Chapter 2

Random Variables

2.1 Definition

■ Basic Definition: A random variable X is a variable that takes a definite set of real values with a definite set of probabilities. If X can assume at most a finite or, countably infinite number of values, X is said to be a discrete random variable, otherwise it is said to be a continuous random variable.

▶ Suppose a coin is thrown 3 times and we are concern about the number of heads. Then we can define a random variable $X=$ Number of heads as

$$X = \begin{cases} 0 & \text{with probability } \frac{1}{8} \\ 1 & \text{with probability } \frac{3}{8} \\ 2 & \text{with probability } \frac{3}{8} \\ 3 & \text{with probability } \frac{1}{8} \end{cases}$$

▶ A random variable is associated with a probability distribution whereas there is no such probability concept in a real-valued function of a real variable.

■ General Definition I: Let $(\Omega, \mathcal{S})$ be a sample space. A finite, single-valued function X which maps Ω into $\mathbb{R}$ is called a random variable if the inverse images under X of all Borel sets in $\mathbb{R}$ are events i.e. $X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{S},\ \forall B \in \mathcal{B}$.

▶ A random variable is a measurable function defined on a probability space whose outcomes are real numbers.

▶ A real-valued function on $(\Omega, \mathcal{S})$ is a random variable iff any class $\mathbb{U}$ of subsets of $\mathbb{R}$ which generates $\mathcal{B}$ (see Chapter 1, Borel Set) is an event.

▶ General Definition II: A real-valued function X on $(\Omega, \mathcal{S})$ is a random variable iff $(-\infty, x],\ x \in \mathbb{R}$ is an event i.e. $X^{-1}(-\infty, x] = \{\omega : X(\omega) \in (-\infty, x]\} = \{X \leq x\} \in \mathcal{S}$.

▶ Note that, If X is a random variable, the sets $\{X = x\}$, $\{a < X \leq b\}$, $\{X < x\}$, $\{a \leq X < b\}$, $\{a < X < b\}$, $\{a \leq X \leq b\}$, etc. are all events. Moreover, we could have used any of these intervals (i.e. any sets generates $\mathcal{B}$) to define a random variable.

▶ Note that, $X : \Omega \to \mathbb{R}$ is a random variable if $X^{-1}(B) \in \mathcal{S},\ \forall B \in \mathcal{B}$. Remember that, it is possible to find subsets of $\mathbb{R}$ which do not belong to $\mathcal{B}$, so there exist real-valued functions defined on Ω which are not random variables. Random variables are Borel measurable.

Example 2.1. *Let* $\Omega = \{H, T\}$ *and* $\mathcal{S}$ *be the class of all subsets of* Ω. *Define* X *by* $X(H) = 1,\ X(T) = 0$. *Then show that* X *is a random variable.*
$\Rightarrow$ *Now*

$$X^{-1}(-\infty, x] = \begin{cases} \phi & \text{if } x < 0 \\ \{T\} & \text{if } 0 \le x < 1 \\ \{H, T\} & \text{if } 1 \le x \end{cases}$$

Therefore, $\forall x \in \mathbb{R},\ X^{-1}(-\infty, x] \in \mathcal{S} \Rightarrow X$ *is a random variable.*

[**Do It Yourself**] **2.1.** *Let* $\Omega = \{HH, HT, TH, TT\}$ *and* $\mathcal{S}$ *be the class of all subsets of* Ω. *Define* X *by* $X(\omega) =$ *No. of heads in* ω, $\omega \in \Omega$. *Then show that* X *is a random variable. Find the domain, range of* X *and draw the mapping.*

[**Do It Yourself**] **2.2.** *Let* Ω *be the outcome of tossing a coin thrice and* $\mathbb{S}$ *be the class of all subsets of* Ω. *Define* X *by* $X(\omega) =$ *No. of heads in* ω, $\omega \in \Omega$. *Then show that* X *is a random variable. Find the domain, range of* X *and draw the mapping. What are the values that* X *assigns to points of* Ω? *Also find the events* $\{1.2 \le X < 2.1\}$, $\{X < 1.5\}$, $\{X \le 2.5\}$, $\{0.2 \le X \le 3.1\}$.
[$\underline{Hint}:\ \Omega = \{HHH, HHT, \cdots, TTT\},\ X(\omega_1) = 3, \cdots, X(\omega_8) = 0,$ *write* $X(\omega)$ *function e.g.* $X(\omega) = 3,\ if\ \omega = \omega_1, X(\omega) = 2,\ if\ \omega = \omega_2, \omega_3, \omega_4,$ *so on and find the required events*]

[**Do It Yourself**] **2.3.** *A die is tossed two times. Let* X *be the sum of face values on the two tosses and* Y *be the absolute value of the difference in face values. What is* Ω? *What values do* X *and* Y *assign to points of* Ω? *Check to see whether* X *and* Y *are random variables.*

[**Do It Yourself**] **2.6.** *Let* $\Omega = [0, 1]$ *and* $\mathcal{S} = \mathcal{B}[0, 1]$. *Define* X *by* $X(\omega) = \omega$, $\omega \in \Omega$. *Then show that* X *is a random variable. Find the domain, range of* X *and draw the mapping. Also find the events* $\{0.2 \le X < 0.5\}$, $\{X < 0.5\}$.
[$\underline{Hint}:$ *Since any Borel subset of* $[0, 1]$ *is an event* $\Rightarrow X^{-1}(B) \in \mathcal{S},\ \forall B \in \mathcal{B};\ X(\omega) = \omega,\ \omega \in [0, 1] \Rightarrow X^{-1}(\omega) = \omega \Rightarrow X^{-1}[0.2, 0.5) = [0.2, 0.5)$]

Example 2.2. *Let* X *be a random variable. Is* $|X|$ *also an RV? If* X *is a random variable that takes only nonnegative values, is* $\sqrt{X}$ *also an RV?*
$\Rightarrow X$ *is a random variable* $\Rightarrow \{\omega : X(\omega) \le x\} \in \mathcal{S}$. *Also* $\{\omega : -x \le X(\omega) \le x\} \in \mathcal{S}$ *as* $[-x, x]$ *generates* $\mathcal{B}$.
Now $\underline{\text{for } x < 0}$, $\{\omega : |X(\omega)| \le x\} = \phi \in \mathcal{S}$.
Now $\underline{\text{for } x \ge 0}$, $\{\omega : |X(\omega)| \le x\} = \{\omega : -x \le X(\omega) \le x\} = \{\omega : -x \le X(\omega) \le x\} \in \mathcal{S}$.
Therefore $|X|$ *is a random variable.*
$\square\ \{\omega : \sqrt{X(\omega)} \le x\} = \{\omega : X(\omega) \le x^2\} \in \mathcal{S}$. *Therefore* $\sqrt{X}$ *is a random variable.*

[**Do It Yourself**] **2.8.** *Let* X *be a random variable. Then* $i)\ X^2,\ ii)\ aX + b,\ iii)\ \frac{1}{X},\ \{X = 0\} = \phi$ *are Random Variable?*

2.2 Probability Distribution of a RV

■ The random variable X defined on the probability space $(\Omega, \mathcal{S}, P)$ induces a probability space $(\mathbb{R}, \mathcal{B}, Q)$ through $Q(B) = P\{X^{-1}(B)\} = P\{\omega : X(\omega) \in B\}\ \forall B \in \mathcal{B}\}$.
▶ Here the set function Q or, PX^{-1} is the probability distribution of X.
▶ Since Q is a set function it is difficult to handle and therefore we will use General Definition II. Here $PX^{-1}(-\infty, x]$ or, $Q(-\infty, x]$ is a point function.

2.2.1 Distribution Function

■ Let X be a random variable defined on $(\Omega, \mathcal{S}, P)$. Define a point function $F(.)$ on $\mathbb{R}$ by $F(x) = Q(-\infty, x] = P\{\omega : X(\omega) \leq x\}$ for all $x \in \mathbb{R}$. Here the function F is called the distribution function of X.
▶ The distribution function of X is $F(x) = P(X \leq x)$.

■ A real-valued function F defined on $\mathbb{R}$ that is
i) non-decreasing $[x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)]$,
ii) right continuous $[F(x+0) = \lim_{h\to 0+} F(x+h) = F(x)]$ and
iii) satisfies $F(-\infty) = 0$ and $F(+\infty) = 1$,
is called a distribution function (DF).
▶ The set of discontinuity points of a DF F is at most countable.

Example 2.3. *Let $\Omega = \{H, T\}$ and $\mathcal{S}$ be the class of all subsets of Ω. Define X by $X(H) = 1$, $X(T) = 0$ and P assign equal masses to $\{H\}$ and $\{T\}$. Then find the distribution function of the random variable X.*
$\Rightarrow$ *Now $P(X = 1) = \frac{1}{2}$, $P(X = 0) = \frac{1}{2}$, so*

$$F(x) = PX^{-1}(-\infty, x] = P(X \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \end{cases}$$

[Do It Yourself] 2.9. *Let $\Omega = \{HH, HT, TH, TT\}$ and $\mathcal{S}$ be the class of all subsets of Ω. Define X by $X(\omega) =$ No. of heads in ω, $\omega \in \Omega$. Then find the DF of X, assuming that the coin is fair.*

[Do It Yourself] 2.10. *Let Ω be the outcome of tossing a fair coin thrice and $\mathcal{S}$ be the class of all subsets of Ω. Define X by $X(\omega) =$ No. of heads in ω, $\omega \in \Omega$. Then find the DF of X.*

Example 2.4. *Let $\Omega = [0, 1]$, $\mathcal{S} = \mathcal{B}[0, 1]$. Define X by $X(\omega) = \omega$, $\omega \in \Omega$. For every subinterval I of $[0, 1]$, let $P(I)$ be the length of the interval, the find the DF of X.*
$\Rightarrow$ *Here*

$$F(x) = P(\omega : X(\omega) \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ P([0, x]) & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \end{cases} = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \end{cases}$$

[Do It Yourself] 2.12. *Check whether the following function is a DF? Also find* $P(\{X > 1/4\}),\ P(\{1/3 < X \leq 3/8\})$ *or, simply* $P(X > 1/4),\ P(1/3 < X \leq 3/8)$.

$$F(x) = \begin{cases} 0 & \textit{if } x < 0 \\ x & \textit{if } 0 \leq x < \frac{1}{2} \\ 1 & \textit{if } x \geq \frac{1}{2} \end{cases}$$

[$\underline{Hint}$: F *is* $\ddagger$ *(derivative)*; $lim_{x \to 0^+} F(x) = 0 = F(x), lim_{x \to \frac{1}{2}^+} F(x) = 1 = F(\frac{1}{2});\ F(-\infty) = 0,\ F(\infty) = 1;\ P(\{X \leq \frac{1}{4}\} \cup \{X > \frac{1}{4}\}) = 1 \Rightarrow P(\{X \leq \frac{1}{4}\}) + P(\{X > \frac{1}{4}\}) = 1$]

[Do It Yourself] 2.13. *Check whether the following function is a DF?*

$$F(x) = \begin{cases} 0 & \textit{if } x \leq 1 \\ 1 - \frac{1}{x} & \textit{if } 1 < x \end{cases}$$