1.4 Set Theoretic Notation of Events

- Here we will deal with 3 arbitrary events A, B, C.
- At least one event occur: $A \cup B \cup C$.
- All these event occur: $A \cap B \cap C$.
- Only A occur: $A \cap B^c \cap C^c$.
- Both A, B but not C occur: $A \cap B \cap C^c$.
- At least two occur: $(A \cap B) \cup (A \cap C) \cup (B \cap C)$.
- Exactly one occur: $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$.
- Exactly two occur: $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$.
- None of the event occur: $A^c \cap B^c \cap C^c$.
- At least one vs. None: $(A \cup B \cup C) \cup (A^c \cap B^c \cap C^c) = \Omega$ and they are disjoint. Therefore $P(At\ least) = 1 - P(None)$.
- Mutually Exclusive Events: $A_1, \cdots, A_n$ are mutually exclusive events if $A_i \cap A_j = \phi,\ \forall i \neq j = 1(1)n$.
- Exhaustive Events: $A_1, \cdots, A_n$ are exhaustive events if $\cup_{i=1}^{n} A_i = \Omega$.

[Do It Yourself] 1.82. *For any two events A_1, A_2, show that $P(A_1 \cap A_2) \leq \min\{P(A_1), P(A_2)\}$.*
$[Hint:\ P(A_1 \setminus A_2) = P(A_1) - P(A_1 \cap A_2) \Rightarrow P(A_1 \cap A_2) \leq P(A_1)]$

Example 1.24. *An integer is chosen at random from $1, 2, \cdots, 50$. What is the probability that the selected integer is divisible by 7 or, 10?*

$\Rightarrow$ *Let A_1 denote the selection of an integer divisible by 7 and A_2 denote the selection of an integer divisible by 10.*
So $P(A_1) = \frac{7}{50}, P(A_2) = \frac{5}{50}$. Now $(A_1 \cap A_2)$ denote the selection of an integer divisible by 7 and 10 $\Rightarrow A_1 \cap A_2 = \phi \Rightarrow P(A_1 \cap A_2) = 0$.
Therefore, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = \frac{12}{50} = \frac{6}{25}$.

[Do It Yourself] 1.83. *Three cards are drawn at random from a full pack of cards. Find the probability that at least one of them is king.*
$[Ans:\ 1 - \binom{48}{3}/\binom{52}{3}]$

[Do It Yourself] 1.84. *13 cards are drawn at random from a full pack of cards. Find the probability that all are red or, black.*
$[Hint:\ P(B) = \frac{\binom{26}{13}}{\binom{52}{13}} = P(R),\ Find\ P(B \cup R)]$

[Do It Yourself] 1.86. *Two fair dice are tossed independently and it is found that one face is odd and the other one is even. Then find the probability that the sum is less than 6?*
$[Hint:\ P(X+Y<6|X=Odd,\ Y=Even) = \frac{P(X+Y<6 \cap X=Odd,\ Y=Even)}{P(X=Odd,\ Y=Even)}$
$= \frac{P(X=1)P(Y=2)+P(X=1)P(Y=4)+P(X=3)P(Y=2)}{\frac{1}{2}\cdot\frac{1}{2}} = \frac{3/36}{1/4}]$

[Do It Yourself] 1.87. *Two fair dice are tossed independently n times. What is the probability of obtaining double six at least one. Also determine the minimum number of throw required for the probability greater than 0.5.*

[Do It Yourself] 1.88. *An MBA Mr. Thanos applies for a job in 2 planet Titan and Earth. The probability of being selected in planet Titan and Earth are respectively* 0.4 *and* 0.7. *The probability of at least one of his applications being rejected is* 0.8. *Find the probability that he will be selected in i) At least one of the planet ii) Planet Earth only.*
$[Ans:\ 0.9, 0.5]$

Example 1.25. *There are* n *objects marked* $1, 2, \cdots, n$ *and also* n *placed marked* $1, 2, \cdots, n$. *The objects are distributed over the places, one object being allotted to each place. What is the probability that none of the objects occupies the place corresponding to itself?*
$\Rightarrow$ *Let* A_i *denotes the event that* i^{th} *object placed on* i^{th} *cell,* $i = 1(1)n$.
Then $P(A_i) = \frac{(n-1)!}{n!},\ i = 1(1)n$.

Now, $A_i \cap A_j$ *denotes the event that* i^{th} *and* j^{th} *object placed on* i^{th} *and* j^{th} *cell.*
Then $P(A_i \cap A_j) = \frac{(n-2)!}{n!},\ i, j = 1(1)n,\ (i < j)$*; and so on.*
Now, $P(At\ least\ one\ event\ A_i\ occur)$ *is*

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j) + \sum_{i<j<k}^{n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P\Big(\bigcap_{i=1}^{n} A_i\Big)$$

$$= 1 - \binom{n}{2}\frac{1}{n(n-1)} + \binom{n}{3}\frac{1}{n(n-1)(n-2)} + \cdots + (-1)^{n+1}\frac{1}{n!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}.$$

Therefore, $P(None\ of\ the\ event\ A_i\ occur) = 1 - P(At\ least\ one\ event\ A_i\ occur) = \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$.

[Do It Yourself] 1.90. *Find the probability that in a bridge game at least one of the players will get a complete suit of cards.*
$[\underline{Hint}:$ *Let* A_i *be the event that* i^{th} *player gets a complete suit of cards,* $P(A_i) = \frac{4}{\binom{52}{13}}, P(A_iA_j) = \frac{4.3}{\binom{52}{13}\binom{39}{13}}, So\ on]$

[Do It Yourself] 1.94. *Let* E *and* F *be two independent events with* $P(E|F) + P(F|E) = 1,\ P(E \cap F) = 2/9,\ P(F) < P(E)$. *Then* $P(E)$ *equals*
(A) $1/3$. *(B)* $1/2$. *(C)* $2/3$. *(D)* $3/4$.

[Do It Yourself] 1.95. *Let* E *and* F *be two events with* $0 < P(E), P(F) < 1$ *and* $P(E|F) > P(E)$. *Which of the following statements is (are) true?*
(A) $P(F|E) > P(F)$. *(B)* $P(E|F^c) > P(E)$. *(C)* $P(F|E^c) < P(F)$. *(D)* E *and* F *are independent.*

[Do It Yourself] 1.97. *Let* E *and* F *be two events with* $P(E) = 0.7, P(F) = 0.4$ *and* $P(E \cap F^c) = 0.4$. *Then* $P(F|E \cup F^c)$ *is equal to*
(A) $1/2$. *(B)* $1/3$. *(C)* $1/4$. *(D)* $1/5$.

[Do It Yourself] 1.98. *Let* E *and* F *be two mutually disjoint events. Further, let* E *and* F *be independent of* G. *If* $p = P(E) + P(F)$ *and* $q = P(G)$, *then* $P(E \cup F \cup G)$ *is*
(A) $1 - pq$. *(B)* $q + p^2$. *(C)* $p + q^2$. *(D)* $p + q - pq$.

[Do It Yourself] 1.105. *Let E and F be any two independent events with $0 < P(E) < 1$ and $0 < P(F) < 1$. Which one of the following statements is NOT TRUE?*
(A) P(Neither E nor F occurs)=$(P(E)-1)(P(F)-1)$. (B) P(Exactly one of E and F occurs)=$P(E)+P(F)-P(E)P(F)$. (C) P(E occurs but F does not occur)=$P(E)-P(E\cap F)$. (D) P(E occurs given that F does not occur)=$P(E)$.

[Do It Yourself] 1.106. *A system comprising of n identical components works if at least one of the components works. Each of the components works with probability 0.8, independent of all other components. Find the minimum value of n for which the system works with probability at least 0.97.* [$\underline{Hint}$: $P(A_i) = 0.8, P(\cup A_i) \geq 0.97 \Rightarrow 1-(0.2)^n \geq 0.97$]

[Do It Yourself] 1.107. *A system consisting of n components functions iff at least one of n component functions works. Suppose that the all n components of the system function independently, each with probability $3/4$. If the probability of the functioning of the system is $63/64$ then the value of n is*
(A) 2. (B) 4. (C) 3. (D) 5.

1.4.1 Conditional Probability

Let A, B be two events such that $P(B) > 0$, then conditional probability of A given B is defined as $\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)}}$.

- ▶ $P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$.
- ▶ $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)$.
- ▶ $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)\cdots P(A_n|A_1A_2\cdots A_{n-1})$. [Prove]

Theorem 1.6. *$\underline{\text{Total probability}}$: Let $A_1, A_2, \cdots, A_n$ be $\underline{\text{exhaustive and mutually exclusive}}$, none of which has zero probability. Then for any event B, $\boxed{P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)}$.*

Proof. [$\underline{Hint}$: $Here\ A_i \cap A_j = \phi,\ \cup A_i = \Omega \Rightarrow B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$ *all are disjoint*] □

[Do It Yourself] 1.111. *An urn contains 7 blue and 5 green balls. Two balls are drawn without replacements. What is the probability that the 2^{nd} ball is blue if it is given that 1^{st} ball is green?*

$$[Hint:\ P(B) = 7/12,\ P(G) = 5/12,\ P(B|G) = \frac{P(BG)}{P(G)} = \frac{\frac{\binom{7}{1}\binom{5}{1}}{\binom{12}{2}}}{5/12}]$$

[Do It Yourself] 1.112. *An urn contains 6 red and 4 black balls. Two balls are drawn without replacements. What is the probability that the 2^{nd} ball is red if it is given that 1^{st} ball is red?*

[Do It Yourself] 1.113. *An urn contains 7 blue and 5 green balls. Three balls are drawn without replacements. What is the probability that the 3^{rd} ball is green if it is given that $1^{st}, 2^{nd}$ ball are blue?*

[Do It Yourself] 1.117. *An urn contains a red and b blue balls, another urn contains c red and d blue balls. One ball is transferred from* 1^{st} *to* 2^{nd} *urn and then one ball is drawn from the second urn. Find the probability that it is a blue ball.*
$[\underline{Hint}:\ P(B_2) = P(B_2 \cap R_1) + P(B_2 \cap B_1) = P(R_1)P(B_2|R_1) + P(B_2)P(B_2|B_1)]$

[Do It Yourself] 1.118. *Two players play a game as follows. Taking turns, they draw the balls out of an urn containing a white and b black balls, one ball at a time. He who pickup the first white ball wins. What is the probability that the players who start the game wins?*
$[\underline{Hint}:\ P(W) = P(W_1) + P(B_1B_2W_3) + P(B_1B_2B_3B_4W_5) + \cdots]$

[Do It Yourself] 1.122. *Suppose there are r red balls and g green balls, n balls are drawn from the urn and it is found that all of them are green. What is the probability that another ball drawn from the remaining is also green?*
$[\underline{Hint}:\ Reduced\ sample\ space:\ r\ red,\ g-n\ green]$

[Do It Yourself] 1.123. *Let* A_1, A_2, A_3 *be three events. Find the probabilities in terms of* $P(A_i), P(A_i \cap A_j), P(A_1 \cap A_2 \cap A_3)$*: i) At least one event, ii) Exactly one event, iii) Exactly two events, iv) Atleast two events.*

[Do It Yourself] 1.124. *Independent trials consisting of rolling a fair die are performed. The probability that* 2 *appears before* 3 *or,* 5 *is*
(A) $\frac{1}{2}$ *(B)* $\frac{1}{3}$ *(C)* $\frac{1}{4}$ *(D)* $\frac{1}{5}$.
$[\underline{Hint}:\ A_i = Appear\ of\ 1,4,6\ in\ i^{th}\ trial \Rightarrow P(A_i) = \frac{1}{2},\ P(B_i) = \frac{1}{6};\ P(B_1)+P(A_1B_2)+P(A_1A_2B_3)+\cdots = \frac{1}{6}(1+\frac{1}{2}+\frac{1}{2^2}+\cdots)]$

- Some important series formula:
- Note that: $(1+x)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^k,\ if\ |x|<1.$
- Note that: $(1+x)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}x^k = \sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k,\ if\ |x|<1.$
- Note that: $(1-x)^{-n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k = \sum_{k=0}^{\infty}\binom{n+k-1}{n-1}x^k,\ if\ |x|<1.$
- Note that: $(a+x)^{-n} = a^{-n}(1+x/a)^{-n} = \sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k a^{-n-k},\ if\ |x|<|a|.$