{\frac{2C_3D}{C_3}}$ for all the prices (as Q_{opt} are

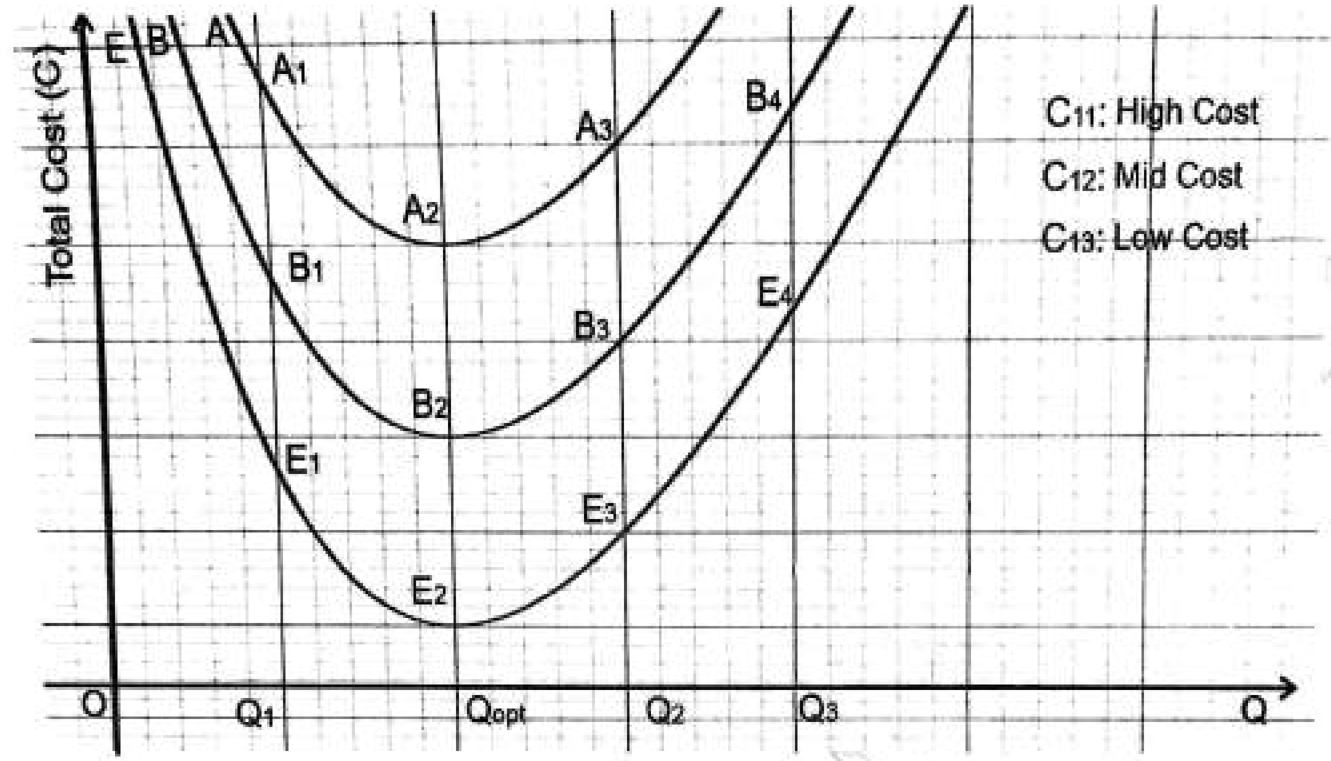


Figure 4.8: Inventory Situation with uniform demand rate and instantaneous order replenishment. Quantity discount applicable with three rates $C_{11} = 11$ (*Green*), $C_{12} = 10$ (*Blue*), $C_{13} = 9$ (*Red*). The other parameter values are $D = 1, C_2 = 20, C_3 = 5$,

independent of C_1). So there are three possibiliteis : i) $Q_{opt} \in (0, Q_1]$, ii) $Q_{opt} \in (Q_1, Q_2]$, iii) $Q_{opt} \in (Q_2, Q_3]$.
The situation is graphically explained by taking some hypothetical values of the parameters (see Fig. 4.8).
► Note that, cost function C minimizes at Q_{opt} i.e. either $AA_1A_2A_3$ (green) curve or, $BB_1B_2B_3B_4$ (blue) curve or, $EE_1E_2E_3E_4$ (red) curve satisfies the Q_{opt}.
► Case I: If $Q_{opt} \in [0, Q_1)$
So the Q_{opt} is valid for the quantity discount cost C_{11} i.e. total cost for C_{11} and Q_{opt} is $C_\alpha = DC_{11} + \frac{C_2 D}{Q_{opt}} + \frac{C_3 Q_{opt}}{2}$. However, we also need to check the total cost for the next two available discount price C_{12}, C_{13} with the minimum quantity Q_1 i.e. $C_\beta = DC_{12} + \frac{C_2 D}{Q_2} + \frac{C_3 Q_2}{2}$ and $C_\gamma = DC_{13} + \frac{C_2 D}{Q_2} + \frac{C_3 Q_2}{2}$.
Next we select Q_{opt} or, Q_2 or, Q_3 at which $\min\{C_\alpha, C_\beta, C_\gamma\}$ occurs. Hence either Q_{opt} or, Q_1 or, Q_2 will be the optimum order quantity.
► Case II: If $Q_{opt} \in [Q_1, Q_2)$
So the Q_{opt} is valid for the quantity discount cost C_{12} blue curve (Fig. 4.8) i.e. total cost for C_{12} and Q_{opt} is $C_\alpha = DC_{12} + \frac{C_2 D}{Q_{opt}} + \frac{C_3 Q_{opt}}{2}$. However, we also need to check the total cost for the next available discount price C_{13} with the minimum quantity Q_2 i.e. $C_\beta = DC_{13} + \frac{C_2 D}{Q_2} + \frac{C_3 Q_2}{2}$.
Next we select Q_{opt} or, Q_2 at which $\min\{C_\alpha, C_\beta\}$ occurs. Hence either Q_{opt} or, Q_2 will be

the optimum order quantity.

▶ Case III: If $Q_{opt} \in [Q_2, \infty)$

So the Q_{opt} is valid for the quantity discount cost C_{13} i.e. total cost for C_{13} and Q_{opt} is $C_\alpha = DC_{12} + \frac{C_2 D}{Q_{opt}} + \frac{C_3 Q_{opt}}{2}$. Hence Q_{opt} will be the optimum order quantity.

□ In a similar way, it can be generalized upto k breaks for the discount cost C_1.

Example 4.3. *Find the optimal order quantity for a product when the annual demand for the product is 500 units. The storage (holding) cost per unit per year is 10% of the unit cost and the ordering cost per order is Rs. 180. The unit costs (in Rs.) are [JNTU 2011]*

$$C_1 = \begin{cases} 25 & if Q \in [0, 500) \\ 24.80 & if Q \in [500, 1500) \\ 24.60 & if Q \in [1500, 3000) \\ 24.40 & if Q \in [3000, \infty) \end{cases}$$

□ *It is given that, Annual demand rate i.e.* $D = 500$, *Ordering cost per order i.e.* $C_2 = 180$ *and holding cost per unit per year i.e.* $C_3 = 0.1 \times$ *unit cost.*

Now for unit price 24.4, we have $Q_{opt} = \sqrt{\frac{2C_2 D}{C_3}} = \sqrt{\frac{2\times 180\times 500}{0.1\times 24.4}} = 271.6$.

This is not feasible as $Q = 271.6$ *is not available for cost 24.4.*

For unit price 24.6, we have $Q_{opt} = \sqrt{\frac{2\times 180\times 500}{0.1\times 24.6}} = 270.5$, *(not feasible).*

For unit price 24.8, we have $Q_{opt} = \sqrt{\frac{2\times 180\times 500}{0.1\times 24.8}} = 269.4$, *(not feasible).*

For unit price 25, we have $Q_{opt} = \sqrt{\frac{2\times 180\times 500}{0.1\times 25}} = 268.3$, *(feasible).*

Note that, Q_{opt} *is the value at which total cost C minimizes.*

So the total annual cost for 268.3 unit with price 25 is $C^1 = DC_{11} + \frac{C_2 D}{Q^1_{opt}} + \frac{C_3 Q^1_{opt}}{2} = 500 \times 25 + \frac{180\times 500}{268.3} + \frac{0.1\times 25\times 268.3}{2} = 13,170.82$.

We also compare the above price with minimum quantity with next lower price break i.e. the total annual cost for 500 unit with price 24.8 is $C^2 = DC_{12} + \frac{C_2 D}{Q_2} + \frac{C_3 Q_2}{2} = 500 \times 24.8 + \frac{180\times 500}{500} + \frac{0.1\times 24.8\times 500}{2} = 13,200$.

We again compare the above price with minimum quantity with next lower price break i.e. the total annual cost for 1500 unit with price 24.6 is $C^3 = DC_{13} + \frac{C_2 D}{Q_3} + \frac{C_3 Q_3}{2} = 500 \times 24.6 + \frac{180\times 500}{1500} + \frac{0.1\times 24.6\times 1500}{2} = 14,205$.

Finally, we compare the above price to the minimum quantity with next lower price break i.e. the total annual cost for 3000 unit with price 24.4 is $C^4 = DC_{14} + \frac{C_2 D}{Q_4} + \frac{C_3 Q_4}{2} = 500 \times 24.4 + \frac{180\times 500}{3000} + \frac{0.1\times 24.4\times 3000}{2} = 15,890$.

As the minimum cost is $13,170.82$ *it implies the optimum order quantity is* 268.3 *i.e.* 269 *units.*

Example 4.4. *Find the optimal order quantity for a product for which the price breaks as follows [CheU 2002]*

$$C_1 = \begin{cases} 10 & if Q \in [0, 500) \\ 9.25 & if Q \in [500, 750) \\ 8.75 & if Q \in [750, \infty) \end{cases}$$

The monthly demand for the product is 200 units, storage cost is 2% of the unit cost and cost of ordering is Rs. 100.

☐ *It is given that, Annual demand rate i.e.* $D = 200$ *units, Ordering cost per order i.e.* $C_2 = 100$ *and holding cost per unit per month i.e.* $C_3 = 0.02 \times$ *unit cost.*
Now for unit price 8.75, *we have* $Q_{opt} = \sqrt{\frac{2C_2D}{C_3}} = \sqrt{\frac{2\times100\times200}{0.02\times8.75}} = 478.1$.
This is not feasible as $Q = 478.1$ *is not available for cost Rs.* 8.75.
For unit price 9.25, *we have* $Q_{opt} = \sqrt{\frac{2\times100\times200}{0.02\times9.25}} = 465$, *(not feasible).*
For unit price 10, *we have* $Q_{opt} = \sqrt{\frac{2\times100\times200}{0.02\times10}} = 447.2$, *(feasible).*
Note that, Q_{opt} *is the value at which total cost* C *minimizes.*
So the total annual cost for 447.2 *unit with price* 10 *is* $C^1 = DC_{11} + \frac{C_2D}{Q^1_{opt}} + \frac{C_3Q^1_{opt}}{2} = 200\times10 + \frac{100\times200}{447.2} + \frac{0.02\times10\times447.2}{2} = 2089.44$.
We also compare the above price with minimum quantity with next lower price break i.e. the total annual cost for 500 *unit with price* 9.25 *is* $C^2 = DC_{12} + \frac{C_2D}{Q_2} + \frac{C_3Q_2}{2} = 200\times 9.25 + \frac{100\times200}{500} + \frac{0.02\times9.25\times500}{2} = 1936.25$.
Finally, we compare the above price to the minimum quantity with next lower price break i.e. the total annual cost for 3000 *unit with price* 24.4 *is* $C^4 = DC_{14} + \frac{C_2D}{Q_4} + \frac{C_3Q_4}{2} = 200\times8.75 + \frac{100\times200}{750} + \frac{0.02\times8.75\times750}{2} = 1842.29$.
As the minimum cost is 1842.29 *it implies the optimum order quantity is* 750 *units.*

4.4 ABC Inventory System

☐ ABC analysis (or, Always Better Control Analysis) is an inventory management technique that determines the value of inventory items based on their importance to the business. ABC ranks items on demand, cost and risk data, and inventory mangers group items into classes based on those criteria. This helps business leaders understand which products or services are most critical to the financial success of their organization.
▶ The most important stock keeping units (SKUs), based on either sales volume or profitability, are Class A items, the next-most important are Class B and the least important are Class C. Some companies may choose a classification system that breaks products into more than just those three groups (A-F, for example).

4.4.1 Pareto's Principle (The 80/20 Rule)

☐ The Pareto principle was developed by Italian economist Vilfredo Pareto in 1896. Pareto observed that 80% of the land in Italy was owned by only 20% of the population. He also witnessed this happening with plants in his garden 20% of his plants were bearing 80% of the fruit. This relationship is best mathematically described as a power law distribution between two quantities, in which a change in one quantity results in a relevant change into the other.

▶ Some examples based on Pareto's Principle:

▷ 80% of a company's profits come from 20% of customers.

▷ 20% of a plant contains 80% of the fruit.

▷ 80% of exam preparation is done at last 20% of the study holidays.

▷ 80% of all wealth in the world lies with only 20% of the population.

▷ 20% of grocery items amounts to 80% total bill.

▷ 20% of the sports people win 80% of matches.

▷ 80% of the crimes are committed by 20% of the population.

▶ The 80/20 rule is not a formal mathematical equation, but more a generalized phenomenon that can be observed in economics, business, time management, and even sports.

◘ The basis of the Pareto principle states that 80% of results come from 20% of actions. If you have any kind of work that can be segmented into smaller portions, the Pareto principle can help you identify what part of that work is the most influential.

▶ Imagine you work at an ecommerce company. You take a look at 100 of your most recent customer service complaints, and notice that the bulk of the complaints come from the fact that customers are receiving damaged products. Your team calculates the amount of refunds given for your damaged products and finds that approximately 80% of refunds given were for damaged products. Your company wants to avoid processing refunds for broken products, so you make this problem a priority solution. Your team decides to update packaging to protect your products during shipping, which resolves the issue of customers receiving damaged products. Here we have used Pareto's principle for decision making.

Question 4.3. *How ABC Analysis Relates to the Pareto Principle?*

★ *The Pareto Principle says that most results come from only 20% of efforts or causes in any system. Based on Pareto's 80/20 rule, ABC analysis identifies the 20% of goods that deliver about 80% of the value.*

Therefore, most businesses have a small number of 'A' items, a slightly larger group of 'B' and a big group of 'C' items. The Pareto Principle may not always be completely accurate. However, analysis shows that valuable things do tend to bend toward an 80/20 distribution. ABC analysis identifies the 'A' items where most of a business's revenue comes from with relatively little effort. □

4.4.2 ABC Analysis

◘ Conduct ABC inventory analysis by multiplying the annual sales of a certain item by its cost. The results tell you which goods are high priority and which yield a low profit, so you know where to focus human and capital resources.

Question 4.4. *Write short notes on ABC Analysis.*

★ *The ABC analysis is based on Paretos law that a few high usage value items constitute a major part of the capital invested in inventories, whereas bulk of items having low usage value constitute insignificant part of the capital.*

It classify all the inventory items into three categories based on their usage values. Items of high usage value but small in number are classified as 'A' items and would be under strict control of top level management. Items of moderate value and size are classified as 'B' items and would attract reasonable attention of the middle level management and 'C' items are large in number but require little capital and would be under simple control. ABC analysis is also known as 'proportional value analysis'.
Usually, inventory items in most organisations show the following distribution patterns:
A : 5 − 10% of the total number of items accounting for 70 − 80% of the annual usage value,
B : 10 − 20% accounting for about 15 − 20% of the annual usage value, and
C : 70 − 80% of the number of items accounting for 5 − 15% of the annual usage value.
In view of the several hundred and even thousand items stocked in most inventory situations, the ABC analysis may be carried out on a sample. Once a random sample has been obtained, the following steps may be performed for the ABC analysis:

1. *Annual Usage Value: Calculate the annual usage value for every item in the sample: Annual Usage Value = Annual Requirement Units × Unit Cost.*
2. *Ordering: Arrange the above Annual Usage Values in descending order.*
3. *Cumulative Annual Usage & Items: Next, the cummulative total number of items and the annual usage value.*
4. *Cumulative Percentage Annual Usage & Items: Convert the cummulative totals of items and annual usage values into percentages.*
5. *Plot Lorentz Curve: Plot the two cumulative percentages. The curve obtained is called ABC distribution curve or, Pareto curve or, Lorentz curve.*
6. *Select Cut-off Points in Lorentz Curve: Mark the cut-off points X and Y where the curve changes its slope, dividing it into three segments A, B and C. These segments A, B and C for the sample are then generalised over the entire population of stock items.*

Under ABC analysis, an organisation would devote much time and effort in the control of 'A' items. It has higher inventory costs and be procured in smaller lots. 'B' items are usually placed under statistical control and attract periodic control of the management. (s, S) inventory control system might be used for these items. 'C' items require very little capital and hence have low inventory carrying costs and should be bought in bigger lots so that there are fewer orders and hence lower acquisition cost and also to take advantage of quantity discounts for bulk purchases. A fixed-order quantity system may be used for such items. □

Example 4.5. *Twelve items kept in inventory are listed below. Which items should be classified as A, B and C items? What percentage of items is in each class? What percent-*

age of total annual value is in each class? [CA 1983]

Items with required units and cost per unit					
Item No.	*Units*	*Cost (Rs.)*	*Item No.*	*Units*	*Cost (Rs.)*
1	7000	5	7	60000	0.2
2	24000	3	8	3000	3.5
3	1500	10	9	300	8
4	600	22	10	29000	0.4
5	38000	1.5	11	11500	7.1
6	40000	0.5	12	4100	6.2

⇒ *We will perform the following steps for the ABC analysis:*

1. *Annual Usage Value*: *Calculate the annual usage value for every item in the sample:*
Annual Usage Value = Annual Requirement Units × Unit Cost

2. *Ordering*: *Arrange the above Annual Usage Values in descending order.*

3. *Cumulative Annual Usage & Items*: *Next, the cummulative total number of items and the annual usage value.*

4. *Cumulative Percentage Annual Usage & Items*: *Convert the cummulative totals of items and annual usage values into percentages.*

5. *Plot Lorentz Curve*: *Plot the two cumulative percentages. The curve obtained is called ABC distribution curve or, Pareto curve or, Lorentz curve.*

6. *Select Cut-off Points in Lorentz Curve*: *Mark the cut-off points X and Y where the curve changes its slope, dividing it into three segments A, B and C. These segments A, B and C for the sample are then generalised over the entire population of stock items.*

First we will perform steps 1, 2 in the following table:

Item No.	*Units*	*Cost (Rs.)*	*Annual Usage*	*Ranking*
1	7000	5	35000	4
2	24000	3	72000	2
3	1500	10	15000	7
4	600	22	13200	8
5	38000	1.5	57000	3
6	40000	0.5	20000	6
7	60000	0.2	12000	9
8	3000	3.5	10500	11
9	300	8	2400	12
10	29000	0.4	11600	10
11	11500	7.1	81650	1
12	4100	6.2	25420	4

Now to perform steps 3,4 we first write the table with respect to ranking and then find cumulative and cumulative percentage.

Item No.	*Annual Usage*	*Cumulative Annual Usage (C_3)*	*Percentage Annual Usage ($C_4 = C_3/355770$)*	*Cumulative Items*	*Percentage Items (C_6)*
11	81650	81650	22.95	1	8.33
2	72000	153650	43.18	2	16.66
5	57000	210650	59.20	3	25
1	35000	245650	69.04	4	33.33
12	25420	271070	76.19	5	41.66
6	20000	291070	81.81	6	50
3	15000	306070	86.03	7	58.33
4	13200	319270	89.74	8	66.66
7	12000	331270	93.11	9	75
10	11600	342870	96.37	10	83.33
8	10500	353370	99.32	11	91.66
9	2400	355770	100	12	100

Next to perform step 5 i.e. to draw Lorentz curve, we plot percentage items (C_6) in x-axis and percentage annual usage (C_4) in y-axis. Then we connected the points to visualize the curve (see Fig. 4.9). To perform step 6, we have identified the cut-off points X and Y where the curve changes its slope, dividing it into three segments A, B and C. Here the item number 11, 2, 5 are classified as A, item number 1, 12, 6, 3 are classified as B and item number 4, 7, 10, 8, 9 are classified as C. In the above table we have classified these three groups as red, green and blue colours.

☐ *Using the above table we observe that: In class A, 25% items present; In class B, 33.33% (= 58.33 − 25) items present; In class C, 41.67% (= 100 − 58.33) items present;*

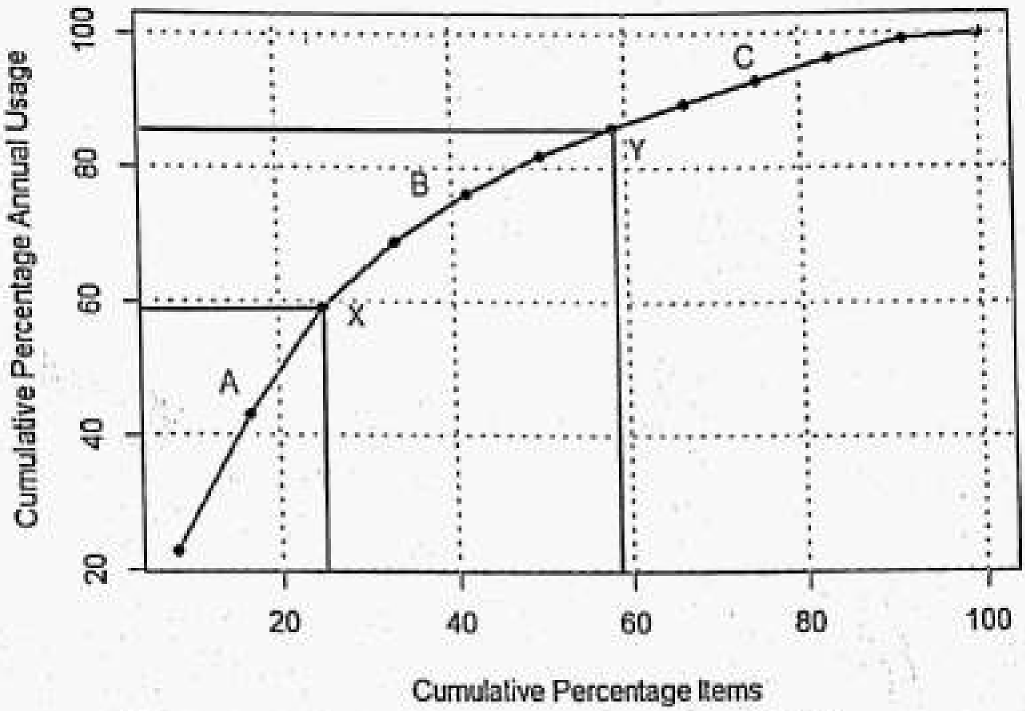


Figure 4.9: Lorentz curve for the example 4.5.

□ *Using the above table we observe that: In class A,* 59.2% *of total annual value is present; In class B,* 26.83% (= 86.03 − 59.20) *items present; In class C,* 13.97% (= 100 − 86.03) *items present;*

[Do It Yourself] 4.2. *The following information is known about a group of items. Classify the items as A, B and C. [CA 1980]*

Items with annual consumption units and cost per unit					
Item No.	*Units*	*Cost (Rs.)*	*Item No.*	*Units*	*Cost (Rs.)*
1	*30000*	*0.1*	*6*	*220000*	*0.1*
2	*280000*	*0.15*	*7*	*15000*	*0.05*
3	*3000*	*0.1*	*8*	*80000*	*0.05*
4	*110000*	*0.05*	*9*	*60000*	*0.15*
5	*4000*	*0.05*	*10*	*8000*	*0.1*

[Do It Yourself] 4.3. *The following information is known about a group of items. Which items should be classified as A, B and C items? What percentage of items is in each class? What percentage of total annual value is in each class?. [ICWA 1991]*

Chapter 2

Game Theory & Networking

2.1 Game theory

In the world of business, conflicts of interest often arise between competitors operating in the same field. To illustrate this, consider two businessmen, A and B, who are players in a game of business. Each has several executives, with A having A_1, A_2, and A_3, and B having B_1, B_2, B_3, and B_4. To control their respective businesses, both players may only utilize the services of one executive at a time. The selection of a particular executive is a strategy, and choosing one strategy, ignoring the strategy taken by the opponent, is a pure strategy. There are a total of seven executives to choose from, so there are seven possible pure strategies for each player. Here we make two assumptions: (i) A is a maximizing player, while B is a minimizing player; and (ii) the total gain of one player is exactly equal to the total loss of the other, resulting in a net gain of zero i.e. zero sum game.

2.1.1 Rectangular game

The possible outcomes of the game can be represented in a matrix, known as a payoff matrix, where each cell represents the gain or loss for each player, depending on the strategies chosen. To illustrate this, let us assume that the gain or loss for each player is measured in terms of rupee. Suppose that the payoff matrix is as follows:

	B_1	B_2	B_3	B_4
A_1	$10, -10$	$-5, 5$	$-2, 2$	$-3, 3$
A_2	$-5, 5$	$8, -8$	$-4, 4$	$-1, 1$
A_3	$-2, 2$	$-4, 4$	$6, -6$	$-3, 3$

In this matrix, the first number in each cell represents the gain for player A, while the second number represents the loss for player B. For example, if A chooses A_1 and B chooses B_1, then A gains Rs. 10, while B loses Rs. 10.

■ In general, if the player A takes m pure strategies and B takes n pure strategies, then the game is called two person zero sum game or, $m \times n$ rectangular game (zero sum as the total gain of one player is exactly equal to the total loss of the other). If $m = n$ then the game is called a square game.

■ As the pay-off matrix of the player B (minimizing payer) is the negative of the pay-off matrix of A (maximizing player). So we can write the pay-off matrix (*say* M) in terms of A (maximizing player) is:

	B_1	B_2	B_3	B_4
A_1	10	-5	-2	-3
A_2	-5	8	-4	-1
A_3	-2	-4	6	-3

► Similarly the pay-off matrix of B is $-M$.

► In general, the pay-off matrix of A is

	B_1	B_2	$\cdots$	B_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{12}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

There are $m \times n$ elements a_{ij} in the pay-off matrix which are the gains obtained by A from the pure moves of A and $B's$ i.e. A_i and B_j respectively for $[i = 1, \cdots, m;\ j = 1, \cdots, n]$.

■ **Pay-off Matrix** : A pay-off matrix is a real matrix (a_{ij}) indicates the gain of the maximization player (Row player) for using the i^{th} and i^{th} move of the row (A) and column (B) players respectively.

2.1.2 Minimax-Maximin Principle

■ To find the optimal strategies for each player, we use minimax theorem, which states that in a zero-sum game, the optimal strategy for a maximizing player (i.e. A) is to choose the strategy that maximizes their minimum gain, while the optimal strategy for a minimizing player is to choose the strategy that minimizes their maximum loss.

► When the equality condition holds, we say that the game problem is solved and the value of the game is: 'maximum of the minimum gains' for A = 'minimum of the maximum losses' for B. Assuming the existence of the value of the game, if the value of the game be the element at of the pay-off matrix, the point (k, l) is called the saddle point of the pay-off matrix.

► There may be more than one saddle point in a pay-off matrix of a game with pure strategy. The principle of determination of the value of a game and saddle point or points is known as the maximin minimax principle. Again if the value of the game be V_{kl} for A and B's pure moves A_k and B_l respectively then the strategies A_k and B_l taken by both players are called the optimal strategies for the players.

Example 2.1. *Find the value of the following game (pure strategy) for a 3 × 4 pay-off matrix*

	B_1	B_2	B_3	B_n
A_1	4	6	−2	1
A_2	3	3	4	2
A_3	4	5	5	1

for the maximizing player A by using maximin-minimax principle.

⇒ *Here A's pure moves are A_1, A_2 and A_3 and B's pure moves are B_1, B_2, B_3 and B_4 Now each player has every right to select a pure move according to his own suitability.*

Now for A's pure move A_1, the minimum gain is −2 units (whatever strategy may be taken by B). Similarly, for A's pure move A_2 and A_3 minimum possible gains are 2 and 1 unit respectively. Therefore, as A has every right to select a move which will maximize his minimum gains, he will definitely select the pure move (strategy) A_2 and for that his gain will be 2 units. His gain will not be less than 2 units at any case.

Now considering B's point of view, if B selects his pure move B_1 the maximum loss will be 4 units (independent of A's move). Similarly, for B's pure moves B_2, B_3 and B_4 the maximum amount of losses will be 6, 5 and 2 units respectively. Therefore, as the ultimate aim of B is to minimize his maximum losses, then B will definitely select the pure strategy B_4 and the minimum loss will be 2 units. His amount of loss cannot be increased any further. But it is interesting to note that the 'maximum of the minimum gains' for A is equal to the 'minimum of the maximum losses' for B. Hence in this pay-off matrix or, game the value of game exists and is 2 units. The optimal strategies are A_2 and B_4 for A and B respectively and $(2,4)^{th}$ position of the pay-off matrix is the saddle point of the pay-off matrix.

▶ In general, there exists no saddle point and the value of the game cannot be determined by the above principle.
▶ There may be more than one saddle point in a particular game and in that case optimal strategies are not unique.

[Do It Yourself] 2.1. *Solve the given games: i)*

	B_1	B_2	B_3
A_1	4	−2	1
A_2	3	4	2
A_3	−3	4	0

and ii)

	B_1	B_2	B_3
A_1	6	3	−3
A_2	−2	1	2
A_3	5	4	6

by using maximum and minimum principle.

[Do It Yourself] 2.2. *Show that the game*

	B_1	B_2	B_3
A_1	6	0	−4
A_2	−3	2	−1
A_3	4	−3	5

does not have any saddle point.

[Do It Yourself] 2.3. *Show that the game*

	B_1	B_2	B_3	B_4
A_1	4	2	3	5
A_2	−2	−1	4	−3
A_3	5	2	3	3
A_4	4	0	0	1

have multiple saddle points. [CU 90,93]

[Do It Yourself] 2.4. *Each of two players A and B shows one, two and three fingers simultaneously. The player B pays to A an amount equal to the two times total number of fingers shown, on the other hand A pays to B equal to the product of the numbers of finger shown. Form the pay-off matrix.*

▫ **Fair Game**: If the value of the game be zero, i.e., no loss or gain for any player then the game is called a fair game.

■ Strictly Determinate Game : If the value of game is a non-zero quantity then the game is called as strictly determinate game. If the value of game is positive (negative) then the game is in favour of the player A (B).

Theorem 2.1. *Suppose $f(x, y)$ be a real valued function of x, y defined for $x \in A$, $y \in B$, where $A, B \subseteq \mathbb{R}$. Now if both* $\max_{x\in A}\min_{y\in B} f(x,y)$ *and* $\min_{y\in B}\max_{x\in A} f(x,y)$ *exist then show that* $\max_{x\in A}\min_{y\in B} f(x,y) \leq \min_{y\in B}\max_{x\in A} f(x,y)$.

Proof. As $f(x,y) \leq \max_{x\in A} f(x,y)$ and $\min_{y\in B} f(x,y) \leq f(x,y) \Rightarrow \min_{y\in B} f(x,y) \leq \max_{x\in A} f(x,y)$.
Now, $\min_{y\in B} f(x,y)$ is independent of $y \Rightarrow \min_{y\in B} f(x,y) \leq \min_{y\in B}\max_{x\in A} f(x,y)$ ... (1).
Now from (1) we have, $\max_{x\in A}\min_{y\in B} f(x,y) \leq \min_{y\in B}\max_{x\in A} f(x,y)$. □

[Do It Yourself] 2.5. *Let $[a_{ij}]_{m\times n}$ be the pay-off matrix for a two-person zero-sum game. Then using Theorem 2.1 prove that,* $\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}$, *where* $i = 1(1)m$, $j = 1(1)n$. [Hint : $A = \{1, 2, \cdots, m\}$, $B = \{1, 2, \cdots, n\}$, $f(i,j) = a_{ij}$]

2.1.3 Mixed Strategy

■ Generally, a game problem cannot be solved by using pure strategies. However, it can be solved by using another technique known as mixed strategy. This technique can solve all two-person zero-sum games when $\nexists$ saddle point.

■ In pure strategy, both players A and B select only a single move at a time say A_i and B_j, at their discretion, irrespective of the move taken by other. In mixed strategy both players A and B select their m and n moves simultaneously.

► Define a set of m positive quantities $p_1, p_2, \cdots, p_m$ with $\sum_{i=1}^{m} p_i = 1$. Now to obtain a least possible gain A may utilize the service of the moves $A_1, A_2, \ldots, A_m$ in such way that A_i performs p_i times of the total work will be done by the moves $A_1, A_2, \ldots, A_m$. Similarly if $q_1, q_2, \cdots, q_n$ be a set of positive quantities such that $\sum_{j=1}^{n} q_j = 1$ then B may utilize the services of the moves $B_1, B_2, \cdots, B_n$ in such way that B_j performs q_j times of the total work will be done by the moves $B_1, B_2, \ldots, B_n$.

► The quantities p_i and q_j associated with the i^{th}, j^{th} move of A and B respectively are called the probabilities of the respective moves.

► We now define two variable vectors $\underset{\sim}{p} = (p_1, p_2, \cdots, p_m)$ in E^m and $\underset{\sim}{q} = (q_1, q_2, \cdots, q_n)$ in E^n. It is always possible to determine some particular value of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ say p^* and q^* such that the value of the game can be determined. Here it is possible to determine the same value of 'maximum of the minimum expected gain for A and the minimum of the maximum expected loss for B'.

■ Pay off function : Let $[a_{ij}]_{m\times n}$ be the pay-off matrix for a two-person zero-sum game. Then the pay-off function or, mathematical expectation of a game which is defined as $E(\underset{\sim}{p}, \underset{\sim}{q}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} p_i q_j$, where $\underset{\sim}{p}, \underset{\sim}{q}$ are the mixed strategies for A and B respectively.

► In matrix notation, $E(\underset{\sim}{p}, \underset{\sim}{q}) = \underset{\sim}{p}' A \underset{\sim}{q}$, where $A = [a_{ij}]$ the pay-off matrix.

► If B takes his pure j^{th} move only then the expected gain of A is given by $E_j(p) = \sum_{i=1}^{m} a_{ij}p_i,\ j = 1, 2, \cdots, n.$
► Similarly for particular i^{th} pure move of A only, the expected loss of B is given by $E_i(q) = \sum_{j=1}^{n} a_{ij}q_j,\ i = 1, 2, \cdots, m.$
■ For any p, A is sure that his expected winning will be at least $\min_q E(p, q)$. He then maximizes the expression over p, so that his expected winnings will be at least $\max_p \min_q E(p, q)$.

Example 2.2. *Find the value of the game*

	B_1	B_2
A_1	2	3
A_2	4	−1

algebraically by using mixed strategies.

⇒ *The problem has no saddle point for pure strategy in the pay-off matrix for A.*
Let us try to solve the problem by using mixed strategies $p = (p_1, p_2)$ *and* $q = (q_1, q_2)$ *with* $p_1 + p_2 = 1,\ p_1, p_2 > 0$ *and* $q_1 + q_2 = 1,\ q_1, q_2 > 0$ *for A and B respectively. Assuming the existence of the value of game we have*
$E_1(p) = 2p_1 + 4p_2 = 2p_1 + 4(1 - p_1),\ E_2(p) = 3p_1 - p_2 = 3p_1 - (1 - p_1).$
To determine the optimal values of p_1, p_2 *we have,* $2p_1 + 4(1 - p_1) = v = 3p_1 - (1 - p_1).$
Solving we get, $p_1^* = 5/6(> 0)$. *So,* $p_2^* = 1 - p_1^* = 1/6$ *and value of the game is* $v = 2p_1^* + 4(1 - p_1^*) = 7/3.$
Again considering from the B's point of view we have,
$E_1(q) = 2q_1 + 3q_2 = 2q_1 + 3(1 - q_1),\ E_2(q) = 4q_1 - q_2 = 4q_1 - (1 - q_1).$
To determine the optimal values of q_1, q_2 *we have,* $2q_1 + 3(1 - q_1) = v = 4q_1 - (1 - q_1).$
Solving we get $q_1^* = 2/3(> 0)$ *and* $q_2^* = 1 - q_1^* = 1/3$ *and the value of the game is* $v = 2q_1^* + 3(1 - q_1^*) = 7/3.$
Hence the optimal strategies are $p^* = (5/6, 1/6)$ *and* $q^* = (2/3, 1/3)$ *and* $v = 7/3$.

► It can be verified that $E(q^*, p^*) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_i^*q_j^* = 2 * \frac{5}{6} * \frac{2}{3} + 4 * \frac{1}{6} * \frac{2}{3} + 3 * \frac{5}{6} * \frac{1}{6} - 1 * \frac{1}{6} * \frac{1}{3} = \frac{7}{3}.$

[Do It Yourself] 2.6. *Find the value of the following 2×2 games by using mixed strategies.*

	B_1	B_2
A_1	4	2
A_2	1	5

	B_1	B_2
A_1	a	$-b$
A_2	$-c$	d

with $a, b, c, d > 0$.

2.1.4 Graphical Method

■ We already solved the 2 × 2 game without any saddle point through algebraically. Although it is not possible to easily solve any $m \times n$ game algebraically. However, by using graph, it is possible to reduce any rectangular game of order $2 \times n$ or, $m \times 2$ to a

2×2 game and then solve it by algebraic method.

► A $2 \times n$ games looks like

	B_1	B_2	$\cdots$	B_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}

Without any saddle point, let the mixed strategies used by A be $p = (p_1, p_2)$ and B be $q = (q_1, q_2, \cdots, q_n)$. Then the net expected gain of a A when B plays his pure strategy B_j is given by $E_j(p) = a_{1j}p_1 + a_{2j}p_2 = a_{1j}p_1 + a_{2j}(1 - p_1)$, $j = 1, 2, \cdots, n$. As $p_1 + p_2 = 1$, so both p_1 and p_2 must lie in the open interval $(0, 1)$ [because if either p_1 or $p_2 = 1$ the game reduces to a game of pure strategy which is against our assumption]. Hence $E_j(p)$ is a linear function of either p_1 or p_2. Considering $E_j(p)$ as a linear function of p_1 (say), we have from the limiting values $(0, 1)$ of p, $E_j(p) = a_{2j}$ if $p_1 = 0$ and $E_j(p) = a_{1j}$ if $p_1 = 1$. Therefore, Hence $E_j(p)$ represents a line segment joining the points $(0, a_{2j})$ and $(1, a_{1j})$.

► Steps: 1. Draw two parallel vertical lines with distance one unit length. Left one represents the line $p_1 = 0$ and the right one is $p_1 = 1$.

2. Draw n line segments joining the points $(0, a_{2j})$ and $(1, a_{1j})$, $j = 1, 2, \cdots, n$. The lower envelope of these line segments is the minimum expected gain of A as a function of p_1 and the highest point of the lower envelope will give the maximum of minimum gain of A.

3. The line segments passing through the point corresponding to B's two pure moves say B_k and B_l are the critical moves for B which will maximize the minimum expected gain of A.

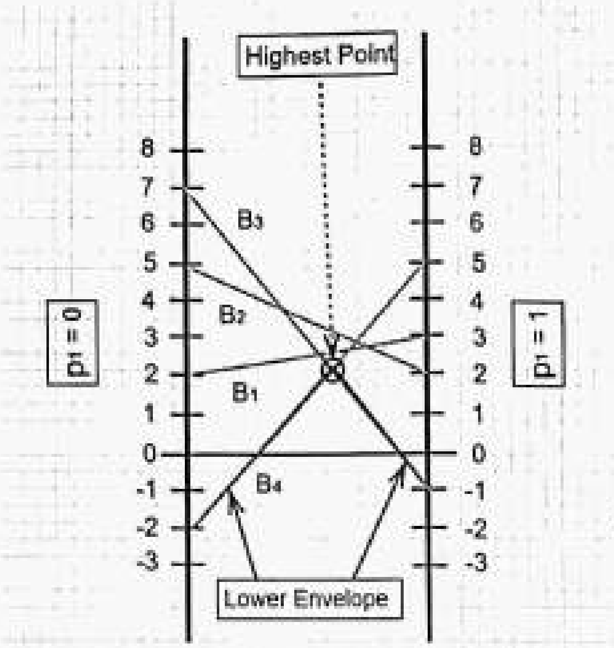
4. Now after finding 2×2 pay-off matrix corresponding to A's moves A_1 and A_2 and B's moves B_k and B_l, we can solve the pay-off matrix algebraically and find the value of the game.

Example 2.3. *Reduce the following $2 \times n$ game*

	B_1	B_2	B_3	B_4
A_1	3	2	-1	5
A_2	2	5	7	-2

into a 2×2 game using graphical method and hence solve it algebraically.

⇒ *The given game does not have any saddle point, let the mixed strategies used by A be $p = (p_1, p_2)$ with $p_1 + p_2 = 1$ and both p_1, p_2 lie in the open interval $(0, 1)$. Also the mixed strategies used by B be $q = (q_1, q_2, q_3, q_4)$ with $q_1 + q_2 + q_3 + q_4 = 1$.*



Draw two vertical lines $p_1 = 0$, $p_1 = 1$ at unit distance apart. Draw a line segment joining $(0, 2)$, $(1, 3)$. This line correspomding to strategy B_1. It represents the expected gain of A due to B's pure moves B_1. Similarly, we can draw lines B_2, B_3, B_4. The lower envelop is the cyan colour segment and the highest point here is the intersection of B_3 and B_4. Therefore, the given 2×4 game can be solved by solving the 2×2 game with A_1, A_2 and B_3, B_4. So the new 2×2 game with strategies of B are $q = (0, 0, q_3, q_4)$ is

	B_3	B_4
A_1	-1	5
A_2	7	-2

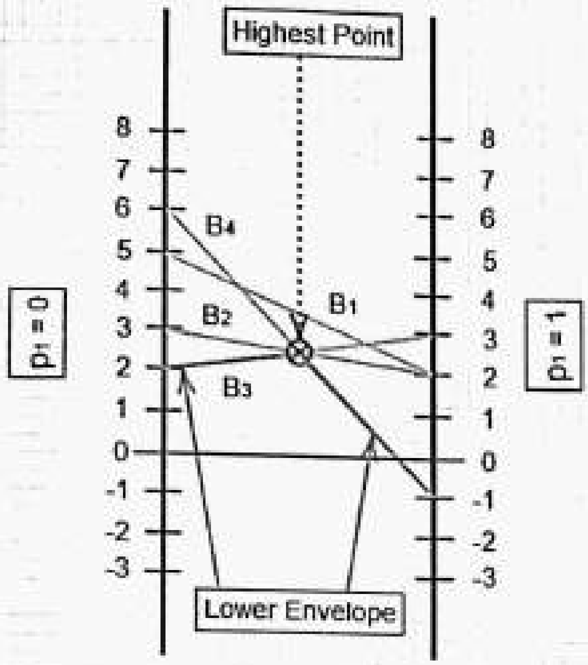
The problem has no saddle point for pure strategy in the pay-off matrix for A.
Let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_1, p_2)$ and $\underset{\sim}{q} = (q_3, q_4)$ with $p_1 + p_2 = 1$, $p_1, p_2 > 0$ and $q_3 + q_4 = 1$, $q_3, q_4 > 0$ for A and B respectively. Assuming the existence of the value of game we have
$E_1(p) = -1p_1 + 7p_2 = -p_1 + 7(1 - p_1)$, $E_2(p) = 5p_1 - 2p_2 = 5p_1 - 2(1 - p_1)$.
To determine the optimal values of p_1, p_2 we have, $-p_1 + 7(1 - p_1) = v = 5p_1 - 2(1 - p_1)$. Solving we get, $p_1^* = 3/5 (> 0)$. So, $p_2^* = 1 - p_1^* = 2/5$ and value of the game is $v = -p_1^* + 7(1 - p_1^*) = 11/5$.
Again considering from the B's point of view we have,
$E_1(q) = -1q_3 + 5q_4 = -q_3 + 5(1 - q_3)$, $E_2(q) = 7q_3 - 2q_4 = 7q_3 - 2(1 - q_3)$.
To determine the optimal values of q_3, q_4, we have, $-q_3 + 5(1 - q_3) = v = 7q_3 - 2(1 - q_3)$. Solving we get $q_3^* = 7/15 (> 0)$ and $q_4^* = 1|q_1^* = 8/15$ and the value of the game is $v = -q_3 + 5(1 - q_3) = 11/5$.
Hence the optimal strategies are $p^* = (3/5, 2/5)$ and $q^* = (0, 0, 7/15, 8/15)$ and $v = 11/5$.

Example 2.4. *Reduce the following $2 \times n$ game*

	B_1	B_2	B_3	B_4
A_1	2	2	3	-1
A_2	5	3	2	6

into a 2×2 game using graphical method and hence solve it algebraically.
⇒ *The given game does not have any saddle point, let the mixed strategies used by A be*

$p = (p_1, p_2)$ with $p_1 + p_2 = 1$ and both p_1, p_2 lie in the open interval $(0,1)$. Also the mixed strategies used by B be $q = (q_1, q_2, q_3, q_4)$ with $q_1 + q_2 + q_3 + q_4 = 1$.



Draw two vertical lines $p_1 = 0$, $p_1 = 1$ at unit distance apart. Draw a line segment joining $(0,5)$, $(1,2)$. This line correspomding to strategy B_1. It represents the expected gain of A due to B's pure moves B_1. Similarly, we can draw lines B_2, B_3, B_4. The lower envelop is the cyan colour segment and the highest point here is the intersection of B_2, B_3 and B_4. Therefore, the given 2×4 game can be solved by solving the 2×2 game with A_1, A_2 and B_2, B_3; B_2, B_4; B_3, B_4. We can discard B_2, B_4 as same sign slope. So the two new 2×2 game with strategies of B are $q = (0, q_2, q_3, 0)$ and $q = (0, 0, q_3, q_4)$ are respectively

	B_2	B_3
A_1	2	3
A_2	3	2

and

	B_3	B_4
A_1	3	-1
A_2	2	6

. The problem has no saddle point for pure strategy in the pay-off matrix for A.

For the First matrix, let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_1, p_2)$ and $\underset{\sim}{q} = (q_2, q_3)$ with $p_1 + p_2 = 1$, $p_1, p_2 > 0$ and $q_2 + q_3 = 1$, $q_2, q_3 > 0$ for A and B respectively. Assuming the existence of the value of game we have

$E_1(p) = 2p_1 + 3p_2 = 2p_1 + 3(1 - p_1)$, $E_2(p) = 3p_1 + 2p_2 = 3p_1 + 2(1 - p_1)$.

To determine the optimal values of p_1, p_2 we have, $2p_1 + 3(1 - p_1) = v = 3p_1 + 2(1 - p_1)$. Solving we get, $p_1^* = 1/2 (> 0)$. So, $p_2^* = 1 - p_1^* = 1/2$ and value of the game is $v = 2p_1^* + 3(1 - p_1^*) = 5/2$.

Again considering from the B's point of view we have,

$E_1(q) = 2q_2 + 3q_3 = 2q_2 + 3(1 - q_2)$, $E_2(q) = 3q_2 + 2q_3 = 3q_2 + 2(1 - q_2)$.

To determine the optimal values of q_2, q_3, we have, $2q_2 + 3(1 - q_2) = v = 3q_2 + 2(1 - q_2)$. Solving we get $q_2^* = 1/2 (> 0)$ and $q_3^* = 1|q_1^* = 1/2$ and the value of the game is $v = 2q_2 + 3(1 - q_2) = 5/2$.

Hence the optimal strategies are $p^* = (1/2, 1/2)$ and $q^* = (0, 1/2, 1/2, 0)$ and $v = 5/2$.

For the second matrix, let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_1, p_2)$ and $\underset{\sim}{q} = (q_3, q_4)$ with $p_1 + p_2 = 1$, $p_1, p_2 > 0$ and $q_3 + q_4 = 1$, $q_3, q_4 > 0$ for A and B respectively. Assuming the existence of the value of game we have

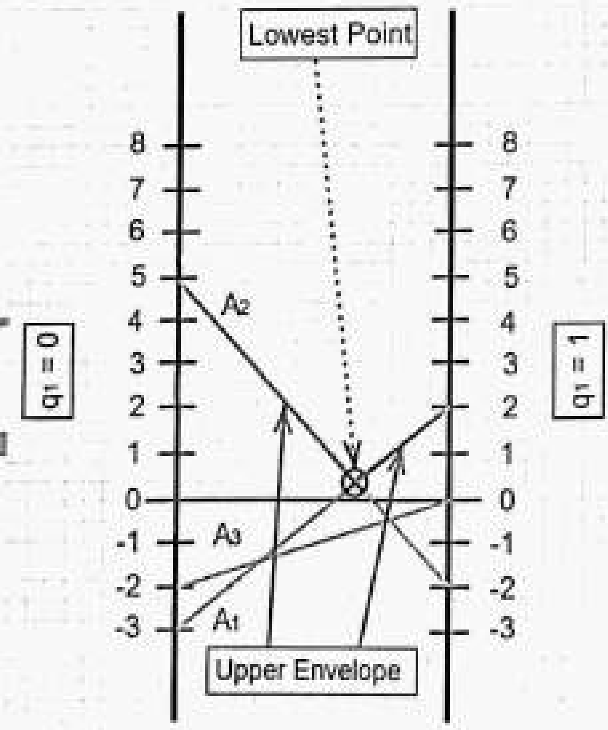
$E_1(p) = 3p_1 + 2p_2 = 3p_1 + 2(1 - p_1)$, $E_2(p) = -1p_1 + 6p_2 = -p_1 + 6(1 - p_1)$.
To determine the optimal values of p_1, p_2 *we have,* $3p_1 + 2(1 - p_1) = v = -p_1 + 6(1 - p_1)$.
Solving we get, $p_1^* = 1/2(> 0)$. *So,* $p_2^* = 1 - p_1^* = 1/2$ *and value of the game is* $v = 3p_1^* + 2(1 - p_1^*) = 5/2$.
Again considering from the B's point of view we have,
$E_1(q) = 3q_3 - 1q_4 = 3q_3 - (1 - q_3)$, $E_2(q) = 2q_3 + 6q_4 = 2q_3 + 6(1 - q_3)$.
To determine the optimal values of q_3, q_4, *we have,* $3q_3 - (1 - q_3) = v = 2q_3 + 6(1 - q_3)$.
Solving we get $q_3^* = 7/8(> 0)$ *and* $q_4^* = 1|q_1^* = 1/8$ *and the value of the game is* $v = 3q_3 - (1 - q_3) = 5/2$.
Hence the optimal strategies are $p^* = (1/2, 1/2)$ *and* $q^* = (0, 0, 7/8, 1/8)$ *and* $v = 5/2$.
Here multiple optimal solution exists.

■ We can solve $m \times 2$ games in a similar way.

Example 2.5. *Reduce the following* 3×2 *game*

	B_1	B_2
A_1	2	−3
A_2	−2	5
A_3	0	−2

into a 2×2 *game using graphical method and hence solve it algebraically.*
⇒ *The given game does not have any saddle point, let the mixed strategies used by* B *be* $q = (q_1, q_2)$ *with* $q_1 + q_2 = 1$ *and both* q_1, q_2 *lie in the open interval* $(0, 1)$. *Also the mixed strategies used by* A *be* $p = (p_1, p_2, p_3)$ *with* $p_1 + p_2 + p_3 = 1$.



Draw two vertical lines $q_1 = 0$, $q_1 = 1$ *at unit distance apart. Draw a line segment joining* $(0, -3)$, $(1, 2)$. *This line correspomding to strategy* A_1. *It represents the expected gain of* B *due to* A*'s pure moves* A_1. *Similarly, we can draw lines* A_2, A_3. *The upper envelop is the cyan colour segment and the lowest point here is the intersection of* A_1 *and* A_2. *Therefore, the given* 3×2 *game can be solved by solving the* 2×2 *game with* B_1, B_2 *and* A_1, A_2. *So the new* 2×2 *game with strategies of* A *are* $p = (p_1, p_2, 0)$ *is*

	B_3	B_4
A_1	2	−3
A_2	−2	5

The problem has no saddle point for pure strategy in the pay-off matrix for A.

Let us try to solve the problem by using mixed strategies $\underline{p} = (p_1, p_2)$ *and* $\underline{q} = (q_1, q_2)$ *with* $p_1 + p_2 = 1,\ p_1, p_2 > 0$ *and* $q_1 + q_2 = 1,\ q_1, q_2 > 0$ *for A and B respectively. Assuming the existence of the value of game we have*

$E_1(p) = 2p_1 - 2p_2 = 2p_1 - 2(1 - p_1),\ E_2(p) = -3p_1 + 5p_2 = -3p_1 + 5(1 - p_1)$.

To determine the optimal values of p_1, p_2 *we have,* $2p_1 - 2(1 - p_1) = v = -3p_1 + 5(1 - p_1)$.

Solving we get, $p_1^* = 7/12 (> 0)$. *So,* $p_2^* = 1 - p_1^* = 5/12$ *and value of the game is* $v = 2p_1^* - 2(1 - p_1^*) = 1/3$.

Again considering from the B's point of view we have,

$E_1(q) = 2q_1 - 3q_2 = 2q_1 - 3(1 - q_1),\ E_2(q) = -2q_1 + 5q_2 = -2q_1 + 5(1 - q_1)$.

To determine the optimal values of q_1, q_2, *we have,* $2q_1 - 3(1 - q_1) = v = -2q_1 + 5(1 - q_1)$.

Solving we get $q_1^* = 2/3 (> 0)$ *and* $q_2^* = 1|q_1^* = 1/3$ *and the value of the game is* $v = 2q_1 - 3(1 - q_1) = 1/3$.

Hence the optimal strategies are $p^* = (7/12, 5/12, 0)$ *and* $q^* = (2/3, 1/3)$ *and* $v = 1/3$.

[Do It Yourself] 2.7. *Reduce the games i)*

	B_1	B_2	B_3	B_4
A_1	2	2	3	−1
A_2	4	3	2	6

, *ii)*

	B_1	B_2	B_3	B_4
A_1	1	2	−3	7
A_2	2	5	4	−6

into a 2 × 2 game using graphical method and hence solve it algebraically. [CU 92, 98]

[Do It Yourself] 2.8. *Reduce the games i)*

	B_1	B_2
A_1	2	7
A_2	3	5
A_3	11	2

, *ii)*

	B_1	B_2
A_1	1	−3
A_2	3	5
A_3	−1	6
A_4	4	1
A_5	2	2
A_6	−5	0

into a 2 × 2 game using graphical method and hence solve it algebraically. [CU 88, 86]

1.1.5 Dominance Property

■ The principle of dominance in Game Theory states that if one strategy of a player dominates over the other strategy in all conditions then the later strategy can be ignored.
► The concept of dominance is useful in two-person zero-sum games when there ∄ a saddle point.
► Row Dominance: If all the elements of a row (i) are less than or equal to the corresponding elements of any other row j, then the row (i) is dominated by row (j) and can be deleted from the matrix.
► Column Dominance: If all the elements of a column (k) are greater than or equal to the corresponding elements of any other column (l), then the column k is dominated by the column j and can be deleted from the pay-off matrix.
► Using this property the game may be reduced to 2×2 or, $2 \times n$ or, $m \times 2$ game so that it can be solved easily.

Example 1.6. *Reduce the given 4×5 game*

	B_1	B_2	B_3	B_4	B_5
A_1	10	5	5	20	4
A_2	11	15	10	17	25
A_3	7	12	8	9	8
A_4	5	13	9	10	5

by using dominance property and hence solve it. [CU 94]

⇒ *As row A_2 dominates both the A_3 and A_4. Therefore, by using dominance property we can discard rows A_3, A_4 and the resulting matrix is*

	B_1	B_2	B_3	B_4	B_5
A_1	10	5	5	20	4
A_2	11	15	10	17	25

The column B_3 dominates B_1, B_2, B_4. Therefore, by using dominance property we can discard columnsB_1, B_2, B_4 and the resulting matrix is

	B_3	B_5
A_1	5	4
A_2	10	25

Again row A_2 dominates A_1. Therefore, by using dominance property we can discard *rows A_1 and the resulting matrix is*

	B_3	B_5
A_2	10	25

In this matrix, column B_3 dominates B_5. So the value of the game is 10 at (A_2, B_3) and $(2,3)$ is the saddle point.

1.1.6 Modified Dominance Property

► Row Modified Dominance: If all the elements of a row (i) are less than or equal to the corresponding elements of the convex combination of some other rows, then the row (i) is dominated by all those rows and can be deleted from the pay-off matrix.

► Column Modified Dominance: If all the elements of a column (k) are greater than or equal to the corresponding elements of the convex combination of some other columns, then column k is dominated by all those columns and can be deleted from the pay-off matrix.

Example 1.7. *Reduce the given 4×4 game*

	B_1	B_2	B_3	B_4
A_1	1	2	−1	2
A_2	3	1	2	3
A_3	0	3	2	1
A_4	−2	1	1	−1

by using dominance/ modified dominance property and hence solve it.

⇒ *As row A_2 dominates $(\geq)$ A_4. Therefore, by using dominance property we can discard row A_4 and the resulting matrix is*

	B_1	B_2	B_3	B_4
A_1	1	2	−1	2
A_2	3	1	2	3
A_3	0	3	2	1

The column B_1 dominates $(\leq)$ B_4. Therefore, by using dominance property we can discard column B_4 and the resulting matrix is

	B_1	B_2	B_3
A_1	1	2	−1
A_2	3	1	2
A_3	0	3	2

Again convex combination of rows i.e. $\frac{1}{2}A_2 + \frac{1}{2}A_3$ *dominates* ($\geq$) A_1. *Therefore, by using modified dominance property we can discard row* A_1 *and the resulting matrix is*

	B_1	B_2	B_3
A_2	3	1	2
A_3	0	3	2

Also convex combination of columns i.e. $\frac{1}{2}B_1 + \frac{1}{2}B_2$ *dominates* ($\leq$) B_3. *Therefore, by using modified dominance property we can discard column* B_3 *and the resulting matrix is*

	B_1	B_2
A_2	3	1
A_3	0	3

The problem has no saddle point for pure strategy in the pay-off matrix for A.
For the First matrix, let us try to solve the problem by using mixed strategies $\underset{\sim}{p} = (p_2, p_3)$ *and* $\underset{\sim}{q} = (q_1, q_2)$ *with* $p_2 + p_3 = 1$, $p_2, p_3 > 0$ *and* $q_1 + q_2 = 1$, $q_1, q_2 > 0$ *for A and B respectively. Assuming the existence of the value of game we have*
$E_1(p) = 3p_2 + 0p_3 = 3p_2$, $E_2(p) = 1p_2 + 3p_3 = p_2 + 3(1 - p_2)$.
To determine the optimal values of p_2, p_3 *we have,* $3p_2 = v = p_2 + 3(1 - p_2)$.
Solving we get, $p_2^* = 3/5(> 0)$. *So,* $p_3^* = 1 - p_2^* = 2/5$ *and value of the game is* $v = 3p_2^* = 9/5$.
Again considering from the B's point of view we have,
$E_1(q) = 3q_1 + 1q_2 = 3q_1 + (1 - q_1)$, $E_2(q) = 0q_1 + 3q_2 = 3(1 - q_1)$.
To determine the optimal values of q_2, q_3, *we have,* $3q_1 + (1 - q_1) = v = 3(1 - q_1)$.
Solving we get $q_1^* = 2/5(> 0)$ *and* $q_2^* = 1|q_1^* = 3/5$ *and the value of the game is* $v = 3q_2^* = 9/5$.
Hence the optimal strategies are $p^* = (0, 3/5, 2/5, 0)$ *and* $q^* = (2/5, 3/5, 0, 0)$ *and* $v = 9/5$.

[Do It Yourself] 1.9. *Solve the game problem by reducing it into* 2×2 *problem using dominance property. [CU 87]*

	B_1	B_2	B_3	B_4	B_5
A_1	0	0	0	0	0
A_2	4	2	0	2	1
A_3	4	3	1	3	2
A_4	4	3	4	−1	2

[Do It Yourself] 1.10. *Reduce the following game to 2×2 game by modified dominance property and then solve it. [CU 90]*

	B_1	B_2	B_3	B_4
A_1	3	2	4	0
A_2	3	4	2	4
A_3	4	2	4	0
A_4	0	4	0	8

[Do It Yourself] 1.11. *Use dominance and modified dominance property to reduce the pay-off matrix given by*

	B_1	B_2	B_3	B_4
A_1	3	−1	1	2
A_2	−2	3	2	3
A_3	2	−2	−1	2

into a 2×2 and find the mixed strategies for A and B and the value of the game. [CU 84]

1.1.7 Game theory to LPP

Theorem 1.2. *Show that every two person zero sum game problem can be converted into to a LPP.*

Proof. Hi □

Theorem 1.3. *If we add a fixed number (positive/ negative) to each element of a pay-off matrix the optimal strategies rematn same, while the value of the game will be increased by that number.*

Proof. Suppose the pay-off matrix is $[\ a_{ij}\ ]_{n \times m}$ and let the mixed strategies taken by A and B are $p = (p_1, p_2, \cdots, p_m)$ with $\sum_{i=1}^{m} p_i = 1,$ *with* $p_i \geq\ 0\ \forall i$ and $q = (q_1, q_2, \cdots, q_n)$ with $\sum_{j=1}^{m} q_j = 1,\ q_j \geq 0,\ \forall j$ respectively.

So, the pay-off function of the game is given by $E(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_iq_j$.
So after adding a number k to each element of the pay-off matrix, the pay-off function of the new game is given by: $E^k(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+k)p_iq_j = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_iq_j + k\sum_{i=1}^{m}\sum_{j=1}^{n} p_iq_j = E(p,q) + k$ as $\sum p_i = 1, \sum q_j = 1$ ... (1)
WLOG we can assume that $(a_{ij}+k) > 0\ \forall i,j$. Then the value of the second game exist and unique if it is optimal strategies be p^*q^* for the second game, then its value of the game is $E^k(p^*,q^*) = \max_p \min_q E(p,q) = E^k(p^*,q^*) + k$
Again, $\max_p \min_q E^k(p,q) = \max_p \min_q E(p,q) + k$.
Therefore, it implies $\max_p \min_q E(p,q) = E^k(p^*,q^*)$ ... (2)
Similarly, we can get $\min_q \max_p E(p,q) = E^k(p^*,q^*)$ ... (3)
So, from (2) and (3) we can say that the value of original game also exists and unique, the optimal strategies for both games remain same and from (1) we can say that value of the second game is only increased by the number k.
In a similar way, we can also show that the relation is also valid if $a_{ij}+k \leq 0,\ \forall i,j$. □

Example 1.8. *Solve the game problem*

	B_1	B_2
A_1	2	1
A_2	−1	3

by converting it into LPP.

⇒ *The value of the game may not be positive. Adding 2 to each element we get a pay-off matrix whose values will be essentially positive and solving the new problem we can find the value of original problem. So the pay off matrix transform to*

	B_1	B_2
A_1	4	3
A_2	1	5

Let the optimal strategies for A is $p^* = (p_1^*, p_2^*)$ *and B is* $q^* = (q_1^*, q_2^*)$. *Now considering B's problem it can be reduced to*

$$\begin{aligned} &\text{Maximize, } q_0 = q_1' + q_2' \\ &\text{Subject to,} \\ &4q_1' + 3q_2' \leq 1 \\ &q_1' + 5q_2' \leq 1 \\ &q_1', q_2' \geq 0. \end{aligned}$$

If $Max\ q_0 = \frac{1}{v^*}$, *then the value of the game of the original problem will be* $v^* - 2$ *and the optimal solution* $q_j^* = q_j'^* v^*,\ p_i^* = p_i'^* v^*,\ i,j = 1,2$.

Introducing the slack variables q_3', q_4' *one in each constraint, we get the following converted equations.*

$$\begin{aligned} 4q_1' + 3q_2' + q_3' \qquad &= 1 \\ q_1' + 5q_2' \qquad + q_4' &= 1 \end{aligned}$$

Here the coefficient matrix contain a unit basis. The adjusted objective function z is given by: Maximize, $q_0 = q_1' + q_2' + 0.q_3' + 0.q_4'$.
Now we will construct the simplex table to solve the given LPP.

		c_j	1	1	0	0	
Basis	B	b	a_1	a_2	a_3	a_4	*Min Ratio*
a_3	0	1	4	3	1	0	$1/3 = 0.33$
a_4	0	1	1	(5)	0	1	$1/5 = 0.2 \leftarrow$
$z_j - c_j$		0	-1	*-1*	0	0	
a_3	0	2/5	(17/5)	0	1	$-3/5$	$(2/5)/(17/5) = 2/17 \leftarrow$
a_2	1	1/5	1/5	1	0	1/5	$(1/5)/(1/5) = 1$
$z_j - c_j$		1/5	*-4/5*	0	0	1/5	
a_1	1	2/17	1	0	5/17	$-3/17$	
a_2	1	3/17	0	1	$-1/17$	4/17	
$z_j - c_j$		5/17	0	0	4/17	1/17	

So Max $q_0 = \frac{5}{17} = \frac{1}{v^*}$ *(say) at* $q_1'^* = \frac{2}{17},\ q_2'^* = \frac{3}{17}$.
The value of the original game is $v^* - 2 = \frac{17}{5} - 2 = \frac{7}{5}$ *at* $q_1^* = q_1'^* v^* = \frac{2}{5}$ *and* $q_2^* = q_2'^* v^* = \frac{3}{5}$.
Now using the duality theory, we have $p_1'^* = \frac{4}{17}$ *and* $p_2'^* = \frac{1}{17}$. *So,* $p_1^* = p_1'^* v^* = \frac{4}{5}$ *and* $p_2^* = p_2'^* v^* = \frac{1}{5}$.
Therefore, the optimal strategies are $p^* = (\frac{4}{5}, \frac{1}{5})$, $q^* = (\frac{2}{5}, \frac{3}{5})$ *and the value of game is* $v = \frac{7}{5}$.

[Do It Yourself] 1.12. *Determine the optimum strategies of A and B from the pay-off matrix*

	B_1	B_2
A_1	2	7
A_2	3	5
A_3	11	2

using LPP. [CU 90]

[Do It Yourself] 1.13. *Solve the game problem by LPP [CU 91]*

	B_1	B_2	B_3
A_1	1	-1	-1
A_2	-1	-1	3
A_3	-1	2	-1