2.3 Functions of a RV

■ Let X is a RV defined on $(\Omega, \mathcal{S}, P)$ and g be a Borel-measurable function on $\mathbb{R}$. Then $g(X)$ is also an RV.
► Note: $\{g(X) \leq y\} = \{X \in g^{-1}(-\infty, y]\} \in \mathcal{S}$ as $g^{-1}(-\infty, y]$ is a Borel set.

2.3.1 Univariate Discrete Transformations

Example 2.9. *If X be a poisson random variable. Then find the pmf of $Y = X^2 + 1$. Also find the pmf of $Y = aX + b$ and $Y = \sqrt{X}$.*
⇒ Since X is a poisson random variable i.e. $X \sim Poi(\lambda)$. So the pmf of X is

$$P(X = x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & \text{if } x = 0, 1, 2, \cdots;\ \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $Y = X^2 + 1$, so $y = x^2 + 1$ maps $A = \{0, 1, 2, \cdots\}$ onto $B = \{1, 2, 5, 10, \cdots\}$.
The inverse map is $x = \sqrt{y-1}$ unique.
So, $P(Y = y) = P(X^2 + 1 = y) = P(X = \sqrt{y-1}) = \frac{e^{-\lambda}\lambda^{\sqrt{y-1}}}{\sqrt{y-1}!}$, $y = 1, 2, 5, 10, \cdots$.
□ *Since $Y = aX + b$, so $y = ax + b$ maps $A = \{0, 1, 2, \cdots\}$ onto $B = \{b, a+b, 2a+b, \cdots\}$.*
The inverse map is $x = \frac{y-b}{a}$ unique.
So, $P(Y = y) = P(aX + b = y) = P(X = \frac{y-b}{a}) = \frac{e^{-\lambda}\lambda^{\frac{y-b}{a}}}{\frac{y-b}{a}!}$, $y = b, a+b, 2a+b, \cdots$.
□ *Since $Y = \sqrt{X}$, so $y = \sqrt{x}$ maps $A = \{0, 1, 2, \cdots\}$ onto $B = \{0, 1, \sqrt{2}, \sqrt{3}, \cdots\}$.*
The inverse map is $x = y^2$ unique.
So, $P(Y = y) = P(\sqrt{X} = y) = P(X = y^2) = \frac{e^{-\lambda}\lambda^{y^2}}{y^2!}$, $y = 0, 1, \sqrt{2}, \sqrt{3}, \cdots$.

$$P(Y = y) = \begin{cases} \frac{e^{-\lambda}\lambda^{y^2}}{y^2!} & \text{if } x = 0, 1, \sqrt{2}, \sqrt{3}, \cdots;\ \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

[Do It Yourself] 2.51. *If X be a binomial random variable. Then find the pmf of $Y = X^2 + 1$. Also find the pmf of $Y = a + bX$ and $Y = \sqrt{X}$.*
[*Hint* : $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x};\ x = 0, 1, 2, \cdots, n;\ 0 \leq p \leq 1$]

[Do It Yourself] 2.54. *Let X be a continuous random variable with probability density function $f(x) = \frac{1}{2}e^{-|x-1|}; -\infty < x < \infty$. Find the value of $P(1 < |X| < 2)$.*
[*Hint* : $P(1 < |X| < 2) = P(-2 < X < -1) + P(1 < X < 2)$]

2.3.2 Univariate Continuous Transformations

Transformation Rule 1 (One to one) : Let X is a CRV with pdf $f(x)$ and $y = g(x)$ be differentiable $\forall x$ and either $g'(x) > 0,\ \forall x$ or, $g'(x) < 0,\ \forall x$. Then $Y = g(X)$ is also a CRV with pdf: $g(y) = f(x)_{|x \to y} |\frac{dx}{dy}|$.

[Do It Yourself] 2.56. *Let X be a* <u>*nonnegative*</u> *CRV with PDF $f(x)$, then find the pdf of X^{α} $(\alpha > 0)$.*
$[\underline{Hint}: \ y = x^{\alpha} \ is \ diff \ \forall x, \ and \ > 0 \ \forall x; \ so \ g(y) = f(x)_{|x \to y}|\frac{dx}{dy}|]$

[Do It Yourself] 2.57. *Let X have the density $f(x) = 1, \ 0 < x < 1$, and $= 0$ otherwise. Find the pdf of i) e^X, ii) $-2\ln X$.*

Transformation Rule 2 (Many to one) : Let X is a CRV with pdf $f(x)$ and $y = g(x)$ be diff. $\forall x$ and $g'(x)$ is continuous and $i)$ $\exists$ multiple inverses $x_1(y), x_2(y), \cdots, x_k(y)$ such that $g[x_n(y)] = y$ and $g'[x_n(y)] \neq 0$ for $n = 1, 2, \cdots, k$ or, $\nexists$ any x such that $g(x) = y, \ g'(x) \neq 0$, i.e. $k = 0$. Then $Y = g(X)$ is also a CRV with pdf: $g(y) = \sum_{n=1}^{k} f(x_n)_{|x_n \to y}|\frac{dx_n}{dy}|$.

[Do It Yourself] 2.58. *Let X have the density $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \ -\infty < x < \infty$. Find the pdf of i) X^2, ii) $|X|$. [These are many to one functions]*
$[\underline{Hint}: \ x_1 = -\sqrt{y}, \ x_2 = \sqrt{y} \Rightarrow g(y) = f(-\sqrt{y})|-\frac{1}{2\sqrt{y}}| + f(\sqrt{y})|\frac{1}{2\sqrt{y}}| = \frac{f(-\sqrt{y})+f(\sqrt{y})}{2\sqrt{y}}]$

[Do It Yourself] 2.59. *Let X have the density $f(x) = \frac{2x}{\pi^2}, \ 0 < x < \pi$. Find the pdf of i) $\sin X$, ii) $\frac{1}{X^2}$. [You can draw the graph if needed, Check the support also]*
$[\underline{Hint}: \ x_1 = \sin^{-1} y, \ x_2 = \pi - \sin^{-1} y; \ One - one]$

[Do It Yourself] 2.61. *The probability density function of a random variable X is* $f(x) = \begin{cases} \frac{|x|}{2}, & -1 \leq x \leq 1, \\ \frac{3-x}{4}, & 1 < x \leq 3, \\ 0, & otherwise \end{cases}$. *Find the cumulative distribution function and the probability density function of $Y = |X|$. Also, find the median of the distribution of Y.*
$[\underline{Hint}: \ Y \ has \ range \ 0 \ to \ 3. \ F(y) = P(Y \leq y) = P(-y < X < y), \ 0 < y < 1; \ P(-y < X < y), \ 1 < y < 3; \ F(y) = 1, \ y \geq 3]$.

[Do It Yourself] 2.62. *Let X be a random variable having probability density function* $f(x; x_0, \alpha) = \begin{cases} \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}}, & x > x_0, \\ 0, & x \leq x_0 \end{cases}$ *where $\alpha > 0, x_0 > 0$. If $Y = \ln\left(\frac{X}{x_0}\right)$, then $P(Y > 3)$ is*
(A) $e^{-3\alpha x_0}$ (B) $1 - e^{-3\alpha x_0}$ (C) $e^{-3\alpha}$ (D) $1 - e^{-3\alpha}$
$[\underline{Hint}: \ P(Y > 3) = P(X > x_0 e^3)]$.

2.4 Two Dimension Random Variables

■ Definition: A real-valued function $\underset{\sim}{X} = (X_1, X_2)$ defined on $(\Omega, \mathcal{S})$ into $\mathbb{R}^2$ is a two-dimensional random variable (or, vector) if the inverse image of every $2-dimensional$ interval $I = \{(-\infty, x_1] \times (-\infty, x_2] \ : (x_1, x_2) \in \mathbb{R}^2\}$ is also in $\mathcal{S}$ i.e. $\underset{\sim}{X}^{-1}(I) = \{\omega : X_1(\omega) \in (-\infty, x_1], X_2(\omega) \in (-\infty, x_2]\} = \{X_1 \leq x_1, \ X_2 \leq x_2\} \in \mathcal{S}$.
► Suppose the outcome of a pair of dice is (x, y), where x, y denotes the face value on the first and second die respectively. Mathematically, we will use the two-dimensional random variables to handle such random experiments.

Example 2.11. *Let $\Omega = \{HH, HT, TH, TT\}$ and $\mathcal{S}$ be the class of all subsets of Ω. Define X_1 by number of heads and X_2 by number of tails. Then show that $\underset{\sim}{X} = (X_1, X_2)$ is a random vector.*

$\Rightarrow$ *Now*

$$X_1^{-1}\{(-\infty, x_1]\} = \begin{cases} \phi & \text{if } x_1 < 0 \\ \{TT\} & \text{if } 0 \leq x_1 < 1 \\ \{HT, TH, TT\} & \text{if } 1 \leq x_1 < 2 \\ \Omega & \text{if } 2 \leq x_1 \end{cases}$$

$$X_2^{-1}\{(-\infty, x_2]\} = \begin{cases} \phi & \text{if } x_2 < 0 \\ \{HH\} & \text{if } 0 \leq x_2 < 1 \\ \{HH, HT, TH\} & \text{if } 1 \leq x_2 < 2 \\ \Omega & \text{if } 2 \leq x_2 \end{cases}$$

Therefore

$$\underset{\sim}{X}^{-1}\{(-\infty, x_1] \times (-\infty, x_2]\} = \begin{cases} \phi & \text{if } x_1 < 0,\ x_2 \in \mathbb{R} \\ \phi & \text{if } x_1 \in \mathbb{R},\ x_2 < 0 \\ \phi & \text{if } 0 \leq x_1 < 1,\ 0 \leq x_2 < 1 \\ \phi & \text{if } 0 \leq x_1 < 1,\ 1 \leq x_2 < 2 \\ \{TT\} & \text{if } 0 \leq x_1 < 1,\ 2 \leq x_2 \\ \phi & \text{if } 1 \leq x_1 < 2,\ 0 \leq x_2 < 1 \\ \{HT, TH\} & \text{if } 1 \leq x_1 < 2,\ 1 \leq x_2 < 2 \\ \{HT, TH, TT\} & \text{if } 1 \leq x_1 < 2,\ 2 \leq x_2 \\ \{HH\} & \text{if } 2 \leq x_1,\ 0 \leq x_2 < 1 \\ \{HH, HT, TH\} & \text{if } 2 \leq x_1,\ 1 \leq x_2 < 2 \\ \Omega & \text{if } 2 \leq x_1,\ 2 \leq x_2 \end{cases}$$

Therefore, $\forall (x_1, x_2) \in \mathbb{R}^2$, $\underset{\sim}{X}^{-1}\{(-\infty, x_1] \times (-\infty, x_2]\} \in \mathcal{S} \Rightarrow \underset{\sim}{X} = (X_1, X_2)$ is a random vector.

2.4.1 Distribution Function

■ The DF of $\underset{\sim}{X} = (X_1, X_2)$ is $\boxed{F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2),\ \forall (x_1, x_2) \in \mathbb{R}^2}$.

▶ Marginal DF of x_1 is $F(x_1, \infty) = P(X_1 \leq x_1) = F_{X_1}(x_1),\ \forall x_1 \in \mathbb{R}$.

▶ Marginal DF of x_2 is $F(\infty, x_2) = P(X_2 \leq x_2) = F_{X_1}(x_2),\ \forall x_2 \in \mathbb{R}$.

▶ $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = P(x_1 < X \leq x_2, Y \leq y_2) - P(x_1 < X \leq x_2, Y \leq y_1) = P(X \leq x_2, Y \leq y_2) - P(X \leq x_1, Y \leq y_2) - P(X \leq x_2, Y \leq y_1) + P(X \leq x_1, Y \leq y_1) = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$. [Easy Draw]

▶ Two RVs X, Y are said to be independent if $\boxed{F(x, y) = F_X(x) F_Y(y),\ \forall (x, y) \in \mathbb{R}^2}$.

Theorem 2.1. *A function $F(,)$ of two variables is a DF of some two-dimensional RV if and only if it satisfies the following conditions:*

1. *F is nondecreasing with respect to both arguments i.e. $F(x+h,y) \geq F(x,y)$ and $F(x,y+k) \geq F(x,y)$ for $h,k > 0$.*
2. *F is right continuous with respect to both arguments i.e. $F(x+0,y) = F(x,y+0) = F(x,y)$.*
3. *$F(-\infty,y) = F(x,-\infty) = 0,\ \forall x,y$; and $F(+\infty,+\infty) = 1$.*
4. *For every $(x_1,y_1),(x_2,y_2)$ with $x_1 < x_2$ and $y_1 < y_2$ the inequality $F(x_2,y_2) - F(x_2,y_1) + F(x_1,y_1) - F(x_1,y_2) \geq 0$.*

Example 2.12. *Check if F is DF or, not.*

$$F(x,y) = \begin{cases} 0 & \text{if } x < 0, \text{ or, } x+y < 1, \text{ or, } y < 0 \\ 1 & \text{otherwise} \end{cases}$$

$\Rightarrow$ *The line $x+y=1$ cuts $X-$axis at A and $Y-$axis at B. The right of the region $YBAX$ where $F(x,y)=1$.*
$\square$ *Fix y, then $F(x,y)$ is non decreasing w.r.t. x. Again, Fix x, then $F(x,y)$ is non decreasing w.r.t. y. [Verified easily from the graph]*
$\square$ *Fix $y = \frac{1}{3}$, then $F(\frac{2}{3}+0, \frac{1}{3}) = 1 = F(\frac{2}{3}, \frac{1}{3})$ i.e. right continuous w.r.t. x. Here, we check at $x = \frac{2}{3}$ as $F(x,y)$ has a jump on the boundary $YBAX$. Similarly, we can show that F is right continuous w.r.t. y.*
$\square$ *$F(-\infty,y) = F(x,-\infty) = 0,\ \forall x,y$; and $F(+\infty,+\infty) = 1$ also holds.*
$\square$ *Take $x_1 = 0.1, x_2 = 1.1$; $y_1 = 0.1, y_2 = 1.1$, So $F(x_2,y_2) - F(x_2,y_1) + F(x_1,y_1) - F(x_1,y_2) = 1 - 1 + 0 - 1 \ngeq 0$.*

[Do It Yourself] 2.65. *Check if F is DF or, not.*

$$F(x,y) = \begin{cases} 1 & \text{if } x+y \leq 1 \\ 0 & \text{if } x+y > 1 \end{cases}; \quad G(x) = \begin{cases} 1 & \text{if } x+2y \geq 1 \\ 0 & \text{if } x+2y < 1 \end{cases}$$

[Do It Yourself] 2.66. *Suppose (X,Y) is a bivariate RV with DF F and the marginals are F_X, F_Y. If $\alpha = \frac{F_X(x)+F_Y(y)}{2}$, $\beta = \sqrt{F_X(x)F_Y(y)}$, then show that the joint DF $F(x,y)$ satisfies $2\alpha - 1 \leq F(x,y) \leq \beta$.*
[Hint : $A = \{X \leq x\}$, $B = \{Y \leq x\} \Rightarrow F(x,y) = P(AB), F_X(x) = P(A), F_Y(y) = P(B)$. $P(AB) \leq P(A), P(AB) \leq P(B) \Rightarrow P(AB) \leq \sqrt{P(A)P(B)}$]

[Do It Yourself] 2.67. *For DFs F, F_1, F_2 show that: $1 - \sum[1 - F_i(x_i)] \leq F(x_1,x_2) \leq \min F_i(x_i)$, for all real numbers x_1, x_2 if and only if F_i's are marginal DFs of F.*