Chapter 3

Expectations and Generating Functions

3.1 Expectation or, Mean

■ If X is a discrete RV with pmf $p_k = P(X = k),\ k = 1, 2, \cdots$, then the expectation of X or, EX exists and $EX = \sum_{k=1}^{\infty} kP(X = k) = \sum kp_k$, if $\sum |k|p_k < \infty$ i.e. convergent.
▶ If X is a discrete RV with pmf $p_k = P(X = k),\ k = 1, 2, \cdots$, then the expectation of $g(X)$ or, $E[g(X)]$ exists and equals $\sum g(k)p_k$, if $\sum |g(k)|p_k < \infty$ i.e. convergent.

■ Some series properties-I: *i)* $\sum \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$. *ii)* $\sum \frac{1}{n^2}$ is convergent and equals $\frac{\pi^2}{6}$. *iii)* $\sum (-1)^n \frac{1}{n}$ is convergent. *iv)* Absolute convergence $\Rightarrow$ Convergence i.e. $\sum |a_n| < \infty \Rightarrow \sum a_n < \infty$.

■ Some series-II: *i)* $(1-x)^{-1} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$. *ii)* $(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$. *iii)* $(1-x)^{-3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$. *iv)* $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$. *v)* $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$. ★ *i), ii), iii), v)* converges if $|x| < 1$.

[Do It Yourself] 3.1. *Show that EX does not exist for the pmf:*
$P(X = x) = \frac{6}{\pi^2} \frac{1}{x^2},\ x = 1, 2, \cdots$.

■ If X is a continuous RV with pdf f, then the expectation of X or, EX exists and $EX = \int_{-\infty}^{\infty} xf(x)\ dx$, if $\int |x|f(x)\ dx < \infty$ i.e. convergent.
▶ If X is a continuous RV with pdf f, then the expectation of $g(X)$ or, $E[g(X)]$ exists and equals $\int g(x)f(x)\ dx$, if $\int |g(x)|f(x)\ dx < \infty$ i.e. convergent.

[Do It Yourself] 3.4. *Show that EX does not exist for Cauchy pdf $f(x) = \frac{1}{\pi(1+x^2)},\ x \in \mathbb{R}$.*
[*Hint*: *Show* $\int xf(x)\ dx = 0$, *but* $\int |x|f(x)\ dx$ *does not exists*]

[Do It Yourself] 3.5. *Let X have the uniform distribution on the first N natural numbers i.e. $P(X = i) = \frac{1}{N},\ i = 1, 2, \cdots, N$. Then find EX, EX^2, EX^3.*

[Do It Yourself] 3.7. *If X be a RV with pdf $f(x) = \frac{2}{x^3},\ if\ x \geq 1$. Then find EX, EX^2, EX^3.*

[Do It Yourself] 3.8. *Feluda and Bomkesh plays a coin-tossing game. Feluda gets Rs/- 1 if head turns up and gives Rs/- 1 if tail turns up. If the probability of getting head is p then find the expected gain of Feluda.*
[*Hint*: $X = Gain\ of\ Feluda,\ P(X = 1) = p,\ P(X = -1) = 1 - p \Rightarrow EX = 2p - 1$]

[Do It Yourself] 3.9. *If X is a RV with pdf $f(x) = \frac{1}{\theta},\ if\ 0 < x < \theta$. Show that EX exists and then find EX.*

[Do It Yourself] 3.12. *An urn contains a white and b black balls. Balls are taken one by one with replacement until the first black ball is drawn. What is the expected number of white balls preceding the first black ball?*
[Hint : $P(B) = \frac{b}{a+b} = p, P(W) = \frac{a}{a+b} = q,\ X = RV$ *denoting the no. of white balls* $\Rightarrow P(X = x) = pq^x,\ x = 0, 1, \cdots \Rightarrow EX = \frac{q}{p}$]

[Do It Yourself] 3.13. *A target is made of 3 concentric circles of radii* $\frac{1}{\sqrt{3}}, 1, \sqrt{3}$ *(cm) respectively. Shots within the inner circle counts 4 points, next ring counts 3 points, outer ring 2 points and outside radii* $\sqrt{3}$ *count 0 points. Let X be the distance of the hit from the center (cm) and the pdf of X is*

$$f(x) = \begin{cases} \frac{2}{\pi(1+x^2)} & \text{if } x > 0 \\ 0 & \text{if o.w.} \end{cases}$$

What is the expected value of the score in i) One shot, ii) Four shot.
[Hint : $Y = Hit\,points,\ P(Y = 4) = \int_0^{1/\sqrt{3}} \frac{2}{\pi(1+x^2)}dx,\ P(Y = 3) = \int_{1/\sqrt{3}}^{1} \frac{2}{\pi(1+x^2)}dx; 4.EY$]

[Do It Yourself] 3.14. *Find the expected number of throws of a fair die until a 6 is obtained.*

[Do It Yourself] 3.19. *Let X be a continuous random variable with the probability density function*

$$f(x) = \begin{cases} ax^2, & 0 < x < 1, \\ bx^{-4}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

Where a and b are positive real numbers. If $E(X) = 1$*, then find* $E(X^2)$*.*

[Do It Yourself] 3.20. *Let X be a random variable with probability mass function*

$$P(X = n) = \begin{cases} \frac{1}{10}, & n = 1, 2, \cdots, 10 \\ 0, & \text{otherwise.} \end{cases}$$

Then find $E(max\{X, 5\})$*.*
[Hint : $EY = \sum_{x=1}^{10} max\{x, 5\}P(X = x) = \sum_{x=1}^{5} 5P(X = x) + \sum_{x=6}^{10} xP(X = x)$]

3.1.1 Expectation Properties

Theorem 3.1. *Prove the following Theorems: If X is a random variables such that* $EX < \infty$*. Also a, b are finite real numbers then:*

1. $E(a + bX)$ *exists and* $E(a + bX) = a + bEX$*.*
2. $|E(X)| \leq E|X|$*.*
3. *If X is a bounded RV then* EX^r *exists.*

$\Rightarrow$ *If X is continuous RV and* $EX < \infty \Rightarrow E(a+bX) = \int(a+bx)f(x)\,dx = a\int f(x)\,dx + b\int xf(x)\,dx = a + bEX$*. Proceed similarly when X is discrete RV.*
□ *If X is continuous RV and* $EX < \infty \Rightarrow |EX| = |\int xf(x)\,dx| \leq \int |x|f(x)\,dx = E|X|$*.*
□ *X is a bounded RV* $\Rightarrow \exists \alpha \in \mathbb{R}$ *s.t.* $P(|X| \leq \alpha) = 1$*. So* $\int |x|^r f(x)\,dx \leq \alpha^r < \infty$*.*

[Do It Yourself] 3.21. *If a random variable X takes non-negative integer values, then show that* $EX = \sum_{x=0}^{\infty} P(X > x)$*, provided the series converges. Hence show that* $EX = \sum_{x=0}^{\infty}[1 - F(x)]$*.*

[Do It Yourself] 3.22. *If X, Y are non-negative continuous RV with DF $F(x), G(x)$ respectively. Further it is given that $F(x) \geq G(x),\ \forall x \geq 0$. Then show that $EX \leq EY$, provided expectations exists.*
$[\underline{Hint}:\ Easy]$

Example 3.4. *If X is a random variable with mean EX. Then show that i) $EX = \int_0^\infty [1 - F(x) - F(-x)]\ dx = \mu$. ii) Also show that $EX^2 = \int_0^\infty 2x[1 - F(x) - F(-x)]\ dx$.*
$\Rightarrow$ *Hint:* $\mu = \int_{-\infty}^{\infty} xf(x)\ dx = \int_0^\infty xf(x)\ dx + \int_{-\infty}^0 xf(x)\ dx = \int_0^\infty (1 - F(x))dx - \int_0^\infty zf(-z)dz = \int_0^\infty (1 - F(x))dx + \int_0^\infty x\ dF(-x) = \int_0^\infty (1 - F(x))dx - \int_0^\infty F(-x)dx$.
□ *Easy*

[Do It Yourself] 3.23. *Find EX for the random variable X with distribution function*

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - (1 - x)^n & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

$[\underline{Hint}:\ Here\ X\ is\ a\ non-negative\ random\ variable]$

[Do It Yourself] 3.25. *A random variable X has probability density function $f(x) = \alpha x e^{-\beta^2 x^2},\ x > 0, \alpha > 0, \beta > 0$. If $E(X) = \frac{\sqrt{\pi}}{2}$, determine α and β.*
$[\underline{Hint}:\ There\ will\ be\ two\ equations]$

3.1.2 Moments

■ If EX^k exists, then EX^k is the k^{th} moment of (the distribution function of) X about the origin.
▶ This is also called k^{th} raw moment and denoted by $m_k = EX^k,\ k \in \mathbb{N}$.
▶ If $E|X|^\alpha < \infty$ for some positive real number α, we call $E|X|^\alpha$ the α^{th} absolute moment of X.

■ Let k be a positive integer and c be a constant. If $E(X - c)^k$ exists, we call it the moment of order k about the point c.
▶ If we take $c = EX = \mu$, then $E(X - \mu)^k$ is the central moment of order k. The k^{th} order central moment is denoted by $\mu_k = E(X - \mu)^k$.

■ The term $E(X - \mu)^2$ is called the variance of X. We write $\sigma^2 = var(X) = E(X - \mu)^2$.
▶ The quantity σ is called the standard deviation (SD) of X.
■ $E(X - c)^2$ is minimum for $c = EX$. So, $V(X) = E(X - EX)^2 \leq E(X - c)^2$.
▷ $E(X-c)^2 = E(X-EX+EX-c)^2 = E(X-EX)^2 + E[(X-EX)(EX-c)] + E(EX-c)^2 = E(X - EX)^2 + (EX - c)^2$. Minimum when $c = EX$.
▶ $\mu_n = E(X - \mu)^n = m_n - \binom{n}{1}\mu m_{n-1} + \binom{n}{2}\mu^2 m_{n-2} - \cdots + (-1)^n \mu^n$.
▶ $V(X) = \mu_2 = E(X - EX)^2 = E(X - \mu)^2 = EX^2 - \mu^2 = EX^2 - E^2X$.
▶ $E[X(X - 1)(X - 2)\cdots(X - k + 1)]$ is called factorial moments of order k.

[Do It Yourself] 3.27. *If X is a non-negative RV and $EX = 0$. Then show that $P(X = 0) = 1$.*
$[\underline{Hint}:\ \sum x_i P(X = x_i) = 0,\ \forall i \Rightarrow x_i = 0,\ \forall i \Rightarrow P(X = 0) = 1]$

[Do It Yourself] 3.29. *Show that $Var(X) = 0$ iff X is degenerate.*
[$\underline{Hint}$: $EX = a,\ EX^2 = a^2 \Rightarrow V(X) = 0;\ E(X-\mu)^2 = 0 \Rightarrow \sum_{x_i}(x_i - \mu)^2 P(X = x_i) = 0 \Rightarrow x_i = \mu,\ \forall i \Rightarrow P(X = \mu) = 1$].

[Do It Yourself] 3.31. *Let X be a continuous random variable with the probability density function $f(x) = \frac{1}{(2+x^2)^{3/2}},\ x \in \mathbb{R}$. Then $E(X^2)$*
(A) equals 0 (B) equals 1 (C) equals 2 (D) does not exist

[Do It Yourself] 3.32. *Pareto's distribution with parameters a, b (both are positive) is defined by the PDF*

$$f(x) = \begin{cases} \frac{ba^b}{x^{b+1}} & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}$$

Show that the moment of order n exists iff $n < b$. If $b > 2$, find the mean and the variance of the distribution.

[Do It Yourself] 3.33. *Poisson distribution with parameters $\lambda(> 0)$ is defined by the PMF: $P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!},\ x = 0, 1, 2, \cdots$. Find EX, EX^2, EX^3. What is mean and variance?*
[$\underline{Hint}$: *Since factorial is in pmf, we will use factorial moments*]

3.1.3 Symmetric Random Variable

■ A random variable X is <u>symmetric about a point</u> 'a' if $P(X \geq a + x) = P(X \leq a - x)$, $\forall x$.
▶ A RV X is symmetric about a point 'a' if $1 - F(a + x) + P(X = a) = F(a - x)$, $\forall x$.
▶ A <u>CRV X is symmetric</u> about a point 'a' iff $f(a - x) = f(a + x)$, $\forall x$.
▶ A RV X is symmetric about a point 'a' and $E|X| < \infty$, then $EX = a$.
▶ A RV X is symmetric about a point 'a' and $E|X| < \infty$, then $Median(X) = a$.

■ A number x is called a <u>quantile of order p</u> (or $(100p)^{th}$ percentile) for the RV X (or, for the DF F of X) if $P(X \leq x) \geq p,\ P(X \geq x) \geq 1 - p,\ 0 < p < 1$.
▶ Quantile of order p for the RV X is denoted by $\zeta_p(X)$.
▶ If x is a <u>quantile of order p</u> for the RV X with DF $F \Rightarrow p \leq F(x) \leq p + P(X = x)$.
▶ If X is a continuous RV and x is a quantile of order $p \Rightarrow \boxed{F(x) = p}$.
▶ We can easily find the <u>median</u> by putting $p = 1/2$.

[Do It Yourself] 3.37. *Let X is a continuous random variable with pdf f. If it is symmetric about the point 'a' and EX exists, then show that $EX = a$.*
[$\underline{Hint}$: $EX = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty}(x-a)f(x)dx + a = \int_{-\infty}^{\infty} uf(u+a)du + a = \int_{-\infty}^{\infty} uf(a-u)du + a = \int_{-\infty}^{\infty}(a-z)f(z)dz + a = 2a - \int_{-\infty}^{\infty} zf(z)dz$]

[Do It Yourself] 3.41. *The probability density function $f(x)$ of a random variable X is symmetric about 0. Then find $\int_{-2}^{2}\int_{-\infty}^{x} f(u)dudx$.*
[$\underline{Hint}$: $I = \int_{-2}^{2}\int_{-\infty}^{x} f(u)dudx = \int_{-2}^{2} F(x)dx = \int_{-2}^{2}[1 - F(-x)]dx = 4 - I$]

[Do It Yourself] 3.43. *Let X be a continuous random variable with the probability density function symmetric about 0. If $V(X) < \infty$, then which of the following statements is true?*
(A) $E(|X|) = E(X)$ (B) $V(|X|) = V(X)$ (C) $V(|X|) < V(X)$ (D) $V(|X|) > V(X)$.
[$\underline{Hint}$: $EX = 0,\ E|X| = \int_{-\infty}^{\infty}|x|f(x)dx = 2\int_0^{\infty}|x|f(x)dx = 2\int_0^{\infty} xf(x) > 0$]