### 2.2.2 Discrete Random Variable

■ A random variable $X$ defined on $(\Omega, \mathcal{S}, P)$ is said to be a discrete RV, if $\exists$ a countable set $E \subseteq \mathbb{R}$ such that $P(X \in E) = 1$.
▶ The points of $E$ which have positive mass are called jump points of the DF of $X$, and their probabilities are called jumps of the DF.
▶ $E \in \mathcal{B}$ as every one-point is in $\mathcal{B} \Rightarrow \{X \in E\}$ is an event.
▶ If $X$ takes the value $x_i$ with probability $p_i (\geq 0)$ i.e. $P(\omega : X(\omega) = x_i) = p_i$. Then $\sum_{i=1}^{\infty} p_i = 1$.

■ Probability Mass Function or, PMF: The collection of positive numbers $\{p_i\}$ satisfying $P(X = x_i) = p_i$, for all $i$ and $\sum_{i=1}^{\infty} p_i = 1$, is called the probability mass function (pmf) of random variable $X$.
▶ $P(X = x_i) = p_i \Rightarrow F(x) = P(X \leq x) = \sum_{x_i \leq x} p_i$.
▶ If $\{p_k\}$ be a collection of nonnegative real numbers with $\sum_{i=1}^{\infty} p_i = 1 \Rightarrow \{p_k\}$ is the PMF of some discrete RV $X$.
▶ Degenerate RV (at the point $a$): The RV $X$ takes only one value with $P(X = a) = 1$

$$P(X = a) = 1 \Rightarrow F(x) = P(X \leq x) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x \geq a \end{cases}$$

### 2.2.3 Random Variable Using Indicator Function

■ Indicator function: $f(x) = I_A(x)$, $x \in \mathbb{R}$.

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

▶ Step function: $f(x) = \sum_{i=0}^{n} \alpha_i I_{A_i}(x)$, $x \in \mathbb{R}$, $A_i$'s are piecewise disjoint interval and $\bigcup_{i=1}^{\infty} A_i = \mathbb{R}$.
▶ A special type of indicator function:

$$\delta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

.

**[Do It Yourself] 2.17.** *Write down the function $\delta(x - c)$.*

■ Indicator function of a set $A$ is

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{o.w.} \end{cases} \Rightarrow I_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{o.w.} \end{cases} \Rightarrow I_{\{X=x_i\}}(\omega) = I_{\{x_i\}}(\omega) = \begin{cases} 1 & \text{if } \omega = x_i \\ 0 & \text{o.w.} \end{cases}$$

Therefore, for discrete case: Random Variable $X(\omega) = \sum_{i=1}^{\infty} x_i I_{[X=x_i]}(\omega)$.

Therefore, for discrete case: DF $F(x) = P(X \leq x) = \sum_{i=1}^{\infty} p_i \delta(x - x_i)$.

**[Do It Yourself] 2.19.** *Show that the DF of a degenerate RV at a is $\delta(x-a)$. Also draw $F(x)$.*

**Example 2.5.** *A box contains good and defective items. If an item drawn is good, we assign the number 1 to the drawing, else the number assign is 0. Let p be the probability of drawing at random a good item. Then define a random variable X and write down its pmf. Also find the distribution function $F(x)$.*
$\Rightarrow$ *Let X be the random variable corresponding to draw an item. So*

$$X = \begin{cases} 1 & \text{if the item is good} \\ 0 & \text{if the item is bad} \end{cases} = \begin{cases} 0 & \text{if the item is bad} \\ 1 & \text{if the item is good} \end{cases}$$

*It is given that $P(X=1)=p$, so $P(X=0)=1-p=p_1$(say). Also $P(X=1)=p_2$ (say) [Ordering].*

*Therefore,* $F(x)=\sum_{i=1}^{\infty} p_i\delta(x-x_i) = p_1\delta(x-0)+p_2\delta(x-1) = \begin{cases} 0 & \text{if } x<0 \\ p_1 & \text{if } 0\le x<1 \\ p_1+p_2 & \text{if } x\ge 1 \end{cases}$

*So the distribution function is:* $F(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-p & \text{if } 0\le x<1 \\ 1 & \text{if } x\ge 1 \end{cases}$

**[Do It Yourself] 2.20.** *If the RV X has the pmf $P(X=k)=\frac{6}{\pi^2}\frac{1}{k^2}$, $k=1,2,\cdots$. Then find the DF $F(x)$.*

**[Do It Yourself] 2.21.** *If the RV X has the pmf $P(X=k)=\frac{1}{k(k+1)}$, $k=1,2,\cdots$. Then find the DF $F(x)$.*

**[Do It Yourself] 2.24.** *Let X be discrete random variable with the probability mass function $p(x)=k(1+|x|)^2$, $x=-2,-1,0,1,2$. Here k is a real constant. Then $P(X=0)$ equals*
*(A) $\frac{1}{9}$. (B) $\frac{2}{27}$. (C) $\frac{1}{27}$. (D) $\frac{1}{81}$.*

### 2.2.4 Continuous Random Variable

■ The random variable $X$ defined on $(\Omega, \mathcal{S}, P)$ with DF $F$. Then $X$ is said to be a continuous RV, if $F$ is absolutely continuous, i.e. if $\exists$ a non-negative function $f(x)$ such that for every real number $x$ we have $F(x)=\int_{-\infty}^{x} f(t)dt$.
▶ The function $f$ is called the probability density function or, PDF of the RV $X$.

■ Properties of PDF: *i)* $f\ge 0$, *ii)* $\int_{-\infty}^{\infty} f(x)\,dx = 1$, *iii)* $a<b\in\mathbb{R}\Rightarrow P(a<X\le b) = F(b)-F(a)=\int_a^b f(t)dt$.
★ *i)* Follows from the definition. *ii)* Follows from $F(\infty)=1$, *iii)* Follows from $P(\{X\le a\}\cup\{a<X\le b\}\cup\{X>b\})=1 \Rightarrow F(a)+P(a<X\le b)+1-F(b)=1.$

▶ If $X$ is a continuous RV with PDF $f \Rightarrow \forall$ Borel set $B \in \mathcal{B}$, $\boxed{P(B) = \int_B f(x)\, dx}$.

▶ If $F$ is absolutely continuous and $f$ is continuous at $x \Rightarrow \boxed{f(x) = F'(x)}$.

▶ Every non-negative real function $f$ that is integrable over $\mathbb{R}$ with $\int_{-\infty}^{\infty} f(x)\, dx = 1$ is the PDF of some continuous RV $X$.

▶ Since $P(X = x) = P(X \leq x) - P(X < x) = F(x) - F(x-)$ it implies $i)$ $F$ has a jump discontinuity at '$a$' iff $P(X = a) > 0$, $ii)$ $F$ is continuous at '$a$' iff $P(X = a) = 0$.

▶ For a continuous RV $P(X = a) = 0$, $\forall a \in \mathbb{R}$.

■ Support of DF: The set of real numbers $x$ for which a DF $F$ increases is called the support of $F$. Let $X$ be the RV with DF $F$, and let $S$ be the support of $F$. Then $P(X \in S) = 1$ and $P(X \in S^c) = 0$.

▶ For discrete RVs, support is the set of all the realizations that have a strictly positive probability of being observed.

▶ For continuous RVs, support is the set of all numbers whose probability density is strictly positive.

**Example 2.6.** *Let $X$ be a RV with DF $F$ as follows:*

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

*Then find i) The density function $f(x)$. ii) The support of $F(x)$. iii) $P(X = 0.5)$, $P(0.5 < X \leq 0.8)$, $P(0.5 \leq X \leq 0.8)$, $P(0.5 < X < 0.8)$.*

$\Rightarrow$ *Differentiating $F(x)$ we get,*

$$f(x) = F'(x) = \begin{cases} 0 & \text{if } x \leq 0,\ x > 1 \\ 1 & \text{if } 0 < x \leq 1 \end{cases}$$

*The function $f$ is not continuous at $x = 0, 1$. We may define $f(0), f(1)$ in any manner. Choosing $f(0) = f(1) = 0$, we have*

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

□ *If the function $F(x)$ is easy then we can draw the graph and easily find the support else we will use differentiation to find the range (support) where $F(x)$ is increasing. Here the support of $F(x)$ is $(0, 1)$. It related to $P(X \in Support) = 1$.*

□ $P(X = 0.5) = 0$, $P(0.5 < X \leq 0.8) = P(0.5 \leq X \leq 0.8) = P(0.5 < X < 0.8) = \int_{0.5}^{0.8} f(x)dx = 0.3$.

**Example 2.7.** *Let $X$ be a RV with triangular pdf $f$ as follows:*

$$f(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & o.w. \end{cases}$$

*Then i) Show that $f(x)$ is a density function. ii) Find $F(x)$ and its support. iii) Draw $f(x)$ and $F(x)$. iv) Find $P(X=1)$, $P(0.2<X<1.7)$, $P(0.2 \le X \le 1.7)$.*
$\Rightarrow$ *Here* $f \ge 0, \forall x$. *Also ii)* $\int_{-\infty}^{\infty} f(x)\ dx = \int_0^1 x\ dx + \int_1^2 (2-x)\ dx = 1$.
□ *DF F is*

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \int_0^x f(t)dt & \text{if } 0 < x \le 1 \\ \int_0^1 t\ dt + \int_1^x f(t)dt & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases} = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{x^2}{2} & \text{if } 0 < x \le 1 \\ 2x - \frac{x^2}{2} - 1 & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

*Here the support is* $(0,2)$. *It can be easily seen from the pdf.*
□ *We can easily draw* $f(x)$ *and* $F(x)$.
□ $P(X=1)=0$, $P(0.2<X<1.7) = P(0.2 \le X \le 1.7) = \int_{0.2}^1 x\ dx + \int_1^{1.7} (2-x)\ dx = 0.935$.

**[Do It Yourself] 2.26.** *If the RV X has the pdf*

$$f(x) = \begin{cases} kx(1-x) & \text{if } 0 < x < 1,\ k > 0 \\ 0 & o.w. \end{cases}$$

*Then find the value of* $k$. *Also find* $P(X > 0.3)$.

**[Do It Yourself] 2.27.** *Does the functions define a pdf for* $\theta > 0$?

$$f(x) = \begin{cases} \theta^2 x e^{-\theta x} & \text{if } x > 0 \\ 0 & o.w. \end{cases}, g(x) = \begin{cases} \frac{x+1}{\theta(\theta+1)} e^{-x/\theta} & \text{if } x > 0 \\ 0 & o.w. \end{cases}$$

*Also find the DF associated with* $f(x), g(x)$ *and* $P(X \ge 2)$.

**[Do It Yourself] 2.28.** *Show that the function:* $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$ *is a pdf. Also find the DF associated with* $f(x)$ *and* $P(X \ge 2)$.

**[Do It Yourself] 2.42.** *The cumulative distribution function of a random variable X is*

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x+k}{5}, & k \le x < k+1, k = 0,1,2. \\ 1, & x \ge 3. \end{cases}$$

*Find: (a)* $P(X=j)$ *for all non-negative integers* $j$. *(b)* $P(X>2)$. *(c)* $P(-1 \le X < 1)$.
$[\underline{Hint}:\ P(X=j) = P(X \le j) - P(X \le j-1) = F(j) - F(j-1)]$

**[Do It Yourself] 2.43.** *The cumulative distribution function of a random variable X is given by*

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{5}(1+x^3), & \text{if } 0 \le x < 1, \\ \frac{1}{5}[3+(x-1)^2], & \text{if } 1 \le x < 2, \\ 1, & \text{if } x \ge 2. \end{cases}$$

*Find* $P(0<X<2), P(0 \le X \le 1)$ *and* $P\left(\frac{1}{2} \le X \le \frac{3}{2}\right)$.
$[\underline{Hint}:\ Easy]$

[Do It Yourself] **2.44.** *Let* $F(x) = \begin{cases} 0, & if\ x < 0 \\ \frac{x^2}{10}, & if\ 0 \le x < 1 \\ \frac{x+2}{8}, & if\ 1 \le x < 2. \\ \frac{c(6x-x^2-1)}{2}, & if\ 2 \le x \le 3 \\ 1, & if\ x > 3. \end{cases}$ *Find the value of c for which F(.) is a cumulative distribution function of a random variable X. Also evaluate*$P(1 \le X < 2)$.

[*Hint* : *Easy*]

## 2.2.5 Decomposition Theorem

■ Every DF $F(x)$ can be decomposed into two parts as $\boxed{F(x) = \alpha F_d(x) + (1-\alpha)F_c(x)}$ with $0 \le \alpha \le 1$. Here $F_d$ and $F_c$ are both DFs of a discrete and a continuous (not necessarily absolutely continuous) RVs. Here $F(x)$ is called mixed DF.

■ Various Event and Corresponding DF: *i)* $P(X \le a) = F(a)$, *ii)* $P(X < a) = F(a^-)$, *iii)* $P(X > a) = 1 - F(a)$, *iv)* $P(X \ge a) = 1 - F(a^-)$, *v)* $P(a < X \le b) = F(b) - F(a)$, *vi)* $P(a < X < b) = F(b^-) - F(a)$, *vii)* $P(a \le X \le b) = F(b) - F(a^-)$, *viii)* $P(a \le X < b) = F(b^-) - F(a^-)$, *ix)* $P(X = a) = P(X \le a) - P(X < a) = F(a) - F(a^-)$.

**Example 2.8.** *Let X be a RV with DF F as follows:*

$$F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{1}{2} & if\ x = 0 \\ \frac{1}{2} + \frac{x}{2} & if\ 0 < x < 1 \\ 1 & if\ x \ge 1 \end{cases}$$

*Then i) What kind of RV is X? ii) Find the support of* $F(x)$*? iii) Write down* $F(x)$ *as a convex combination of discrete and continuous DFs. iv) What can you say about the pdf of* $F(x)$ *as well as its decompositions.*

⇒ *Here* $F(x)$ *has a jump at* $x = 0$ *and F is continuous in the interval* $(0, 1)$. *So F is the DF of an RV X that is neither discrete nor continuous i.e. F is a mixed DF.*

□ *Here* $F(x)$ *is* ↑ *on* $0 < x < 1$ *also* $F(x)$ *has a jump at* $x = 0$. *So the support is* $0 \le x < 1$.

□ *Here* $F(x)$ *has a jump at* $x = 0$ *and F is continuous in the interval* $(0, 1)$.

$$F_d(x) = \begin{cases} 0 & if\ x < 0 \\ 1 & if\ x \ge 0 \end{cases}$$

$$F_c(x) = \begin{cases} 0 & if\ x \le 0 \\ x + 1 & if\ 0 < x < 1 \\ 1 & if\ x \ge 1 \end{cases}$$

*Here* $F(x) = \frac{1}{2}F_d(x) + \frac{1}{2}F_c(x)$, (*Range break*: $x < 0, x = 0, 0 < x < 1, x \geq 1$).
□ *Here* $F_d(x)$ *is a discrete distribution with pmf*

$$P(X = x) = \begin{cases} 1 & \textit{if } x = 0 \\ 0 & o.w. \end{cases}$$

*Note that,* $X$ *is a degenerate RV at* $x = 0$*. We can easily find* $P(X = x)$ *from* $F_d(x)$ *through its jump points.*
*Here* $F_c(x)$ *is a continuous distribution with pdf*

$$f_c(x) = \begin{cases} 1 & \textit{if } 0 < x < 1 \\ 0 & o.w. \end{cases}$$

*Note that,* $X$ *is a continuous RV. We can easily find* $f_c(x)$ *from* $F_c(x)$ *through differentiation.*
*Since* $F(x)$ *is a mixed DF it is neither continuous nor discrete, so there is no questions about its pdf or, pmf.*

**[Do It Yourself] 2.45.** *If the RV* $X$ *has the DF*

$$F(x) = \begin{cases} 0 & \textit{if } x < 0 \\ \frac{x}{2} & \textit{if } 0 \leq x < 1 \\ \frac{3}{4} & \textit{if } 1 \leq x < 2 \\ 1 & \textit{if } x \geq 2 \end{cases}$$

*Then show that* $F(x)$ *can be written as mixture of two distributions. Also find* $P(X \leq 1), P(X = 1), P(1 < X < 2), P(1 \leq X \leq 2), P(1 < X \leq 2)$.
[*Hint* : $F$ *has jump at* $x = 1, 2$. *Take* $F_d(x) = 0(x < 1),\ \frac{1}{2}(1 \leq x < 2), 1(x \geq 2) \Rightarrow F_c = 0(x < 0), x(0 \leq x < 1), 1(1 \leq x < 2), 1(x \geq 2) \Rightarrow F = \frac{1}{2}F_d + \frac{1}{2}F_c$]