3.2 Generating Function

■ The generating function of a random variable X is a function of the form $E[f(t, X)]$, provided the expectation exists. Here t is a parameter.

3.2.1 Probability Generating Function (PGF)

■ PGF is specially meant for a discrete distribution which has its jumps at non-negative integer values of the random variable X.
■ Let X be a RV with pmf $P(X = x) = p_x,\ x = 0, 1, 2, \cdots$, then the PGF is defined by $E(t^X) = \sum_x t^x P(X = x) = \sum_x t^x p_x = p_0 + tp_1 + t^2 p_2 + \cdots$. It is convergent for $|t| < 1$. PGF is denoted by $P_X(t)$ or, $P(t)$.
▶ Note that: $\sum_{x=0}^{\infty} t^x p_x \leq \sum_{x=0}^{\infty} t^x$ is convergent for $|t| < 1$ (Comparison test).
▶ If $EX^r < \infty$ then $EX = P'(1),\ E[X(X-1)] = P''(1),\ E[X(X-1)(X-2)] = P'''(1)$ and so on.
▶ PGF is not applicable for continuous distribution.

Example 3.9. *Consider the pmf of a binomial distribution with parameter n, p i.e. $X \sim Bin(n, p)$. It has pmf $p_x = P(X = x) = \binom{n}{x} p^x q^{n-x},\ x = 0, 1, \cdots, n$. Then find the PGF. Hence find $E(X), E[X(X-1)]$ and $V(X)$.*
$\Rightarrow$ *The PGF is $P_X(t) = E(t^X) = \sum_{x=0}^{n} t^x P(X = x) = \sum_{x=0}^{n} \binom{n}{x} (pt)^x q^{n-x} = (q + pt)^n, \forall t$.*
$E(X) = P'(1) = np(q + pt)^{n-1}_{|t=1} = np(q + p)^{n-1} = np$.
Similarly, $E[X(X-1)] = n(n-1)p^2 \Rightarrow EX^2 = n(n-1)p^2 + np$.
So $V(X) = EX^2 - E^2X = n(n-1)p^2 + np - n^2p^2 = np - np^2 = npq$.

[Do It Yourself] 3.44. *Consider the pmf of a poisson distribution with parameter λ i.e. $X \sim Poi(\lambda)$. It has pmf $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \cdots$. Then find the PGF. Hence find $E(X), E[X(X-1)]$ and $V(X)$.*

[Do It Yourself] 3.45. *Consider the pmf of a geometric distribution with parameter p i.e. $X \sim Geo(p)$. It has pmf $P(X = x) = pq^x,\ x = 0, 1, 2, \cdots$. Then find the PGF. Hence find $E(X), E[X(X-1)]$ and $V(X)$.*

Theorem 3.2. *If X, Y are non-negative integer valued independent RVs with PGFs $P_X(t), P_Y(t)$ respectively. Then show that PGF of $X + Y$ is $P_{X+Y}(t) = P_X(t)P_Y(t)$.*
□ *Let $P_X(t)$ is defined for $|t| < t_1$, $P_Y(t)$ is defined for $|t| < t_2$.*
Take $t_0 = \min\{t_1, t_2\} \Rightarrow P_{X+Y}(t)$ is defined for $|t| < t_0$.
Now $P_{X+Y}(t) = E(t^{X+Y}) = E(t^X)E(t^Y) = P_X(t)P_Y(t)$.
Note that, since X, Y are independent $\Rightarrow t^X, t^Y$ are independent $\Rightarrow E(t^{X+Y}) = E(t^X)E(t^Y)$.

[Do It Yourself] 3.49. *If $X_1 \sim Bin(n_1, p),\ X_2 \sim Bin(n_2, p)$ and X_1, X_2 are independent. Then show that $X_1 + X_2 \sim Bin(n_1 + n_2, p)$.*
[Hint : Show $P_{X_1+X_2}(t) = (q + pt)^{n_1+n_2} \Rightarrow$ It is a PGF of Binomial distribution]

[Do It Yourself] 3.50. *If $X_1 \sim poi(\lambda_1),\ X_2 \sim Poi(\lambda_2)$ and X_1, X_2 are independent. Then show that $X_1 + X_2 \sim Poi(\lambda_1 + \lambda_2)$.*

3.2.2 Moment Generating Function (MGF)

■ Let X be a RV defined on $(\Omega, \mathbb{S}, P)$. The function $M_X(t) = M(t) = E(e^{tX})$ is known as the moment generating function (MGF) of the RV X provided the expectation exists in some neighborhood of the origin i.e. $|t| < \varepsilon$ for some $\varepsilon > 0$.

▶ The MGF uniquely determines a DF and conversely if the MGF exists, it is unique.

▶ The requirement $M(t)$ exists in a neighborhood of zero is a very strong requirement that is not satisfied by some common distributions. So we can't say that MGF always exists.

▶ MGF can be applicable for both discrete and continuous distributions.

▶ If the MGF $M(t)$ of a RV X exists for $|t| < \varepsilon \Rightarrow$ The derivatives of all order exist at $t = 0$ and $\boxed{EX^r = \frac{d^r}{dx^r} M_X(t)_{|t=0}}$. Also, $\boxed{E(X-\mu)^r = \frac{d^r}{dx^r} M_{X-\mu}(t)_{|t=0}}$.

▶ Note that, $M(t) = E(e^{tX}) = 1 + tEX + \frac{t^2}{2!}EX^2 + \frac{t^3}{3!}EX^3 + \cdots = \sum_{r=0}^{\infty} \frac{t^r}{r!} m_r$.

[Do It Yourself] 3.57. *Find the MGF, EX, VX for the following distributions:*
i) $P(X = x) = \frac{6}{\pi^2}\frac{1}{x^2},\ x = 1, 2, 3, \cdots$, *ii)* $Bin(n, p)$, *iii)* $Poi(\lambda)$

[Do It Yourself] 3.58. *Find the MGF, EX, VX for the following distributions:*
i) Gamma(α, β) *distribution* : $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},\ x > 0,\ \alpha, \beta > 0.$
ii) Laplace(α) *distribution* : $f(x) = \frac{\alpha}{2} e^{-\alpha|x|},\ -\infty < x < \infty,\ \alpha > 0.$
iii) Laplace distribution : $f(x) = \frac{1}{2\alpha} e^{-\frac{|x-\beta|}{\alpha}},\ -\infty < x < \infty,\ \alpha > 0, -\infty < \beta < \infty.$
[*Hint* : *ii)* $\frac{\alpha}{2}\int_{-\infty}^{\infty} e^{tx} e^{-\alpha|x|}\, dx = \frac{\alpha}{2}[\int_{-\infty}^{0} e^{tx} e^{-\alpha|x|}\, dx + \int_{0}^{\infty} e^{tx} e^{-\alpha|x|}\, dx]$]

[Do It Yourself] 3.59. *If the MGF of X is $M_X(t)$, then show that the MGF of $aX + b$ is $e^{at} M_X(bt)$.*

Theorem 3.3. *If X, Y are independent RVs with MGFs $M_X(t), M_Y(t)$ respectively. Then show that MGF of $X + Y$ is $M_{X+Y}(t) = M_X(t)M_Y(t)$. Extend this for n RVs.*
☐ *Let $M_X(t)$ is defined for $|t| < \varepsilon_1$, $P_Y(t)$ is defined for $|t| < \varepsilon_2$.*
Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} \Rightarrow M_{X+Y}(t)$ is defined for $|t| < \varepsilon$.
Now $M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY}) = M_X(t)M_Y(t)$.
Note that, since X, Y are independent $\Rightarrow e^{tX}, e^{tY}$ are independent $\Rightarrow E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY})$.
☐ *Easy.*

[Do It Yourself] 3.70. $\boxed{\textit{Important Properties}}$: *If X_1, X_2 are independent RVs. Then using MGF show that*
i) $X_1 \sim Bin(n_1, p),\ X_2 \sim Bin(n_2, p) \Rightarrow X_1 + X_2 \sim Bin(n_1 + n_2, p).$
ii) $X_1 \sim Poi(\lambda_1),\ X_2 \sim Poi(\lambda_2) \Rightarrow X_1 + X_2 \sim Poi(\lambda_1 + \lambda_2).$

[Do It Yourself] 3.72. *The probability mass function of a random variable X is given by $P(X=x) = k\binom{n}{x}$, $x = 0, 1, \cdots, n$, where k is a constant. The moment generating function $M_X(t)$ is*
(A) $\frac{(1+e^t)^n}{2^n}$ *(B)* $\frac{2^n}{(1+e^t)^n}$ *(C)* $\frac{1}{2^n(1+e^t)^n}$ *(D)* $2^n(1+e^t)^n$.
[$\underline{Hint}$: *Show* $k = \frac{1}{2^n}$, *Then* $P(X=x) = \binom{n}{x}(\frac{1}{2})^x(\frac{1}{2})^{n-x}$]

3.2.3 Cumulant Generating Function (CGF)

■ Let X be a RV defined on $(\Omega, \mathcal{S}, P)$. The function $K_X(t) = K(t) = \ln[M_X(t)] = \ln[E(e^{tX})]$ is known as the cumulant generating function (CGF) of the RV X provided the expectation exists and positive in some neighborhood of the origin.

▶ Cumulants are $\boxed{\kappa_r = \frac{d^r}{dx^r} K_X(t)_{|t=0}}$.

▶ There are situations (not always) when work with cumulants are easier than work with moments.

Theorem 3.4. *If X, Y are independent RVs with CGFs $K_X(t), K_Y(t)$ respectively. Then show that CGF of $X+Y$ is $K_{X+Y}(t) = K_X(t) + K_Y(t)$. Extend this for n RVs.*
□ *Easy.*

3.2.4 Characteristic Functions (CF)

■ Let X be a RV defined on $(\Omega, \mathcal{S}, P)$. The complex-valued function ϕ defined on $\mathbb{R}$ by $\phi_X(t) = \phi(t) = E(e^{itX}) = E(\cos tX) + i\,E(\sin tX)$, $t \in \mathbb{R}$, is called the characteristic function (CF) of RV X.
▶ For Discrete RV: $\phi_X(t) = \sum_x (\cos tx + i\ \sin tx)P(X=k)$.
▶ For Continuous RV: $\phi_X(t) = \int_x \cos tx\ f(x)\ dx + i\ \int_x \sin tx\ f(x)\ dx$.
▶ Note that, $\phi(t)$ uniquely determines the DF of RV X.
▶ Although an MGF may not exist for some distributions but a CF always exists.

[Do It Yourself] 3.79. *Show that for a constant c, $\phi_{cX}(t) = \phi_X(ct)$.*

[Do It Yourself] 3.80. *If X_1, X_2 are independent RVs, then show that $\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\phi_{X_2}(t)$. Does the converse is true?*
[$\underline{Hint}$: *Use Cauchy distribution* $f(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty \Rightarrow \phi_X(t) = e^{-|t|}$]

3.3 Bivariate Case

We will discuss the above scenario for bivariate case.

3.3.1 Expectation

▶ If (X, Y) is a bivariate discrete RV with pmf $p_{xy} = P(X=x, Y=y)$, then the expectation of $g(X,Y)$ or, $E[g(X,Y)]$ exists and equals $\sum_{x,y} g(x,y)p_{xy}$, if $\sum |g(x,y)|p_{xy} < \infty$ i.e. convergent.

▶ If (X, Y) is a bivariate continuous RV with pdf $f(x, y)$, then the expectation of $g(X, Y)$ or, $E[g(X, Y)]$ exists and equals $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f(x, y)\, dx\, dy$, if $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x, y)|f(x, y)\, dx\, dy < \infty$ i.e. convergent.

[Do It Yourself] 3.82. *If (X,Y) are jointly distributed RVs. Then show that $E(aX + bY + c) = aE(X) + bE(Y) + c$.*
[Hint : $E(aX + bY + c) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(ax + by + c)f(x, y)dx\, dy$]

[Do It Yourself] 3.83. *X and Y are independent RVs. Then show that $E(XY) = E(X)E(Y)$.*

[Do It Yourself] 3.86. *Let X and Y be two positive integer valued random variables with the joint probability mass function*

$$P(X = m, Y = n) = \begin{cases} g(m)h(n), & m, n \geq 1 \\ 0, & \textit{otherwise.} \end{cases}$$

where $g(m) = \left(\frac{1}{2}\right)^{m-1}, m \geq 1$ and $h(n) = \left(\frac{1}{3}\right)^{n}, n \geq 1$. Then find $E(XY)$.
[Hint : $EXY = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty} xy\left(\frac{1}{2}\right)^{x-1}\left(\frac{1}{3}\right)^{y} = \sum_{x=1}^{\infty} x\left(\frac{1}{2}\right)^{x-1}\sum_{y=1}^{\infty} y\left(\frac{1}{3}\right)^{y} = (1 - \frac{1}{2})^{-2}.\frac{1}{3}(1 - \frac{1}{3})^{-2} = 4.\frac{1}{3}.\frac{9}{4} = 3$]

[Do It Yourself] 3.87. *Let X and Y be continuous random variables with the joint probability density function*

$$f(x, y) = \begin{cases} cx(1 - x), & \textit{if } 0 < x < y < 1 \\ 0, & \textit{Otherwise} \end{cases}$$

where c is a positive real constant. Then $E(X)$ equals
(A) 1/5. (B) 1/4. (C) 2/5. (D) 1/3.

3.3.2 Generating Functions

■ Let (X, Y) be a bivariate RV, then the function $M_{X,Y}(t_1, t_2) = M(t_1, t_2) = E(e^{t_1X + t_2Y})$ is known as the moment generating function (MGF) of the RV (X, Y) provided the expectation exists for $|t_j| < \varepsilon_j$ for some $\varepsilon_j > 0$, $j = 1, 2$.

▶ For discrete case $M(t_1, t_2) = \sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} e^{t_1x + t_2y}P(X = x, Y = y)$ and for continuous case $M(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x + t_2y}f(x, y)dx\, dy$.

▶ The MGF $M(t_1, t_2)$ completely determines the marginal distributions of X and Y. Moreover, $M(t_1, 0) = E(e^{t_1X}) = M_X(t_1)$ and $M(0, t_2) = E(e^{t_2Y}) = M_Y(t_2)$.

▶ If $M(t_1, t_2)$ exists, then moments of all orders of (X, Y) exist and will be found from $E(X^mY^n) = \left.\frac{\partial^{m+n}M(t_1, t_2)}{\partial t_1^m\, \partial t_2^n}\right|_{t_1=0, t_2=0}$.

[Do It Yourself] 3.94. *X and Y are iid $Bin(n,p)$ RVs i.e. pmf $P(X=x)=\binom{n}{x}p^x(1-p)^{n-x},\ x=0,1,\cdots,n$. Then using MGF find the pmf of $Z=X+Y$.*
[*Hint* : $M_X(t)=M_Y(t)=(q+pe^t)^n,\ Now\ M_{X+Y}(t)=M_X(t)M_Y(t)=(q+pe^t)^{2n}\Rightarrow Bin(2n,p)\Rightarrow P(Z=z)=\binom{2n}{z}p^z(1-p)^{2n-z},\ z=0,1,\cdots,2n$]

[Do It Yourself] 3.95. *X and Y are iid $Poi(\lambda)$ RVs i.e. pmf $P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!},\ x=0,1,2,\cdots$. Then using MGF find the pmf of $Z=X+Y$.*

3.3.3 Moments and Correlation

■ Let (X,Y) be a bivariate RV, if $E(|X^jY^k|)<\infty$ then $E(X^jY^k)$ is said to be a raw moment of order $(j+k)$ of (X,Y).
▶ Moreover, $E(X^jY^k)=\sum_x\sum_y x^jy^kP(X=x,Y=y)=\int_x\int_y x^jy^kf(x,y)\ dxdy$.

■ Let (X,Y) be a bivariate RV, if $E[|(X-EX)^j(Y-EY)^k|]<\infty$ then $E[(X-EX)^j(Y-EY)^k]$ is said to be a central moment of order $(j+k)$ of (X,Y).
▶ Moreover, $E[(X-EX)^j(Y-EY)^k]=\sum_x\sum_y(x-EX)^j(y-EY)^kP(X=x,Y=y)=\int_x\int_y(x-EX)^j(y-EY)^kf(x,y)\ dxdy$.

■ Let (X,Y) be a bivariate RV, if $E[|(X-EX)(Y-EY)|]<\infty$ then $E[(X-EX)(Y-EY)]$ is said to be covariance between (X,Y) and denoted by $Cov(X,Y)$.
▶ Covariance: $Cov(a_1X+b_1Y+c_1,a_2X+b_2Y+c_2)=a_1a_2V(X)+(a_1b_2+b_1a_2)Cov(X,Y)+b_1b_2V(Y)$.
▶ Covariance: $Cov(X,Y)=E(XY)-E(X)E(Y)$.
▶ Variance: $V(X)=EX^2-E^2X=E(X^2)-[EX]^2$ and $V(Y)=EY^2-E^2Y=E(Y^2)-[EY]^2$.
▶ Standard Deviation: $Sd(X)=\sqrt{V(X)}$ and $Sd(Y)=\sqrt{V(Y)}$.

▶ $Var(aX+bY+c)=a^2V(X)+b^2V(Y)+2abCov(X,Y)$.
▶ In general, $Var(\sum_{i=1}^n a_iX_i)=\sum_{i=1}^n a_i^2V(X_i)+\sum_{i\neq j=1}^n a_ia_jCov(X_i,X_j)$.

■ Correlation Coefficient between X,Y: $\rho=r_{xy}=\frac{Cov(X,Y)}{\sqrt{V(X)\ V(Y)}}$.
▶ If X,Y are uncorrelated $\Leftrightarrow\rho=0$.
▶ If X,Y are independent $\Rightarrow\rho=0$. If $\rho=0\not\Rightarrow X,Y$ are independent.

[Do It Yourself] 3.98. *Let X and Y have the joint probability density function*

$$f(x,y)=\begin{cases}e^{-y}, & 0<x<y<\infty,\\ 0, & otherwise.\end{cases}$$

Then the correlation coefficient between X and Y equals
(A) $\frac{1}{3}$. *(B)* $\frac{1}{\sqrt{3}}$. *(C)* $\frac{1}{\sqrt{2}}$. *(D)* $\frac{2}{\sqrt{3}}$.
[*Hint* : *Find* $EX,EY,VX,VY,Cov(X,Y)$ *then find* ρ]

[Do It Yourself] 3.102. *Let the random variables X_1 and X_2 have joint probability density function* $f(x_1, x_2) = \begin{cases} \frac{x_1 e^{-x_1 x_2}}{2}, & \text{if } 1 < x_1 < 3, x_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$.

Find the covariance between X_1 and X_2.

$\left[\underline{Hint}: \ Easy\right]$

3.3.4 Moment Inequalities

- Cauchy Schwartz Inequality: If X and Y are RVs then $\boxed{[E(XY)]^2 \leq E(X^2)E(Y^2)}$.
- Jensen Inequality: $\boxed{E[g(X)] \geq g[E(X)]}$ where g is a continuous and convex function.
- $\boxed{E[g(X)] \leq g[E(X)]}$ where g is a continuous and concave function.
- A function g is convex (concave) if $g'' > (<)0$.
- $E[f(X)g(X)] \geq E[f(X)]E[g(X)]$ if f, g are monotone in the same direction.
- $E[f(X)g(X)] \leq E[f(X)]E[g(X)]$ if f, g are monotone in the opposite direction.

[Do It Yourself] 3.105. *Prove the following inequalities (assume that expectations exists)*

1. $EX^2 \geq E^2X$.
2. $E(\frac{1}{X}) \geq \frac{1}{EX}$, $X > 0$.
3. $E(\sqrt{X}) \leq \sqrt{EX}$, $X > 0$.
4. $E(\ln X) \leq \ln(EX)$, $X > 0$.
5. $E(X^{\alpha+\beta}) \geq E(X^\alpha)E(X^\beta)$, $X > 0$ *and* $\alpha, \beta > 0$.
6. $E(\frac{1}{X}) \geq \frac{1}{EX}$, $X > 0$.
7. $E(\frac{1}{X^2}) \geq \frac{1}{EX^2}$, $X > 0$.