2.4.2 Discrete Random Vector

■ Probability Mass Function or, PMF: The collection of positive numbers $\{p_{ij}\}$ satisfying $P(X = x_i, Y = y_j) = p_{ij}$, for all i, j and $\sum_{i,j=1}^{\infty} p_{ij} = 1$, is called the joint probability mass function (pmf) of (X, Y).

▶ $P(X = x_i, Y = y_j) = p_{ij} \Rightarrow F(x, y) = P(X \leq x, Y \leq y) = \sum_{x_i \leq x, y_i \leq y} p_{ij}$.

▶ If $\{p_{ij}\}$ be a collection of nonnegative real numbers with $\sum_{i,j=1}^{\infty} p_{ij} = 1 \Rightarrow \{p_{ij}\}$ is the PMF of some discrete RV (X, Y).

Example 2.13. *A fair die is rolled, and a fair coin is tossed independently. Let X be the face value on the die, and let Y be the number of head appears. Find the DF $F(x, y)$ of the bivariate random variable (X, Y).*

$\Rightarrow$ *Here X can take values $1, 2, \cdots, 6$ and Y can take values $0, 1$. The bivariate random variable (X, Y) can take values: $\{(i, j) : i = 1, \cdots, 6;\ j = 0, 1\}$.*
Since both are fair: $P(X = x, Y = y) = \frac{1}{12}$ where $x = 1, \cdots, 6;\ y = 0, 1$. Therefore $F(x, y) = P(X \leq x, Y \leq y)$ is

$$F(x,y) = \begin{cases} 0 & \text{if } x < 1,\ y \in \mathbb{R} \\ 0 & \text{if } x \in \mathbb{R},\ y < 0 \\ 1/12 & \text{if } 1 \leq x < 2,\ 0 \leq y < 1 \\ 1/6 & \text{if } 1 \leq x < 2,\ 1 \leq y \\ 1/6 & \text{if } 2 \leq x < 3,\ 0 \leq y < 1 \\ 1/3 & \text{if } 2 \leq x < 3,\ 1 \leq y; \\ 1/6 & \text{if } 3 \leq x < 4,\ 0 \leq y < 1 \\ 1/2 & \text{if } 3 \leq x < 4,\ 1 \leq y \\ 1/3 & \text{if } 4 \leq x < 5,\ 0 \leq y < 1 \\ 2/3 & \text{if } 4 \leq x < 5,\ 1 \leq y \\ 5/12 & \text{if } 5 \leq x < 6,\ 0 \leq y < 1 \\ 5/6 & \text{if } 5 \leq x < 6,\ 1 \leq y \\ 1/2 & \text{if } 6 \leq x,\ 0 \leq y < 1 \\ 1 & \text{if } 6 \leq x,\ 1 \leq y \end{cases} = \begin{cases} 0 & \text{if } x < 1,\ y \in \mathbb{R};\ x \in \mathbb{R},\ y < 0 \\ 1/12 & \text{if } 1 \leq x < 2,\ 0 \leq y < 1 \\ 1/6 & \text{if } 1 \leq x < 2,\ 1 \leq y;\ 2 \leq x < 3,\ 0 \leq y < 1 \\ 1/4 & \text{if } 3 \leq x < 4,\ 0 \leq y < 1 \\ 1/3 & \text{if } 2 \leq x < 3,\ 1 \leq y;\ 4 \leq x < 5,\ 0 \leq y < 1 \\ 5/12 & \text{if } 5 \leq x < 6,\ 0 \leq y < 1 \\ 1/2 & \text{if } 3 \leq x < 4,\ 1 \leq y;\ 6 \leq x,\ 0 \leq y < 1 \\ 2/3 & \text{if } 4 \leq x < 5,\ 1 \leq y \\ 5/6 & \text{if } 5 \leq x < 6,\ 1 \leq y \\ 1 & \text{if } 6 \leq x,\ 1 \leq y \end{cases}$$

■ Marginal PMF: Let (X, Y) be a two-dim RV with PMF: $\boxed{p_{ij} = P(X = x_i, Y = y_j)}$.

▶ The marginal PMF of X is $\boxed{p_{i.} = \sum_{j=1}^{\infty} p_{ij} = \sum_{j=1}^{\infty} P(X = x_i, Y = y_j) = P(X = x_i)}$.

▶ The marginal PMF of Y is $\boxed{p_{.j} = \sum_{i=1}^{\infty} p_{ij} = \sum_{i=1}^{\infty} P(X = x_i, Y = y_j) = P(Y = y_j)}$.

▶ Marginal PMF's are univariate, so we can easily find the DF's of marginal distributions.

▶ The RV's X and Y are said to be independent if $\boxed{P(X = x, Y = y) = P(X = x)P(Y = y)}$ i.e. $p_{ij} = p_{i.}p_{.j}$.

Example 2.14. *Consider the Example 2.13, write down the joint PMF of (X,Y) in tabular form or, matrix form. Hence find the marginal PMF of X and Y.*
$\Rightarrow$ *Here X can take values $1,2,\cdots,6$ and Y can take values $0,1$. The bivariate random variable (X,Y) can take values: $\{(i,j):\ i=1,\cdots,6;\ j=0,1\}$.*
Since both are fair: $P(X=x,Y=y)=\frac{1}{12}$ where $x=1,\cdots,6;\ y=0,1$. So the table is:

Y \ X	1	2	3	4	5	6
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

Table 2.1: Joint probability distribution

☐ *The marginal distribution table is as follows:*

Y \ X	1	2	3	4	5	6	P(Y=y)
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
P(X=x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Table 2.2: Marginal probability distribution

The marginal PMF of X, shown in the row representing column totals and the marginal PMF of Y is shown in the column representing row totals.

$$P(X=x)=\begin{cases}1/6 & \text{if } x=1,2,3,4,5,6\\ 0 & O.w.\end{cases}\ ;\ P(Y=y)=\begin{cases}1/2 & \text{if } y=0,1\\ 0 & O.w.\end{cases}$$

[Do It Yourself] 2.69. *A fair coin is tossed three times. Let X = number of heads in three tossings, and Y = difference, in absolute value, between number of heads and number of tails. Write down the joint PMF of (X,Y) in tabular form. Hence find the marginal PMF of X and Y. Are X and Y independent?*

[Do It Yourself] 2.70. *Verify if $f(x,y)=\frac{e^{-2}}{x!(y-x)!};\ x=0,1,\cdots,y,\ y=0,1,\cdots,\infty$ is a joint pmf of (X,Y) or, not. If it is a joint pmf then find the marginal pmf's of X and Y. Are X and Y independent?*

$$[\underline{Hint}:\ \sum_{y=0}^{\infty}\sum_{x=0}^{y}\frac{e^{-2}}{x!(y-x)!}=\sum_{y=0}^{\infty}\sum_{x=0}^{y}\frac{e^{-2}}{y!}\binom{y}{x}=e^{-2}\sum_{y=0}^{\infty}\frac{1}{y!}2^{y}]$$

[Do It Yourself] 2.73. *Let X and Y have the joint probability mass function*
$P(X=n,Y=k)=\left(\frac{1}{2}\right)^{n+2k+1};n=-k,-k+1,\cdots;\ k=1,2,\cdots$. *Then $E(Y)$ equals*
(A) 1 *(B)* 2 *(C)* 3 *(D)* 4.

$$\left[\underline{Hint}:\ P(Y=y)=\sum_{n=-k}^{\infty}\left(\tfrac{1}{2}\right)^{n+2k+1}=\sum_{n=0}^{\infty}\left(\tfrac{1}{2}\right)^{n+k+1}\right]$$

2.4.3 Continuous Random Vector

■ Properties of PDF: *i)* $f \geq 0, \forall (x,y)$; *ii)* $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

★ *i)* Follows from the definition. *ii)* Follows from $F(\infty,\infty) = 1$.

▶ If F is abs. continuous and f is continuous at (x,y) $\Rightarrow$ $f(x,y) = \frac{\partial^2 F(x)}{\partial x \partial y}$.

▶ Every non-negative function f that is integrable over $\mathbb{R}^2$ with $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ is the PDF of some bivariate continuous RV (X,Y).

Example 2.15. *Let (X,Y) be a bivariate RV with joint PDF: $f(x,y) = e^{-(x+y)},\ x,y > 0$. Then find the DF $F(x,y)$.*

$\Rightarrow$ *We know that,* $F(x,y) = \int_{-\infty}^{x}\left[\int_{-\infty}^{y} f(u,v)dv\right]du$. *Therefore,*

$$F(x,y) = \begin{cases} 0 & \text{if } x \leq 0,\ y \in \mathbb{R} \\ 0 & \text{if } x \in \mathbb{R},\ y \leq 0 \\ \int_0^x e^{-u}du \int_0^y e^{-v}dv & \text{if } 0 < x < \infty,\ 0 < y < \infty \end{cases}$$

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & \text{if } x > 0,\ y > 0 \\ 0 & \text{Otherwise} \end{cases}$$

■ Marginal PDF: Let (X,Y) be a two-dim RV with PDF: $f(x,y)$.

▶ The marginal PDF of X is $f_x(x) = \int_{y=-\infty}^{\infty} f(x,y)dy$.

▶ The marginal PDF of Y is $f_y(y) = \int_{x=-\infty}^{\infty} f(x,y)dx$.

▶ Marginal PDF's are univariate, so we can easily find the DF's of marginal distributions.

▶ The RV's X and Y are said to be independent if $f(x,y) = f_x(x)f_y(y)$.

[Do It Yourself] 2.74. *Verify if $f(x,y) = 2;\ 0 < x < y < 1$ is a joint pdf of (X,Y) or, not. If it is a joint pdf then find the marginal pdf's of X and Y. Are X and Y independent?*

[Hint : $\int_{x=0}^{1}\int_{y=x}^{1} 2\ dy\ dx = 1;\ f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)dy = \int_x^1 2dy = 2(1-x);\ 0 < x < 1.\ f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)dx = \int_0^y 2dx = 2y;\ 0 < y < 1.$ *Not independent*]

[Do It Yourself] 2.75. *Verify if $f(x,y) = 2e^{-x-y};\ 0 < x < y < \infty$ is a joint pdf of (X,Y) or, not. If it is a joint pdf then find the marginal pdf's of X and Y. Are X and Y independent?*

[Do It Yourself] 2.78. *Let (X,Y) have the joint PDF f defined by $f(x,y) = \frac{1}{2}$ inside the square with corners at the points $(1,0),(0,1),(-1,0)$ and $(0,-1)$ in the (x,y) plane, and $= 0$ otherwise. Find the marginal PDFs of X and Y.*

[Hint : $f(x,y) = \frac{1}{2},\ if\ |x| + |y| \leq 1$]

2.4.4 Conditional distribution

■ Conditional PMF: Let (X, Y) be a discrete random variable.

▶ If $P(Y = y) > 0$, the function $\boxed{P(X = x|Y = y) = \frac{P(X=x,Y=y)}{P(Y=y)}}$, for fixed y is known as the conditional PMF of X, given $Y = y$.

▶ If $P(X = x) > 0$, the function $\boxed{P(Y = y|X = x) = \frac{P(X=x,Y=y)}{P(X=x)}}$, for fixed x is known as the conditional PMF of Y, given $X = x$.

□ If X, Y are independent $P(X = x|Y = y) = P(X = x)$ for $P(Y = y) > 0$.

□ If X, Y are independent $P(Y = y|X = x) = P(Y = y)$ for $P(X = x) > 0$.

▶ Combining: If X, Y are independent then $P(X = x, Y = y) = P(X = x)P(Y = y)$.

[Do It Yourself] 2.81. *Consider the Example 2.13, write down the conditional PMFs:* $P(X = x|Y = 0)$, $P(X = x|Y = 1)$, $P(Y = y|X = 1)$, $P(Y = y|X = 5)$.

[Do It Yourself] 2.82. *Consider the Example 2.70, write down the conditional PMFs:* $P(X = x|Y = y)$ *for fixed* y, *and* $P(Y = y|X = x)$ *for fixed* x.
$[\underline{Hint}: \ Easy]$

[Do It Yourself] 2.85. *For the trinomial RV* (X, Y) *with PMF as follows:* $P(X = x, Y = y) = \frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y (1 - p_1 - p_2)^{n-x-y}$, *where* $x, y = 0, 1, \cdots, n$ *(with* $x + y \leq n$*),* $0 < p_1 < 1$, $0 < p_2 < 1$ *so that* $p_1 + p_2 \leq 1$. *Show that it is a PMF. Find the marginal PMFs of* X *and* Y *and the conditional PMFs.*
$[\underline{Hint}: \ Easy]$

■ Conditional PDF: Let (X, Y) be a continuous RV [Note that, $P(Y = y) = 0$].

▶ The conditional DF of a random variable X, given $Y = y$, is defined as the limit $F_{X|Y}(x|y) = \lim_{h \to 0+} P(X \leq x|Y \in (y - h, y + h])$, provided the limit exists. We define the conditional density function of X, given $Y = y$ by $f_{X|Y}(x|y)$ as a nonnegative function satisfying $\boxed{F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(t|y)\ dt}$ for all $x \in \mathbb{R}$.

▶ Note that: $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\ dx = F_{X|Y}(\infty|y) = 1$.

▶ Taking $h \to 0+$, we can show that: The conditional PDF of $X|Y = y$ is $\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$.

▶ Similarly, the conditional PDF of $Y|X = x$ is $\boxed{f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}}$.

Example 2.16. *Let* (X, Y) *have the joint PDF*

$$f(x, y) = \begin{cases} c[xy + \frac{x^2}{2}] & \text{if } 0 < x < 1, 0 < y < 2 \\ 0 & \text{Otherwise} \end{cases}$$

Find c, $P(X < 1/2)$, $P(Y < 1/3)$, $P(0.5 < X < 1, 0 < Y < 1)$, $P(Y < 1|X < 1/2)$, $P(X = Y)$, $P(X < Y)$, $P(X + Y < 1)$, $P(XY < 1/2)$.
$\Rightarrow$ *Since* f *is a pdf* $\Rightarrow \int_{y=0}^{2} \int_{x=0}^{1} f(x, y) dx dy = 1 \Rightarrow c = \frac{3}{4}$.
So, $f(x, y) = \frac{3}{4}[xy + \frac{x^2}{2}]$, $0 < x < 1, 0 < y < 2$.

☐ *Marginal distributions are* $f_X(x) = \int_{y=0}^{2} f(x,y)dy = \frac{3}{4}(2x + x^2),\ 0 < x < 1.$
and $f_Y(y) = \int_{x=0}^{1} f(x,y)dy = \frac{3}{4}(\frac{y}{2} + \frac{1}{6}),\ 0 < y < 2.$
Therefore, $P(X < 1/2) = \int_{x=0}^{1/2} f_X(x)dx = \frac{7}{32}$, $P(Y < 1/3) = \int_{y=0}^{1/3} f_Y(y)dy = \frac{1}{16}$.
☐ $P(0.5 < X < 1, 0 < Y < 1) = \int_{y=0}^{1}\int_{x=0.5}^{1} f(x,y)dxdy = 0.25.$

$P(Y < 1|X < 1/2) = \frac{P(X<1/2,Y<1)}{P(X<1/2)} = \frac{0.0625}{7/32} = 0.2857.$
☐ $P(X = Y) = 0$, *since continuous distribution.*
Draw region, $P(X < Y) = \int_{y=0}^{1}\int_{x=0}^{y} f(x,y)dxdy + \int_{y=1}^{2}\int_{x=0}^{1} f(x,y)dxdy = \frac{1}{8} + \frac{11}{16} = \frac{13}{16}.$
$P(X + Y < 1) = \int_{y=0}^{1}\int_{x=0}^{y} f(x,y)dxdy = \frac{1}{8}.$
$P(XY < 1/2) = \int_{y=0}^{1/2}\int_{x=0}^{1} f(x,y)dxdy + \int_{y=1/2}^{2}\int_{x=0}^{1/2y} f(x,y)dxdy = 0.11 + 0.16 = 0.27.$

[Do It Yourself] 2.87. *Let X and Y be continuous random variables with the joint probability density function*

$$f(x,y) = \begin{cases} x+y, & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{Otherwise} \end{cases}$$

Then $P(X + Y > \frac{1}{2})$ *equals*
(A) 23/24. *(B)* 1/12. *(C)* 11/12. *(D)* 1/24.

[Do It Yourself] 2.95. *Let the joint density function of (X,Y) be*
$f(x,y) = \begin{cases} c(x+y), & \text{if } -x < y < x, 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$. *Then find the value of c.*
[$\underline{Hint}$: *Easy*]

[Do It Yourself] 2.96. *Let the joint probability mass function of random variable X and Y be given by* $P(X = m, Y = n) = \frac{e^{-1}}{(n-m)!m!2^n}$, $m = 0, 1, 2, \cdots, n;\ n = 0, 1, 2, \cdots$. *Find the marginal probability mass functions of X and Y. Also, find the conditional probability mass function of X given $Y = 5$, and that of Y given $X = 6$.*
[$\underline{Hint}$: $P(X = m) = \sum_{n=0}^{\infty} \frac{e^{-1}}{(n-m)!m!2^n} = \sum_{n=m}^{\infty} \frac{e^{-1}}{(n-m)!m!2^n} = \frac{e^{-1}}{m!}\sum_{n=0}^{\infty} \frac{1}{n!2^{n+m}}$]

[Do It Yourself] 2.97. *Let X and Y have the joint probability density function*
$f(x,y) = \begin{cases} cxye^{-(x^2+2y^2)}, & \text{if } x > 0, y > 0, \\ 0, & \text{otherwise.} \end{cases}$. *Evaluate the constant c and* $P(X^2 > Y^2)$.
[$\underline{Hint}$: $P(X^2 > Y^2) = P(X > Y)$ *as all positive*]

2.4.5 Independent Random Variables

■ Two RVs X, Y are <u>independent</u> iff $\boxed{F(x,y) = F_X(x)F_Y(y),\ \forall (x,y) \in \mathbb{R}^2}$.

▶ The RV's X and Y are <u>independent</u> iff $\boxed{P(X \in A_x, Y \in A_y) = P(X \in A_x)P(Y \in A_y)}$, for all Borel sets A_x on the $x-axis$ and A_y on the $y-axis$.

▶ The DRV's X and Y are <u>independent</u> iff $\boxed{P(X = x, Y = y) = P(X = x)P(Y = y)}$.

▶ The RV's X and Y are said to be <u>independent</u> iff $\boxed{f(x,y) = f_x(x)f_y(y)}$.
☐ If X, Y are <u>independent</u> $P(X = x|Y = y) = P(X = x)$ for $P(Y = y) > 0$.
☐ If X, Y are <u>independent</u> $P(Y = y|X = x) = P(Y = y)$ for $P(X = x) > 0$.
▶ A degenerate RV is independent of any RV.
▶ If $X_1, X_2, \cdots, X_n$ are independent $\Rightarrow$ <u>every sub-collection</u> $X_{i_1}, X_{i_2}, \cdots, X_{i_k}$ of $X_1, X_2, \cdots, X_n$ is also independent.

[Do It Yourself] 2.99. *Let X and Y are jointly distributed with pdf*

$$f(x,y) = \begin{cases} \frac{1+xy}{4} & \text{if } |x| < 1, \ |y| < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Then show that X and Y are not independent but X^2 and Y^2 are independent. Explain the reason.
[Hint : *Easy;* $P(X^2 \leq x, Y^2 \leq y) = P(-\sqrt{x} \leq X \leq \sqrt{x}, -\sqrt{y} \leq Y \leq \sqrt{y}) = \int_{-\sqrt{x}}^{\sqrt{x}} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dy\,dx = \sqrt{x}\sqrt{y}$; *Show* $F_X^2(x) = \sqrt{x}$]

■ Identically Distributed: Two RVs X, Y are identically distributed if X and Y have the same DF i.e. $F_X(x) = F_Y(y)$.

▶ Two RVs X, Y are independent and identically distributed (iid) if they are independent and identically distributed.

▶ If $P(X = Y) = 1$, we say that X and Y are equivalent RVs.

▶ Note that, equivalent RVs has equal event sets whereas for identical distribution the probability of events are equal.

▶ Two RVs X, Y are exchangeable if $(X,Y) \stackrel{d}{=} (Y,X)$ i.e (X,Y) and (Y,X) are identically distributed.

▶ If X, Y are exchangeable RVs $\Rightarrow X - Y$ has a symmetric distribution.

[Do It Yourself] 2.100. *Let X_1, X_2 be iid RVs with common PMF: $P(X = \pm 1) = \frac{1}{2}$. Take $X_3 = X_1X_2$, show that X_1, X_2, X_3 are pairwise independent but not independent.*
[Hint : $P(X_3 = 1) = P(X_1 = 1, X_2 = 1) + P(X_1 = -1, X_2 = -1) = \frac{1}{2}$, $P(X_3 = -1) = \frac{1}{2}$; $P(X_1 = 1, X_3 = 1) = P(X_1 = 1, X_1X_2 = 1) = P(X_1 = 1, X_2 = 1) = \frac{1}{4} = P(X_1 = 1)P(X_3 = 1)$]

[Do It Yourself] 2.103. *Let $X_1, X_2, \cdots, X_n$ be a set of exchangeable RVs. Then show that $E\left(\frac{X_1+X_2\cdots+X_p}{X_1+X_2\cdots+X_n}\right) = \frac{p}{n}$, for $1 \leq p \leq n$.*
[Hint : $E\left(\frac{X_1+X_2\cdots+X_n}{X_1+X_2\cdots+X_n}\right) = 1 \Rightarrow E\left(\frac{X_1}{X_1+X_2\cdots+X_n}\right) + E\left(\frac{X_2}{X_1+X_2\cdots+X_n}\right) + \cdots + E\left(\frac{X_n}{X_1+X_2\cdots+X_n}\right) = 1 \Rightarrow nE\left(\frac{X_1}{X_1+X_2\cdots+X_n}\right) = 1 \Rightarrow E\left(\frac{X_1}{X_1+X_2\cdots+X_n}\right) = \frac{1}{n}$]

[Do It Yourself] 2.104. *Let (X_1, X_2, X_3) be a RV with joint PMF: $f(x_1, x_2, x_3) = \frac{1}{4}$ if $(x_1, x_2, x_3) \in A$, where $A = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$. Are X_1, X_2, X_3 independent? Are X_1, X_2, X_3 pairwise independent? Are $X_1 + X_2$ and X_3 independent?*
[Hint : $P(X_1 = 0) = f(0,1,0) + f(0,0,1) = \frac{1}{2}$; $P(X_1 = 1) = f(1,0,0) + f(1,1,1) = \frac{1}{2}$]