1.4.2 Bayes' Theorem

Suppose the events $A_1, A_2, \cdots, A_n$ are exhaustive and mutually exclusive, none of them has zero probability. Further, let B be an event which too has non-zero probability then

$$\boxed{P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)},\ i = 1(1)n.}$$

▶ Proof is easy. Hint: $P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}$.

▶ Two popular terms related to Bayes' theorem are : posterior and prior probability. The posterior probability is the probability of the A_i given the evidence B and $P(A_i)$ is the prior probability of occurring A_i.

Example 1.27. *In a group of 20 males and 5 females, 10 males and 3 female are service holders. What is the probability that a person selected at random from the group, is a service holder, given that the selected person a male?*

⇒ *Let A = event that the person selected from the group is a service holder, B = event that the selected person is male.*

Here we have to find $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(Male\ service\ holder)}{20/25} = \frac{10/25}{20/25} = 1/2$.

Example 1.28. *A picnic is arranged to be held on a particular day. The weather forecast says that there is 80% chance of rain on that day. If it rains, the probability of a good picnic is 0.3 and if it does not the probability is 0.9. What is the probability that the picnic will be good?*

⇒ *Let A_1 = event that there will be rain on that day, A_2 = event that there will be no rain on that day, B = event that the picnic will be good.*

Therefore, $P(B) = P(B \cap A_1) + P(B \cap A_2) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) = 0.42$.

Example 1.29. *There are 3 identical boxes have the following proportions of black and white balls: Box I- 5 black and 3 white, Box II- 6 black and 2 white, Box III- 3 black and 5 white. One of the boxes is selected at random and 1 ball drawn randomly from it. Find i) What is the probability that the ball is black? ii) Given that the ball is black, find the probability that it came from Box-III.*

⇒ *We have three boxes with following black and white balls:*

Box-I: 5 black and 3 white.

Box-II: 6 black and 2 white.

Box-III: 3 black and 5 white.

Let A_1 = event that Box-I is selected, A_2 = event that Box-II is selected, A_3 = event that Box-III is selected.

So $P(A_1) = P(A_2) = P(A_3) = 1/3$.

Let B = event of drawing a black ball ⇒ $P(B|A_1) = 5/8, P(B|A_2) = 6/8, P(B|A_3) = 3/8$.

i) Therefore, $P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) = 7/12$.

ii) Therefore, $P(A_3|B) = \frac{P(A_3)P(B|A_3)}{\sum_{j=1}^{3} P(A_j)P(B|A_j)} = 3/14$.

[**Do It Yourself**] **1.126.** *Three urns contain respectively, 4 white 2 black balls; 2 white 4 black balls and 3 white 3 black balls. One of the urns is chosen on the results of two throws of a coin: the first urn if head appears on each throw, the second urn if tail appears on each throw, and third urn in case head appears on one throw and tail on the other. Finally, a ball is drawn at random from the chosen urn. Then i) What is the probability for the ball being white? ii) A ball is taken at random from the chosen urn is found to be white. What is the probability that the first urn was chosen?*
[$\underline{Ans}$: $1/2, 1/3$]

[**Do It Yourself**] **1.127.** *The probabilities of Toyota, Ford and BMW will start a new outlet in Bolpur are respectively* $0.3, 0.5, 0.2$. *The probability that 'Air-taxi' will be introduced in Bolpur if Toyota, Ford and BMW start a new outlet are respectively* $0.4, 0.6, 0.1$. *Given that the 'Air-taxi' has been introduced, find the probability that 'Ford' has started a new outlet in Bolpur.*
[$\underline{Ans}$: 0.68]

[**Do It Yourself**] **1.128.** *It is known that the population of a certain city is* 55% *male and* 45% *female. Suppose that the* 70% *males and* 10% *females smoke. Find the probability that a smoker is male.*

[**Do It Yourself**] **1.129.** *Consider four coins labelled as* $1, 2, 3$ *and* 4. *Suppose that the probability of obtaining a 'head' in a single toss of the* i^{th} *coin is* $i/4$, $i = 1, 2, 3, 4$. *A coin is chosen uniformly at random and flipped. Given that the flip resulted in a 'head', the conditional probability that the coin was labelled either* 1 *or* 2 *equals*
(A) $\frac{1}{10}$. *(B)* $\frac{2}{10}$. *(C)* $\frac{3}{10}$. *(D)* $\frac{4}{10}$.
[$\underline{Hint}$: $P(H|C_i) = i/4,\ P(C_i) = 1/4,\ P(C_1 \cup C_2|H) = P(C_1|H) + P(C_2|H)$]

[**Do It Yourself**] **1.130.** *Student population of a university has 30% Asian, 40% American, 20% European and 10% African students. It is known that 40% of all Asian students, 50% of all American students, 60% of all European students and 20% of all African students are girls. Find the probability that a girl chosen at random from the university is an Asian.*

1.4.3 Statistical Independence of Events

Two events A, B are said to be independent if $P(A|B) = P(A)$ and $P(B|A) = P(B)$.
► Independent Implies: $P(A|B) = P(A \cap B)/P(B) \Rightarrow \boxed{P(A \cap B) = P(A)P(B)}$.
► Mutually Exclusive: $P(A_1 \cap A_2) = 0$.
► Independent: $P(A_1 \cap A_2) = P(A_1)P(A_2)$.
► Mutually Independent: $P(A_1 \cap A_2) = P(A_1)P(A_2)$, $P(A_1 \cap A_3) = P(A_1)P(A_3)$, $P(A_2 \cap A_3) = P(A_2)P(A_3)$; $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.
► Number of Conditions in Mutually Independent: $\binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n - n - 1$.
► Pairwise Independent: $P(A_1 \cap A_2) = P(A_1)P(A_2)$, $P(A_1 \cap A_3) = P(A_1)P(A_3)$, $P(A_2 \cap A_3) = P(A_2)P(A_3)$.
► Number of Conditions in Pairwise Independent: $\binom{n}{2}$.

[Do It Yourself] 1.141. *If A, B are independent events. Then show that $i) A^c, B$ are independent $ii) A, B^c$ are independent $iii) A^c, B^c$ are independent.*
[$\underline{Hint}$: $P(A^cB) = P(B \setminus A) = P(B) - P(AB) = P(B) - P(A)P(B) = P(A^c)P(B)$]

[Do It Yourself] 1.142. *If $P(A^c \cup B^c) = 5/6,\ P(A) = 1/2,\ P(B^c) = 2/3$. Check if A, B are independent or not?*
[$\underline{Hint}$: $P(A \cap B) = 1 - P(A \cap B)^c = 1 - P(A^c \cap B^c) = 1 - P(A^c)P(B^c) = 1/6 = P(A)P(B)$]

[Do It Yourself] 1.143. *If two independent events A and B such that $P(A \cap B^c) = \frac{3}{25}$, $P(A^c \cap B) = \frac{8}{25}$ and $P(A) > 1/2$. Then find $P(A), P(B)$.*

[Do It Yourself] 1.144. *Suppose that all 4 possible outcomes $\{e_1, e_2, e_3, e_4\}$ of an experiment are equally likely. Define the events $A = \{e_1, e_4\}, B = \{e_2, e_4\}, C = \{e_3, e_4\}$. Check if A, B, C are independent or not?*
[$\underline{Hint}$: *Check pairwise and mutual both.*]

[Do It Yourself] 1.145. *Give an example where three events are pairwise independent but not mutually independent.*

Example 1.30. *The probability of solving a problem by 3 students A, B, C are $3/7, 3/8, 1/3$ respectively. If all of them try independently, find the probability that the problem could be solved by one person only. Find also the probability that the problem is not solved.*
$\Rightarrow P(A) = 3/7,\ P(B) = 3/8, P(C) = 1/3 \Rightarrow P(A^c) = 4/7,\ P(B^c) = 5/8, P(C^c) = 2/3$
P(Problem could be solved by one person only)$=P[(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)] = P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C) = P(A)P(B^c)P(C^c) + P(A^c)P(B)P(C^c) + P(A^c)P(B^c)P(C) = 37/84$.
□ *P(Problem is not solved)*$=P(A^c \cap B^c \cap C^c) = 5/21$.

[Do It Yourself] 1.146. *A can solve 70% problems of a statistics book and B can solve 75% problems. What is the probability that a radomly selected problem will be solved if both of them try?*