1.2.8 Sample Space

The sample space of a statistical experiment is a pair $(\Omega, \mathcal{S})$, where

1. Ω is the set of all possible outcomes of the experiment and
2. $\mathcal{S}$ is a σ-field of subsets of Ω.

▶ The elements of Ω are called sample points.
▶ Any set $A \in \mathcal{S}$ is known as an event.
▶ An event is a collection of sample points.
▶ Each one-point set is known as a simple or an elementary event. For example, tossing of coin is a simple event but if we are tossing a coin and rolling a die simultaneously then it will not be called a simple event as two events are occurring simultaneously.
▶ If the set Ω contains only a finite number of points, we say that $(\Omega, \mathcal{S})$ is a finite sample space.
▶ If Ω contains at most a countable number of points, we call $(\Omega, \mathcal{S})$ a discrete sample space.
▶ If Ω contains uncountably many points, we say that $(\Omega, \mathcal{S})$ is an uncountable sample space.
▶ If $\Omega = \mathbb{R}^k$ or some rectangle in $\mathbb{R}^k$ then we say that it a continuous sample space.

Example 1.3. *Suppose we toss a coin one time then find Ω and $\mathcal{S}$.*
□ *The set $\Omega = \{H, T\}$, where H denotes head and T is the tail.*
Also, $\mathcal{S}$ is the class of all subsets of Ω, i.e. $\mathcal{S} = \{\phi, \{H\}, \{T\}, \{H, T\}\} = \{\phi, \{H\}, \{T\}, \Omega\}$.

■ Any set $A \in \mathcal{S}$ is known as an event. For ex. Ω is an event that either head or tail appear. Also ϕ is an event that neither head nor tail appear.

Event: An event is a subset of sample space.
▶ Simple Event: If an event contains only one sample point, it is known as simple / elementary event.
▶ Composite Event:If an event contains more than one sample point, it is known as composite event.

Example 1.4. *Suppose we toss a coin two times then find Ω and $\mathcal{S}$. Also find the event that at least one tail occur.*
□ *The set $\Omega = \{HH, HT, TH, TT\}$, where H denotes head and T is the tail.*
Also, $\mathcal{S}$ is the class of all subsets of Ω, i.e. $\mathcal{S} = \{\phi, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, HT, TT\}, \{HH, TH, TT\}, \{HT, TH, TT\}, \Omega\}$.
□ *The event that at least one tail occur consists of sample points $\{HT, TH, TT\}$.*

[Do It Yourself] 1.10. *A coin is tossed until the first head appears then find Ω and $\mathcal{S}$.*

[Do It Yourself] 1.11. *A die is thrown then find Ω and $\mathcal{S}$.*

Example 1.5. *Two coins are tossed simultaneously. Which one of the following is a simple event?*
a) Getting at least one tail.
b) Getting first head and then tail.
□ *The sample space for tossing two coins is* $\{HH, HT, TH, TT\}$.
The event having at least one tail is $\{HT, TH, TT\}$.
This can be disintegrated to $\{HT\}$, $\{TH\}$ *and* $\{TT\}$ $\Rightarrow$ *it is not a simple event.*
The event of getting head first and then tail is $\{HT\}$ $\Rightarrow$ *this is a simple event.*

Example 1.6. *Suppose we roll a dice two times then find* Ω *and* $\mathcal{S}$. *Also find the event that sum of two face is* 9.
□ *The set* $\Omega = \{(x_1, x_2) : x_1, x_2 \in \{1, 2, 3, 4, 5, 6\}\}$, *where* i *denotes the face numbered* i, *i=1(1)6. Here* Ω *contains* $6^2 = 36$ *elements.*
Also, $\mathcal{S}$ *is the class of all subsets of* Ω. *Here you can imagine* $\mathcal{S}$ *but writing it is a very lengthy job.*
□ *The event that sum of two face is* 9 *consists of sample points* $\{(3, 6), (6, 3), (4, 5), (5, 4)\}$.

[Do It Yourself] 1.12. *A die is rolled* n *times then find* Ω *and* $\mathcal{S}$. *Also find the event that* 2 *shows at least once.*

[Do It Yourself] 1.13. *Four players* A, B, C *and* D *are playing ludo. Write the sample space associated with the win of any two players as first and second. What is the event that i)* B *is first or, second? ii)* D *always first. iii)* A, C *always third and fourth.*
□ <u>*Hint:*</u> *First you can think about* A, B *may win or,* A, C *may win or, so on. Second, i)* B *will be always in the pair. ii)* D *will be always in the first position.*

[Do It Yourself] 1.14. *A club has five members A, B, C, D, and E. It is required to select a chairman and a secretary. Assuming that one member cannot occupy both positions, write the sample space associated with these selections. What is the event that member A is an office holder?*

Example 1.7. *Let A, B are two arbitrary events then find* Ω *and* $\mathcal{S}$. *What is the event that only A occurs? What is the event that at either one of A, B occur? What is the event that both A and B occur? What is the event that none of A, B occur?*
□ *The set* $\Omega = \{A \cap B^c, A \cap B, A^c \cap B, A^c \cap B^c\}$.
Also, $\mathcal{S}$ *is the class of all subsets of* Ω *which contains* $2^{2^2} = 16$ *elements and we already did this.*
□ *The event that only A occurs is* $A \cap B^C$. *The event that at either one of A, B occur is* $A \cup B$. *The event that both A and B occur is* $A \cap B$. *The event that none of A, B occur is* $A^c \cap B^c$.

[Do It Yourself] 1.15. *Let A, B, C be three arbitrary events on a sample space* $(\Omega, \mathcal{S})$. *What is the event that only A occurs? What is the event that at least two of A, B, C occur? What is the event that both A and C, but not B, occur? What is the event that at most one of A, B, C occurs? What is the event that exactly one of A, B, C occurs?*

1.2.10 Borel Set (*in* $\mathbb{R}$)

■ A set that can be obtained from the union, intersection and relative complement of enumerable collection of closed and open sets (for the time being take interval) in $\mathbb{R}$, is known as Borel set.

■ Specifically, In $\mathbb{R}$, every interval of the form $(x, y]$ is a Borel set and the σ-field generated by $(x, y]$ is called Borel σ-field. It is denoted by $\mathcal{B}$.

▶ $\mathcal{B} = \sigma(x, y] = \sigma[x, y) = \sigma(x, y) = \sigma[x, y] = \sigma[x, \infty) = \sigma(x, \infty) = \sigma(-\infty, x] = \sigma(-\infty, x)$.

▶ Any interval or, one point set is a Borel set and the Borel σ-field $\mathcal{B}$ can be generated by any above type of intervals.

▶ If $\Omega = \mathbb{R}^2$ or, $\Omega = [a, b] \times [c, d]$ then $\mathcal{S} = \mathcal{B}_2$ and so on.

[Do It Yourself] 1.17. *Show that the σ-field generated by $(x, y]$ is $\mathcal{B}$ contains all one point sets and all intervals.*

Hint : $(x, y) = \cup_{i=1}^{\infty}(x, y - \frac{1}{n}] \in \mathcal{B}$, $(x - \frac{1}{n}, x + \frac{1}{n})^c \in \mathcal{B} \Rightarrow \cup_{i=1}^{\infty}(x - \frac{1}{n}, x + \frac{1}{n})^c \in \mathcal{B} \Rightarrow \cap_{i=1}^{\infty}(x - \frac{1}{n}, x + \frac{1}{n}) \in \mathcal{B} \Rightarrow \{x\} \in \mathcal{B}$, $[x, y] = (x, y] \cup \{x\}$, $[x, y) = (x, y) \cup \{x\}$, $(-\infty, y) = \cup_{i=1}^{\infty}(-n, y)$, *so on.*

1.2.11 Problems on Uncountable Sample Space

▶ Suppose each point on the circumference of a circle with radius r is a possible outcome of the experiment. Then Ω consists of all points within $[0, 2\pi r)$.

▶ Every one-point set $\{x \in [0, 2\pi r)\}$ is a simple event.

▶ Here $\mathcal{S}$ is taken to be the Borel σ-field of subsets of $[0, 2\pi r)$.

▶ The events of interest are the length of arc traveled by the point.

▶ In discrete sample space, every one-point set is also an event, and $\mathcal{S}$ is the class of all subsets Ω.

▶ In uncountable sample space, not all subsets Ω events. The case of most interest is the one in which $\Omega = \mathbb{R}^p$. Here roughly all sets that have a well-defined volume (or area or length) are events.

Example 1.8. *A rod of length l is thrown onto a flat table, which is ruled with parallel lines at distance $2l$. The experiment consists in noting whether the rod intersects one of the ruled lines. Then how do you construct Ω and $\mathcal{S}$?*

□ *Let r be the distance from the center of the rod to the nearest ruled line and θ be the angle that the axis of the rod makes with this line. So here, $0 \leq r \leq l$, since the nearest ruled line from the center of the rod always lies between 0 and l. Also the range of θ is $0 \leq \theta < \pi$.*

Therefore, if we know (r, θ), we can express the location of the rod on the table. Now if we throw the rod it will either intersect a line or, land within two lines. To construct Ω, we will take all possible outcome of the throw which is equivalent to all possible value of (r, θ) i.e. $\boxed{\Omega = \{(r, \theta) : 0 \leq r \leq l, 0 \leq \theta < \pi\}.}$

$\mathcal{S}$ is the σ-field generated by $[0, l] \times [0, \pi)$ i.e. $\mathcal{S} = \mathcal{B}_2 = \sigma([0, l] \times [0, \pi))$.

1.3 Probability

Let $(\Omega, \mathcal{S})$ be the sample space associated with a statistical experiment. Now we will define a probability set function and study some of its properties.

▶ A set function is a function whose input is a set and the output is usually a number.

▶ Suppose $S = \{1, 2, 3, 7\}$, and a set function f is defined the numbers of elements in S. Here $f : S \to \mathbb{Z}^+$ and $f(S) = 4$.

1.3.1 Probability Axioms

Let $(\Omega, \mathcal{S})$ be a sample space. A set function P defined on $\mathcal{S}$ is called a probability measure (or simply probability) if it satisfies the following conditions:

1. $P(A) \geq 0, \forall\ A \in \mathcal{S}$.
2. $P(\Omega) = 1$.
3. If $A_1, A_2, \cdots$ are disjoint sequence of sets i.e. $A_i \cap A_j = \phi,\ i \neq j$.

   Then $P(\cup_{i=1}^{\infty} A_i) = \displaystyle\sum_{i=1}^{\infty} P(A_i)$.

▶ $P(A)$ is known as the probability of event A.

▶ Property (3) is called countable additivity.

▶ Probability is a function that has a set for an input, and a real number as an output between $[0, 1]$.

1.3.4 Probability Space

The triplet $(\Omega, \mathcal{S}, P)$ is called a probability space.

▶ Let $A \in \mathcal{S}$, we say that the odds for A are a to b if $PA = \frac{a}{a+b}$, and then the odds against A are b to a.

▶ Suppose $\Omega = \{H, T\} \Rightarrow \mathcal{S} = \{\phi, \{H\}, \{T\}, \Omega\}$. Now define P on $\mathcal{S}$ as $P(\{H\}) = \alpha$, $P(\{T\}) = 1 - \alpha \Rightarrow P$ defines a probability on $(\Omega, \mathcal{S})$. Note that, here all 3 axioms are satisfied.

▶ We can make equally likely argument by taking $\alpha = 1/2$.

▶ Suppose $\Omega = \{1, 2, 3, \cdots\} \Rightarrow \mathcal{S}=$ Class of all subsets of Ω. Now define P on $\mathcal{S}$ as $P(\{i\}) = \frac{1}{2^i},\ i = 1, 2, \cdots$. Here $\sum_{i=1}^{\infty} P(\{i\}) = 1 \Rightarrow P$ defines a probability on $(\Omega, \mathcal{S})$.

▶ We can't make equally likely argument for countable number of points.

▶ Suppose $\Omega = (0, \infty) \Rightarrow \mathcal{S} = \mathcal{B}$. Define P for each interval $A \in \mathcal{S}$ as, $P(A) = 2\int_A e^{-2x} dx$. Note that, $P(A) \geq 0$, $P(\Omega) = 1$ and P is countably additive by properties of integrals.

[Do It Yourself] 1.21. *Let Ω be the set of all nonnegative integers and $\mathcal{S}$ the class of all subsets of Ω. Does $(\Omega, \mathcal{S}, P)$ define a probability space?*

(a) *For $A \in \mathcal{S}$, $P(A) = \sum_{x \in A} \frac{e^{-\lambda}\lambda^x}{x!}$, $\lambda > 0$.*

(b) *For $A \in \mathcal{S}$, $P(A) = \sum_{x \in A} p(1-p)^x$, $0 < p < 1$.*

(c) *For $A \in \mathcal{S}$, let $P(A) = 1$ if A has a finite number of elements, and $P(A) = 0$ otherwise.*

$[Ans: Y, Y, N.\ Hint: (\Omega, \mathcal{S})\ is\ a\ sample\ space.\ To\ show\ P\ satisfies\ 3\ axiom]$

[Do It Yourself] 1.22. *Let $\Omega = \mathbb{R}$ and $\mathcal{S} = \mathcal{B}$. In each of the following cases does P define a probability on $(\Omega, \mathcal{S})$?*

(a) *For each interval I, $P(I) = \int_I \frac{1}{\pi(1+x^2)} dx$.*

(b) *For each interval I, $P(I) = 1$ if I is an interval of finite length and $P(I) = 0$ otherwise.*

(c) *For each interval I, $P(I) = 0$ if $I \subseteq (-\infty, 1)$ and $P(I) = \int_I \frac{1}{2} dx$ if $I \subseteq (1, \infty)$.*

$[Ans: Y, N, N.\ Hint: (\Omega, \mathcal{S})\ is\ a\ sample\ space.\ To\ show\ P\ satisfies\ 3\ axiom]$

[Do It Yourself] 1.23. *If $A, B \in \mathcal{S}$, then show that $P(A \setminus B) = P(A) - P(A \cap B)$.*
☐ *<u>Hint:</u> $A = (A \setminus B) \cup (A \cap B)$. Now apply countable additivity.*

[Do It Yourself] 1.24. *Suppose $A, B \in \mathcal{S}$, if $A \subseteq B$ i.e. monotone then show that $P(A) \leq P(B)$ i.e. P is subtractive.*
☐ *<u>Hint:</u> $B = (B \setminus A) \cup (A \cap B) = (B \setminus A) \cup A$. Now apply countable additivity.*

Theorem 1.1. *Addition Rule*:
If $A, B \in \mathcal{S}$, then show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
☐ *<u>Hint:</u> $(A \cup B) = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$. Now apply countable additivity.*

[Do It Yourself] 1.25. *Express i) $A_1 \cup A_2 \cup A_3$, ii) $\cup_{i=1}^{n} A_i$, iii) $\cup_{i=1}^{\infty} A_i$ as union of disjoint sets.*

[Do It Yourself] 1.26. *If $A \in \mathcal{S}$, then show that $P(A^c) = 1 - P(A)$.*

[Do It Yourself] 1.27. *If $A, B \in \mathcal{S}$, then show that $P(A \cup B) \leq P(A) + P(B)$.*

Example 1.9. *If $A_1, A_2, \cdots \in \mathcal{S}$, then show that* $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$.

$\Rightarrow$ *Here we will use mathematical induction to prove the result.*
Now we know, $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$. *[To show]*
So the result is true for $n = 2$.
Again $P(A_1 \cup A_2 \cup A_3) \leq P(A_1 \cup A_2) + P(A_3) \leq P(A_1) + P(A_2) + P(A_3)$. *[Using the result,* $n = 2$*]*
So the result is true for $n = 3$.
Let us assume the result is true for $n = m$ *i.e.* $P(\cup_{i=1}^{m} A_i) \leq \sum_{i=1}^{m} P(A_i)$.
Now $P(\cup_{i=1}^{m+1} A_i) \leq \sum_{i=1}^{m} P(A_i) + P(A_{m+1}) \leq \sum_{i=1}^{m+1} P(A_i)$. *[Using the result,* $n = 2, m$*]*
Therefore by mathematical induction, the result is true for $n \in \mathbb{N}$ *i.e.* $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$.

Theorem 1.2. *Principle of Inclusion-Exclusion (Poincare)*:
If $A_1, A_2, \cdots, A_n \in \mathcal{S}$, *then show that*

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j) + \sum_{i<j<k}^{n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P\Big(\bigcap_{i=1}^{n} A_i\Big).$$

□ *Hint:* $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. *[For* $n = 2$*]*
$P(A_1 \cup A_2 \cup A_3) = P(A_1 \cup A_2) + P(A_3) - P[(A_1 \cup A_2) \cap A_3] = \sum_{i=1}^{3} P(A_i) - P(A_1 \cap A_2) - [P(A_1 \cap A_3) + P(A_2 \cap A_3) - P(\overline{A_1 \cap A_3} \cap \overline{A_2 \cap A_3})] = \sum_{i=1}^{3} P(A_i) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) = \sum_{i=1}^{3} P(A_i) - \sum_{i<j}^{3} P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3)$.

So the result is true for $n = 3$. *Let us assume the result is true for* $n = m$ *i.e.*

$$P\Big(\bigcup_{i=1}^{m} A_i\Big) = \sum_{i=1}^{m} P(A_i) - \sum_{i<j}^{m} P(A_i \cap A_j) + \sum_{i<j<k}^{m} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{m+1} P\Big(\bigcap_{i=1}^{m} A_i\Big)$$

Now, $P\Big(\cup_{i=1}^{m+1} A_i\Big) = P\Big(\cup_{i=1}^{m} A_i \cup A_{m+1}\Big) = P(\cup_{i=1}^{m} A_i) + P(A_{m+1}) - P(\cup_{i=1}^{m} A_i \cap A_{m+1})$

$$=\Big[\sum_{i=1}^{m+1} P(A_i) - \sum_{i<j}^{m} P(A_i \cap A_j) + \sum_{i<j<k}^{m} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{m+1} P\Big(\bigcap_{i=1}^{m} A_i\Big)\Big] -$$

$$\Big[\sum_{i=1}^{m} P(A_i \cap A_{m+1}) - \sum_{i<j}^{m} P(A_i \cap A_j \cap A_{m+1}) + \sum_{i<j<k}^{m} P(A_i \cap A_j \cap A_k \cap A_{m+1}) - \cdots + (-1)^{m+1} P\Big(\bigcap_{i=1}^{m+1} A_i\Big)\Big]$$

$$=\Big[\sum_{i=1}^{m+1} P(A_i) - \sum_{i<j}^{m+1} P(A_i \cap A_j) + \sum_{i<j<k}^{m+1} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{m+2} P\Big(\bigcap_{i=1}^{m+1} A_i\Big)\Big].$$

So the result is true for $n = m + 1$.

Theorem 1.4. *Bonferroni's Inequality*:
If $A_1, A_2, \cdots, A_n \in \mathcal{S}$, *then show that* $\sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j) \leq P\Big(\bigcup_{i=1}^{n} A_i\Big) \leq \sum_{i=1}^{n} P(A_i)$.

□ *Hint: Easy one.*

Theorem 1.5. *Boole's Inequality*:
If $A_1, A_2, \cdots \in \mathcal{S}$, *then show that* $P\Big(\bigcap_{i=1}^{\infty} A_i\Big) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$.

□ *Hint: Easy one.*