Chapter 2

Markov Chains

► **Markov Process** A SP $\{X(t), t \in T\}$ is said to be a Markov Process (MP), if $\forall\, t_1, \cdots, t_n,\ t_1 < t_2 < \cdots < t_n,\ P(\alpha \leq X \leq \beta \mid X(t_1) = x_1, \cdots, X(t_n) = x_n) = P(\alpha \leq X \leq \beta \mid X(t_n) = x_n)$. Here the first member has to be defined.
► **Markov Chain** A discrete parameter MP is known as a Markov Chain (MC).
► **Chain** If a SP has continuously infinite positions on which the process stands, forms a chain.

2.1 Markov Chains

A SP $\{X(t), t \in T\}$ is said to be a MC, if, $P(X_n = j \mid X_{n-1} = i, Xn - 2 = i_1, \cdots, X_0 = i_{n-1}) = P(X_n = j \mid X_{n-1} = i) = p_{ij}$. Here $j, i, i_1, \cdots \in \mathbb{Z}$.
► **Transitional Probability (One Step)**: To a pair of states (i, j) at the <u>two successive trials (one step)</u>, the associated conditional probability p_{ij} is known as the probability of transition from the state i at $(n-1)^{th}$ trial to the state j at n^{th} trial i.e. $P(X_n = j \mid X_{n-1} = i) = p_{ij} = P(X_{n+1} = j \mid X_n = i)$.
► **Transitional Probability (m Step)**: To a pair of states (i, j) at the <u>two non successive trials</u>, the associated conditional probability $p_{ij}^{(m)}$ is known as the probability of transition from the state i at n^{th} trial to the state j at $(n+m)^{th}$ trial i.e. $P(X_{m+n} = j \mid X_n = i) = p_{ij}^{(m)}$.
► **States**: The outcomes are called the states of the MC. Here $X_n = j$ means the process is at state j at n^{th} trial.
► **Initial Probability**: Unconditional probability for the state of MC is called initial probability e.g. $P[X(t) = j] = p_j$ is the initial probability of the process at state j.
► **Homogeneous and non-homogeneous MC**: The transition probability may or, may not be dependent of n. If the transition probability p_{ij} does not dependent on n (i.e. step) implies the MC is said to be homogeneous (or to have stationary transition probabilities).

If p_{ij} is dependent on n, the chain is said to be non-homogeneous.

Example 2.1. *Let, X_n be the RV denotes the outcome of the n^{th} toss $(n = 1, 2, \cdots)$ of a coin where*

$$X_n = \begin{cases} 1 & \text{if Head appears} \\ 0 & \text{if Tail appears} \end{cases}$$

with $P(X_n = 1) = p$ and $P(X_n = 0) = 1 - p$. Here, $X_1, X_2, \cdots$ are independent.
Define, $S_n = X_1 + \cdots + X_n$ i.e. total no. of heads up to n^{th} trial. So S_n is a RV takes values $0, 1, \cdots, n$.
Here, $P(S_{n+1} = j + 1 \mid S_n = j) = p$ and $P(S_{n+1} = j \mid S_n = j) = 1 - p$, so the $(n+1)^{th}$ outcome depends only on n^{th} outcome it implies a Markov Chain.
Also note that the probabilities are not at all affected by the values of $S_1, \cdots, S_{n-1}$. Since $P(S_{n+1} = j) = P(S_n + X_{n+1} = j) = P(X_{n+1} = 1|S_n = j-1)P(S_n = j-1) + P(X_{n+1} = 0|S_n = j)P(S_n = j)$

2.1.2 Transition Probability Matrix

Let $\{X_n, n \geq 0\}$ be a MC with states $1, 2, \cdots, k$. Then all the one step transition probabilities $p_{ij} = P(X_{n+1} = j \mid X_n = i),\ i, j = 1, 2, \cdots, k$, can be written in the form of a square matrix is known as one step transition probability matrix (TPM) of the MC and

it is denoted by P. Therefore,
$$P = \begin{array}{c} \\ X_n{=}1 \\ 2 \\ 3 \\ \vdots \\ k \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} X_{n+1}{=}1 & 2 & 3 & \cdots & k \end{array} \\ \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1k} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2k} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3k} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p_{k1} & p_{k2} & p_{k3} & \cdots & p_{kk} \end{pmatrix} \end{array}.$$

► For TPM, $\sum_{j=1}^{k} p_{ij} = 1,\ i = 1, 2, \cdots, k$, i.e. Row sum = 1.

■ Similarly, m step TPM of the MC is denoted by $P^{(m)}$.

Therefore,
$$P^{(m)} = \begin{array}{c} \\ X_n{=}1 \\ 2 \\ 3 \\ \vdots \\ k \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} X_{n+m}{=}1 & 2 & 3 & \cdots & k \end{array} \\ \begin{pmatrix} p_{11}^{(m)} & p_{12}^{(m)} & p_{13}^{(m)} & \cdots & p_{1k}^{(m)} \\ p_{21}^{(m)} & p_{22}^{(m)} & p_{23}^{(m)} & \cdots & p_{2k}^{(m)} \\ p_{31}^{(m)} & p_{32}^{(m)} & p_{33}^{(m)} & \cdots & p_{3k}^{(m)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p_{k1}^{(m)} & p_{k2}^{(m)} & p_{k3}^{(m)} & \cdots & p_{kk}^{(m)} \end{pmatrix} \end{array}.$$

2.1.3 Random Walk

■ A random walk is a stochastic or, random process, that describes a path that consists of several random steps in one or, more than one dimension.

► Example (1D): A random walk starts at 0 and at each step moves $+1$ or, -1 with equal probability on the integer line.

► Example (2D): A random walk starts at the corner of a square and at each step moves clockwise or, anti-clockwise with equal probability on the corner of a square.

► Example (3D): The path traced by a molecule as it travels in a liquid or, a gas i.e. Brownian motion.

► Example: The price of a fluctuating stock, etc.

► Random walks are a fundamental topic in discussions of Markov processes. We will study several properties, like dispersal distributions, first-passage or hitting times, encounter rates, recurrence or transience to quantify their behavior.

Example 2.2. *Suppose there are six shops named $1, 2, \cdots, 6$ and a customer moves shop i to shop $i+1$ with probability p and i to shop $i-1$ with probability q, here $1 < i < 6$ and $p + q = 1$. Shop 1 and 6 are respectively pizza and wine shop, once customer goes that shop will remain there i.e. absorbing state. Find the one step TPM.*

■ *Let X_n be the position of the people after n moves. So the states of X_n are $1, 2, \cdots, 6$. It is given that, $P(X_{n+1} = 1 \mid X_n = 1) = 1$ and $P(X_{n+1} = 6 \mid X_n = 6) = 1$. Also $P(X_{n+1} = i+1 \mid X_n = i) = p$ and $P(X_{n+1} = i-1 \mid X_n = i) = q$ for $1 < i < 6$.*

Therefore the one step TPM is
$$P = \begin{array}{c} X_{n+1}= \\ X_n=1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

[Do It Yourself] 2.1. *It is known from the Meteorological Department that, for the month of July if today rains the probability of rains tomorrow is 0.76. Also if today does not rain the probability of rains tomorrow is 0.43. Write down the one step TPM for the SP.*

[Do It Yourself] 2.2. *A particle performs a random walk with absorbing barriers, say, as 1 and 4. Whenever it is at any position r $(1 < r < 4)$, it moves to $r+1$ with probability 0.4 or to $(r-1)$ with probability 0.6. But if it reaches to 1 or, 4 it remains there itself. Let X_n be the position of the particle after n moves. The different states of X_n are the different positions of the particle. $\{X_n\}$ is a Markov chain, find its one step TPM.*

[Do It Yourself] 2.3. *Suppose there are two popular online shopping sites (Amazon and Flipkart) have lots of customer. Among them 30% change their preferences from Flipkart to Amazon in every month and 25% change their preferences from Amazon to Flipkart in every month. Let X_n be the preference of the customer in n month. Then for the MC $\{X_n\}$, find its one step TPM.*

[Do It Yourself] 2.4. *Suppose there are three popular online shopping sites (Amazon, Flipkart and Myntra) have lots of customer. Among them 20%, 15% change their preferences from Flipkart to Amazon, Myntra respectively in every month; 24%, 18% change their preferences from Myntra to Amazon, Flipkart respectively in every month and 12%, 9% change their preferences from Amazon to Flipkart, Myntra respectively in every month. Let X_n be the preference of the customer in n month. Then for the MC $\{X_n\}$, find its one step TPM.*

Example 2.3. *Show that a MC is completely defined by initial and transitional probabilities.*

▷ *Suppose* $\{X_n\}$ *is a MC having states* $i_0, i_1, \cdots, i_n$.

Now $P(X_n = i_n, X_{n-1} = i_{n-1}, \cdots, X_1 = i_1, X_0 = i_0) = P(X_n = i_n \mid X_{n-1} = i_{n-1}, \cdots, X_1 = i_1, X_0 = i_0)P(X_{n-1} = i_{n-1}, \cdots, X_1 = i_1, X_0 = i_0) = P(X_n = i_n \mid X_{n-1} = i_{n-1})P(X_{n-1} = i_{n-1}, \cdots, X_1 = i_1, X_0 = i_0) = p_{i_{n-1}i_n}P(X_{n-1} = i_{n-1}, \cdots, X_1 = i_1, X_0 = i_0) = p_{i_{n-1}i_n}p_{i_{n-2}i_{n-1}}P(X_{n-2} = i_{n-2}, \cdots, X_1 = i_1, X_0 = i_0) = \cdots = p_{i_{n-1}i_n}\cdots p_{i_1i_2}P(X_1 = i_1, X_0 = i_0) = p_{i_{n-1}i_n}\cdots p_{i_1i_2}P(X_1 = i_1|X_0 = i_0)P(X_0 = i_0) = [p_{i_{n-1}i_n}\cdots p_{i_1i_2}p_{i_0i_1}]p_{i_0} = [Transition\ Probabilities]\ Initial\ Probability$

Example 2.4. *The TPM of a MC* $\{X_n,\ n = 1, 2, \cdots\}$ *having three states* $1, 2$ *and* 3 *is*

$$P = \begin{pmatrix} 0.2 & 0.5 & 0.3 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{pmatrix} \text{ and the initial distribution is } \Pi_0 = (0.5, 0.4, 0.1).$$

Find $P(X_1 = 1),\ P(X_2 = 1, X_1 = 1),\ P(X_2 = 3),\ P(X_3 = 2, X_2 = 3, X_1 = 2, X_0 = 1)$.

■ $P(X_1 = 1) = P(X_1 = 1, X_0 = 1) + P(X_1 = 1, X_0 = 2) + P(X_1 = 1, X_0 = 3) = P(X_1 = 1 \mid X_0 = 1)P(X_0 = 1) + P(X_1 = 1 \mid X_0 = 2)P(X_0 = 2) + P(X_1 = 1 \mid X_0 = 3)P(X_0 = 3) = p_{11} \times 0.5 + p_{21} \times 0.4 + p_{31} \times 0.1 = 0.2 \times 0.5 + 0.3 \times 0.4 + 0.1 \times 0.1 = 0.23$.

► $P(X_2 = 1, X_1 = 1) = P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1) = p_{11} \times 0.23 = 0.2 * 0.23 = 0.046$.

► $P(X_2 = 3)$ *is easy.*

► $P(X_3 = 2, X_2 = 3, X_1 = 2, X_0 = 1) = P(X_3 = 2 \mid X_2 = 3, X_1 = 2, X_0 = 1)P(X_2 = 3, X_1 = 2, X_0 = 1) = P(X_3 = 2 \mid X_2 = 3)P(X_2 = 3, X_1 = 2, X_0 = 1) = p_{32}P(X_2 = 3 \mid X_1 = 2, X_0 = 1)P(X_1 = 2, X_0 = 1) = p_{32}p_{23}p_{12}P(X_0 = 1) = 0.5 \times 0.4 \times 0.5 \times 0.5 = 0.05$.

[Do It Yourself] 2.5. *The TPM of a MC* $\{X_n,\ n = 1, 2, \cdots\}$ *having three states* $1, 2$

$$\text{and } 3 \text{ is } P = \begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.6 & 0.3 & 0.1 \end{pmatrix} \text{ and the initial distribution is } \Pi_0 = (0.3, 0.4, 0.3).$$

Find $P(X_2 = 1),\ P(X_2 = 1 \mid X_1 = 1),\ P(X_3 = 2, X_2 = 3 \mid X_1 = 2, X_0 = 1),\ P(X_3 = 2, X_2 = 3, X_1 = 2 \mid X_0 = 1)$.

[Do It Yourself] 2.6. *In example 2.1, find the probability that there will be i) rain on* 3^{rd} *day ii) no rain on* 4^{th} *day.*

[<u>*Hint*</u> *i)* $P(X_3 = 1)$ *ii)* $P(X_4 = 0)$]

[Do It Yourself] 2.7. *In example 2.4, find the probability that a customer visit i) Amazon on* 1^{st} *month, Flipkart on* 2^{nd} *month and Myntra on* 2^{nd} *month, ii) Flipkart on* 1^{st} *month, Amazon on* 2^{nd} *month and Myntra on* 3^{rd} *month, iii) Flipkart on* 2^{nd} *month, Amazon on* 3^{rd} *month given Myntra on* 1^{st} *month.*