Example 1.11. *A fair coin is tossed three times. Let A be the event that at least one head shows up in three throws. Then find* $P(A)$.
$\Rightarrow$ *Here* $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
Now, $P(A) = 1 - P(A^c) = 1 - P(\text{no head in three toss}) = 1 - P(TTT) = 1 - 1/8 = 7/8$.

Example 1.12. *A fair die is rolled twice. Let A be the event that the first throw shows a number* ≤ 2, *and B is the event that the second throw shows at least* 5. *Then find* $P(A \cup B)$.

$\Rightarrow$ *Here* $\Omega = \{(i,j) : i,j = 1(1)6\}$, $A = \{(i,j) : i = 1,2; j = 1(1)6\}$, $B = \{(i,j) : i = 1(1)6; j = 5,6\}$, $A \cap B = \{(i,j) : i = 1,2; j = 5,6\}$.
Now, $P(A) = 12/36 = 1/3$, $P(B) = 12/36 = 1/3$, $P(A \cap B) = 4/36 = 1/9$.
So, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 5/9$.

[Do It Yourself] 1.28. *A box contains* 1000 *light bulbs. The probability that there is at least* 1 *defective bulb in the box is* 0.1, *and the probability that there are at least* 2 *defective bulbs is* 0.05. *Then find the probability that i) The box contains no defective bulbs. ii) The box contains exactly* 1 *defective bulb. iii) The box contains at most* 1 *defective bulb.* $[Ans: 0.9, 0.05, 0.95,\ Hint: Take\ X = No.\ of\ defective\ bulbs,\ So\ X = 0, 1, \cdots, 1000]$

[Do It Yourself] 1.29. *Let A and B be two events such that* $A \subsetneq B$. *Then find* $P(A \cap B), P(A \cup B), P(A \setminus B)$*?*

[Do It Yourself] 1.32. *A random experiment has only two possible outcomes, the first occurs with probability* $p^2 + \frac{p}{4}$ *and the second occurs with probability* $\frac{3-p}{4}$. *Then show that the possible values of p are* $\pm\frac{1}{2}$.

1.3.5 Geometric Probability

Let $\Omega \subset \mathbb{R}^n$ be a given set, and $A \subset \Omega$. We are interested in the probability that a 'randomly chosen point' in Ω falls in A. Here 'randomly chosen' means that the point may be any point of Ω and that the probability of its falling in some subset A of Ω is proportional to the measure of A (independently of the location and shape of A). Assuming that both A and Ω have well-defined finite measures (length, area, volume, etc.), here $\boxed{P(A) = \frac{measure(A)}{measure(\Omega)}}$.

[Do It Yourself] 1.33. *A point is picked at random from a square. Let* $\Omega = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 2\}$. *We choose* $\mathcal{S} = \mathcal{B}^2$, *if* $A \in \mathcal{S}$, *then find* $P(A)$ *in each of the following case. Here we take* $P(A)$ *is the area of the set A.*

1. $A = \{(x,y) : 0 \leq x \leq 3/2, 1/2 \leq y \leq 2\}$.
2. *A is a circle with center (1,1) and radius 1 unit.*
3. *A is the set of all points which are at most* $3/2$ *unit distance from the origin.*

$[Ans: 9/16, \pi/4, 9\pi/64]$

Example 1.13. *An interval* $(0, 3)$ *is divided into three intervals by choosing two points at random. What is the probability that the three line segments form a triangle?*
$\Rightarrow$ *Three line segments form a triangle iff the sum of the length of any two line segments is greater than the other one. Let the length of left one is* x*, second one is* $y \Rightarrow$ *third one is* $3-x-y$*. So the conditions are* $x+y > 3-x-y$ *or,* $x+3-x-y > y$ *or,* $y+3-x-y > x$
$\Rightarrow x + y > 3/2$ *or,* $y < 3/2$ *or,* $x < 3/2$*.*
In two dimension we can easily plot and calculate the area is: $\frac{1}{2}.\frac{3}{2}.\frac{3}{2} = \frac{9}{8}$ *sq. unit.*
Again the total area corresponds to : $x > 0, y > 0, 3 - x - y > 0$ *i.e.* $\frac{1}{2}.3.3 = 9/2$ *sq. unit.*
Therefore, the required probability is $1/4$*.*

[Do It Yourself] 1.35. *Two points are chosen at random on a line of unit length. Find the probability that each of the three line segments so formed will have a length* $> 1/4$*.* $[Ans : 1/16]$

[Do It Yourself] 1.37. *The base and the altitude of a right triangle are obtained by picking points randomly from* $[0, a]$ *and* $[0, b]$*, respectively. Show that the probability that the area of the triangle so formed will be less than* $ab/4$ *is* $\frac{1+\ln(2)}{2}$*.*
$[Hint : Take\ x\ in\ [0, a]\ and\ y\ in\ [0, b]\ now\ solve\ the\ easy\ problem]$

[Do It Yourself] 1.38. *A point* X *is chosen at random on a line segment* AB*.* (i) *Show that the probability that the ratio of lengths* AX/BX *is smaller than* $a(a > 0)$ *is* $a/(1+a)$*.* (ii) *Show that the probability that the ratio of the length of the shorter segment to that of the larger segment is less than* $1/3$ *is* $1/2$*.*

[Do It Yourself] 1.39. *Two points are chosen at random on a line segment of length* 9 *cm. Find the probability that the distance between these two points is less than* 3 *cm?*

1.3.6 Finite Sample Spaces

▶ Here $\Omega = \{\omega_1, \cdots, \omega_n\}$ with $P\{\omega_j\} = 1/n$, $j = 1, 2, \cdots, n$.
▶ Here elementary event in Ω is assigned the same probability.
▶ With equally likely assumption: $P(A) = \frac{number\ of\ elementary\ events\ in\ A}{total\ number\ of\ elementary\ events\ in\ \Omega}$.
▶ The above is also known as 'Classical definition of probability'.
▶ A coin is tossed twice. The sample space consists of four points. Under the uniform assignment, each of four elementary events is assigned probability $1/4$.
▶ Three dice are rolled. The sample space consists of 6^3 points. Each one-point set is assigned probability $1/6^3$.

Example 1.14. *What are the probability of obtaining i) an odd number ii) a multiple of 3 in the throw of a fair die?*
$\Rightarrow$ *Let A denote the odd number of points is obtained. Among 6 total events, 3 are elementary events in E i.e. 1,3,5. So* $P(E) = 3/6 = 1/2$.
$\square$ *Easy.*

[Do It Yourself] 1.40. *Two unbiased dice are thrown. Find the probability of obtained i)a total of 8 points ii) at least an ace.*
$[Ans:\ 5/36, 11/36]$

[Do It Yourself] 1.41. *The probability that a teacher will take a surprise test in his class is 2/7. If a student is absent on 3 days, what is the probability that he will miss atleast one test.*

[Do It Yourself] 1.45. *If p and q are chosen randomly from the set* $\{1,2,3,4,5,6,7\}$ *with replacement, find the probability that the equation* $x^2+px+q=0$ *has real but unequal roots.* $[Ans:\ \frac{25}{49}]$

[Do It Yourself] 1.49. *The face cards are removed from a full pack. Out of the remaining cards, 4 are drawn i) What is the probability that they belong to different suit? ii) What is the probability that the 4 cards belongs to different suit and different denominations?*
$[Hint:\ Face\ cards\ are\ J, K, Q]$

[Do It Yourself] 1.50. *If the letters of the word STUDENTS are arranged at random, then find the probability that the vowels will be next to each other.*
$[Hint:\ Total\ events = \frac{8!}{2!2!},\ Elementary\ events = \frac{7!}{2!2!}2!]$

[Do It Yourself] 1.51. *A club consisting 15 married couples chooses a president and then a secretary by random selection. What is the probability that i) Both are men ii) One is man and other is woman iii) President is a man and secretary is an woman.*
$[Ans:\ \frac{15\times14}{30\times29}, \frac{15\times15\times2}{30\times29}, \frac{15\times15}{30\times29}.]$

[Do It Yourself] 1.52. *Among 270 tickets sold in a lottery, 100 are colored red, 90 colored blue and 80 colored green. What is the probability that (i) blue tickets will win both the first and second prizes? (ii) A blue ticket will win first prize and a green ticket the second?*

Example 1.16. *A box contains 7 white and 5 black balls. 3 balls are drawn random. Find the probability that they all of the same colour when i) The balls are drawn at a time ii) One by one without replacement iii) One by one with replacement.*

⇒ *Let A be the event that three same colour balls are drawn at a time.*
Here total number of events $= \binom{12}{3}$.
Among them $\binom{7}{3} + \binom{5}{3}$ *are elementary events related to* A.
Therefore, the required probability $= \frac{\binom{7}{3}+\binom{5}{3}}{\binom{12}{3}}$.
□ *Let A be the event that three same colour balls are drawn one by one without replacement.*
Here total number of events $= 12 \times 11 \times 10$.
Among them $(7 \times 6 \times 5) + (5 \times 4 \times 3)$ *are elementary events related to* A.
Therefore, the required probability $= \frac{(7\times6\times5)+(5\times4\times3)}{12\times11\times10}$.
□ *Let A be the event that three same colour balls are drawn one by one with replacement.*
Here total number of events $= 12 \times 12 \times 12 = 12^3$.

Among them $7^3 + 5^3$ *are elementary events related to* A.
Therefore, the required probability $= \frac{7^3+5^3}{12^3}$.

1.3.7 Permutations & Combinations

▶ Permutation are for list (order matters) whereas combinations are for groups (order does not matter).
▶ Example: We form a group of three people as team leader, manager and technical person out of 7 person then the number of way: $^7P_3 = 7.6.5 = 210$.
▶ Example: We form a group of three people out of 7 person then the number of way: $^7C_3 = \frac{7.6.5}{6} = 35$.
▶ Falling Factorial: $(6)_2 = 6.5$, $(n)_r = n.(n-1).(n-2)\cdots(n-\overline{r-1})$.
▶ Formulaes: *i)* $\binom{n}{x} = \binom{n}{n-x}$, *ii)* $\binom{n}{x}+\binom{n}{x-1} = \binom{n+1}{x}$, *iii)* $\sum_{x=0}^{n}\binom{n}{x} = 2^n$, *iv)* $\sum_{x=0}^{n} x\binom{n}{x} = n2^{n-1}$, *v)* $\sum_{x=0}^{n}\binom{n}{x}^2 = \binom{2n}{n}$, *vi)* $\sum_{x=0}^{k}\binom{m}{x}\binom{n}{k-x} = \binom{m+n}{k}$, *vii)* $\binom{-n}{x} = (-1)^x\binom{n+x-1}{x}$.
▶ We have 6 digits, the number of ways to form a 3 digit number in an increasing order is $\binom{6}{3}$.

[Do It Yourself] 1.54. *How many ways the 7 different object can be arranged in a row?*

[Do It Yourself] 1.55. *In how many different ways can 6 people be seated in 10 seats?*

[Do It Yourself] 1.56. *In a restaurant 16 seats are available, 8 chairs and 8 tools. How many ways, a group of 6 boys and 7 girls can be seated so that boys always get tools and girls always get chairs.*

Example 1.17. *How many ways 6 people be seated at a round table if i) They can seat anywhere, ii) Two particular person always sit together, iii) Two particular people must not seat to each other.*
⇒ *If 1 people be seated anywhere then rest can be seated in 5! ways. So the number of ways are 5!.*
□ *Take two people as one unit so the total number of ways are 4!.2!.*
□ *Take two people as one unit so the total number of ways are 4!.2!. Therefore, two particular people must not seat to each other can be formed in 5! − 4!.2! ways.*

[Do It Yourself] 1.60. *There are three different groups A, B, C with $5, 6, 2$ members respectively will form a line. How many different arrangements are possible if i) All group members stand together. ii) Only group B members stand together.*

[Do It Yourself] 1.61. *Five red marbles, three white marbles and two green marbles are arranged in a row. If all the marbles of the same color are not distinguishable from each other, then how many different arrangement is possible?*

Example 1.18. *Suppose we have 5 vowels and 8 consonants, how many words of length 4 can be found with 2 vowels and 3 consonants.*
$\Rightarrow$ *Two vowels can be choose in ${}^5C_2$ ways, three consonant can be choose in ${}^8C_3$ ways and the 5 letters can be arranged in 5! ways. Therefore, the total number of arrangements are* $5!\,{}^5C_2\,{}^8C_3$.

1.3.9 Rule of Divisibility

- ▶ A number is divisible by 2 if it ends with a $0, 2, 4, 6, 8$.
- ▶ A number is divisible by 3 if the sum of the digits is a multiple of 3.
- ▶ A number is divisible by 4 if the last two digits are a multiple of 4 or, 00.
- ▶ A number is divisible by 5 if it ends with a $0, 5$.
- ▶ A number is divisible by 6 if it is divisible by 2 and by 3.
- ▶ A number is divisible by 9 if the sum of the digits are a multiple of 9.

Example 1.21. *The digits $1, 2, 3, 4, 5, 6$ are written down in random order to give a 6 digit number. What is the probability that i) the number is divisible by 3 ii) the number is divisible by 4.*
$\Rightarrow$ *Let A denote the event that the number is divisible by 3.*
Here total number of events $= 6!$. Here (1+2+3+4+5+6) is always divisible by 3.
It implies 6! are elementary events related to A. Therefore, required probability $= 1$.
□ *Let A denote the event that the number is divisible by 4.*
Here total number of events $= 6!$. Here the last 2 digits which are divided by 4 are 8 namely $(12, 16, 24, 32, 36, 52, 56, 64)$.
It implies $8 \times 4!$ are elementary events related to A. Therefore, required probability $= \frac{8\times 4!}{6!}$.

[Do It Yourself] 1.77. *A four digit number is chosen at random. The probability that there are exactly two zeros in that number is*
(A) 0.73. *(B)* 0.973. *(C)* 0.027. *(D)* 0.27.
$[Hint:\ Total:\ First\ position\ can't\ be\ 0,\ Elementary:\ First\ is\ not\ zero\ \&\ exactly\ two\ zeros]$

[Do It Yourself] 1.78. *The digits $1, 2, 3, 4, 5, 6, 7$ are written down at random order to form a 7 digit number. What is the probability that the number is divisible by 4?*