- <u>Degenarate distribution</u> (One point distribution)

$$P(X = a) = \begin{cases} 1 \\ 0, & o.w \end{cases}$$

- <u>Binomial Dist$\frac{n}{\cdot\cdot}$</u> : $X \sim Bin(n, P)$

$$P(X = x) = \begin{cases} {}^{n}C_{x}\, p^{x} (1-P)^{n-x} ; & x = 0, 1, 2, \ldots n \\ 0, & o.w \end{cases}$$

- <u>Geometric Dist$\frac{n}{\cdot\cdot}$</u> : $X \sim Geo(P)$

$$P(X = x) = \begin{cases} p.q^{x}, & x = 0, 1, \cdots \infty \;.\;, \; q = (1-P) \\ 0, & o.w \end{cases}$$

- <u>Negatbre Binomial Dist$\frac{n}{\cdot\cdot}$</u> : $X \sim Neg.\, bin(r, P)$

$$\cancel{P(X = x)} = \begin{cases} \cancel{\binom{r-1}{x-1} p^{x} q^{r-x}}, & \cancel{x = 1, 2, \ldots r} \\ \cancel{0}, & \cancel{o.w} \end{cases}$$

$$P(X = x) = \begin{cases} \binom{x+r-1}{x} \cdot p^{r} q^{x}, & x = 0, 1, 2, \ldots \infty \\ 0, & o.w \end{cases}$$

- $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$

- $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

- $(1+x)^{-1} = \sum_{r=0}^{\infty} (-1)^r x^r \ , \ |x| < 1$

- $(a+b)^{-r} =$

Negative Binomial

~~Consider a succession of trials~~

X denotes the no. of failure that preceed the rth success.

$\Rightarrow X + r$ is the total no. of replication needed to produce r success

$\Rightarrow$ The last trial results in a success and among the previous $r + x - 1$ trials there are exactly x failures.

HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT,

HTTH, TTHH, THHT, THTH, HTTT, THTT,

TTHT, TTTH, TTTT

- $X \sim Bin(4, \frac{1}{2})$ coin is unbiased

X = no. of heads.

$P(X=2) = \frac{6}{16}$

$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$P(X=3) = \frac{4}{16}$

$P(X=2) = \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$

$P(X=1) = \frac{1}{16}$

$= \boxed{\frac{6}{16}}$

- $X \sim Geo\left(\frac{1}{2}\right)$

$P(X=0) = \frac{1}{16}$ $p = \frac{1}{2}, \; q = \frac{1}{2}$

$P(X=1) =$ H $= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^1 = \frac{1}{4}$ $P(X=x) = pq^x$

TH

TTH

TTTH

- $X \sim Negbin(r, p)$

$r + x$ = total no. of trials

exactly r success and x failure

$P(X=x) = \binom{r+x-1}{x} p^r q^x$, $x = 0, 1, 2, \ldots \infty$

$$(p+q)^{-r} = \sum_{x=0}^{\infty} (-1)^x \binom{-r}{x} p^r q^x \quad , x=0,1,\ldots$$

$$(-1)^x \binom{-r}{x} = \binom{r+x-1}{r-1} = \binom{r+x-1}{x}$$

H = success ,

r = 2 , success

(1) HH

(2) TH | H   TTH | H

HT | H   THT | H

HTT | H

r = 3 , $\boxed{X = 0, 1, 2}$ X = Tail denote

HHH

HTH H

HHT H

THHH

TTH H H

HTTH H

THTH H

Inverse polynomial

$$E(t^x) \cancel{= \sum \binom{n}{x} (p)}$$

$$= \sum t^x \binom{n}{x} p^x q^{n-x}$$

$$= (pt+q)^x$$

↓ pgf of binomial

$$E(e^{tx}) = \sum e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= (pe^t + q)^x \rightarrow \text{mgf of binomial}$$

- $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx \quad , x>0$

$$\Gamma(an) = \frac{\Gamma(n)}{a^n}$$

- $U(0, 0.5) = \frac{1}{0.5-0} = 2$

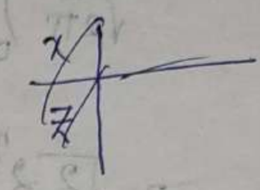
$\therefore, = \cancel{\frac{1}{\sqrt{2\pi}}}$

Pdf of standard normal.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty<x<\infty$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

Let, $\frac{x^2}{2} = z$



or, $x^2 = 2z$

$2x\,dx = 2\,dz$

$\cancel{z = \frac{x^2}{2}}$ $\quad x = \sqrt{2z}$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-z} \cdot \frac{dz}{\sqrt{2z}}$$

$$= \frac{2}{\sqrt{2}\sqrt{2\pi}} \int_0^\infty z^{\frac{1}{2}-1} e^{-z}\,dz$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = 1$$

- Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- $E(x^n) = \int_{-\infty}^{\infty} x^n \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty x^n e^{-x^2/2}\,dx$$

$\frac{x^2}{2} = z$, $x = \sqrt{2z}$

$2x\,dx = 2\,dz$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty x^{n-1} \cdot e^{-x^2/2}\, x\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty 2 z^{\frac{1}{2}(n-1)} \cdot e^{-z}\, dz$$

$$= \frac{\sqrt{2}\, 2^{n/2}}{\sqrt{\pi}\, 2} \int_0^\infty z^{\left(\frac{n}{2} - \frac{1}{2}\right)} e^{-z}\, dz$$

$$= \frac{2^{n/2}}{\sqrt{\pi}} \int_0^\infty z^{\frac{n+1}{2} - 1} e^{-z}\, dz$$

$$= \frac{2^{n/2}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right)$$

- $\Gamma(n+1) = n!$, if n = integer
- $\Gamma(n+1) = n\Gamma(n)$ $\forall n$

for $n = 4$

$\Gamma(5/2)$

$= \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$= \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)$

$= \frac{3}{4}\sqrt{\pi}$

for $n = 6$

$$\Gamma\left(\frac{6+1}{2}\right) = \Gamma\left(7/2\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2} \cdot \frac{3}{4} \cdot \sqrt{\pi}$$

$$= \frac{15}{8}\sqrt{\pi}$$

- <u>Mgf of standard normal dist^{n}</u>

$$E(e^{tx}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tx} \cdot e^{-x^2/2}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{tx - \frac{x^2}{2}}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{\frac{2tx - x^2}{2}} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{(x^2 - 2tx + t^2 - t^2)}{2}} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\{(x-t)^2 - t^2\}}{2}} dx$$

$$= \sqrt{\frac{2}{\pi}} e^{\frac{t^2}{2}} \int_0^\infty e^{-\frac{(x-t)^2}{2}} dx$$

$$\frac{(x-t)^2}{2} = z \quad \Big| \quad \cdot \sqrt{z}\, dx = dz$$

$$(x-t)\, dx = dz$$

$$= \sqrt{\frac{2}{\pi}} e^{t^2/2} \int_0^\infty e^{-z}\, \cancel{dz}\, z^{\frac{1}{2}-1} dz$$

$$= \sqrt{\frac{2}{\pi} \cdot \frac{1}{\sqrt{2}}} e^{t^2/2}\, \Gamma\left(\frac{1}{2}\right)$$

$$= \cancel{}\, e^{t^2/2}$$

~~Mgf~~

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty$$

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\frac{x-\mu}{\sigma} = z$$

$$\Rightarrow x-\mu = \sigma z$$

$$\Rightarrow dx = \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\cancel{\sigma}} \times \cancel{\sigma} \cdot e^{-\frac{z^2}{2}} dz$$

$$\frac{z^2}{2} = t$$

$$z^2 = 2t$$

$$2z\,dz = 2dt$$

$$dz = \frac{dt}{z}$$

$$= \frac{dt}{\sqrt{2t}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} t^{-1/2} e^{-t} dt$$

$$= \frac{2}{2\sqrt{\pi}} \int_{0}^{\infty} t^{1/2-1} e^{-t} dt$$

$$= \frac{1}{\sqrt{\pi}} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1$$

$$E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad \underline{\underline{\text{mgf}}}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{t(\mu+\sigma u)} \cdot e^{-\frac{1}{2}u^2} du \qquad \frac{x-\mu}{\sigma} = u$$

$$dx = \sigma du$$

$$= e^{t\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tu\sigma - \frac{u^2}{2}} du \qquad x = \mu + \sigma u$$

$$= e^{t\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tu\sigma - \frac{u^2}{2} + \frac{t^2\sigma^2}{2} - \frac{t^2\sigma^2}{2}} du$$

$$= e^{t\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{2tu\sigma - u^2 - t^2\sigma^2}{2}} \cdot e^{\frac{t^2\sigma^2}{2}} du$$

$$= e^{t\mu + \frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-t\sigma)^2} du$$

Let, $u - t\sigma = \omega$

$$\Rightarrow du = d\omega$$

$$= e^{t\mu + \frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} d\omega$$

$$= e^{t\mu + \frac{t^2\sigma^2}{2}} \times 2 \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2} d\omega$$

$$= e^{t\mu + \frac{t^2\sigma^2}{2}}$$ (from std normal dis^t $= 1$)

$M_x(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$

$$E(x) = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= e^{t\mu + \frac{t^2\sigma^2}{2}} \cdot (\mu + t\sigma^2) \Big|_{t=0}$$

$$= \mu$$

$$E(x^2) = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0} = e^{t\mu + \frac{t^2\sigma^2}{2}} (\mu + t\sigma^2)^2 + \cancel{(\mu + \sigma} \, e^{t\mu + \frac{t^2\sigma^2}{2}} \cdot \sigma^2 \Big|_{t=0}$$

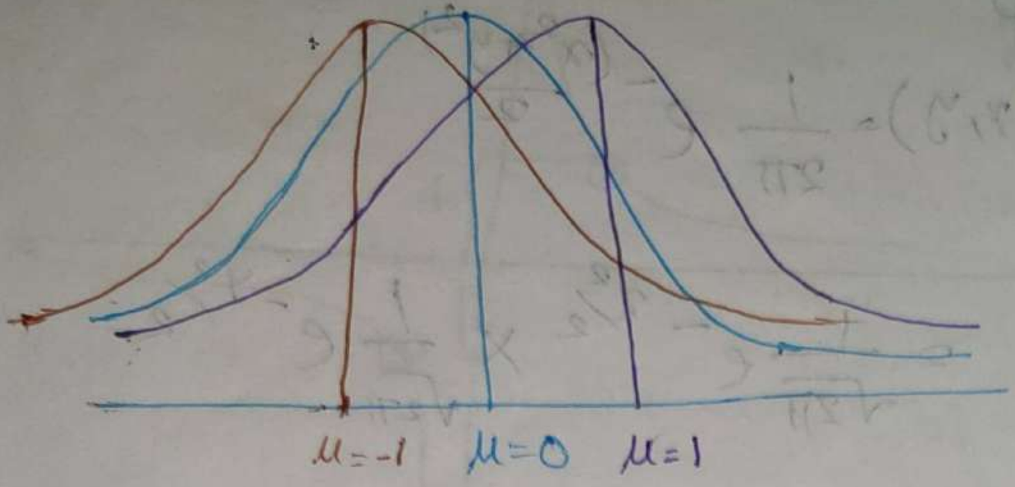
$$= \mu^2 + \sigma^2 = E(x^2)$$

$$V(x) = E(x^2) - \{E(x)\}^2$$
$$= \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$

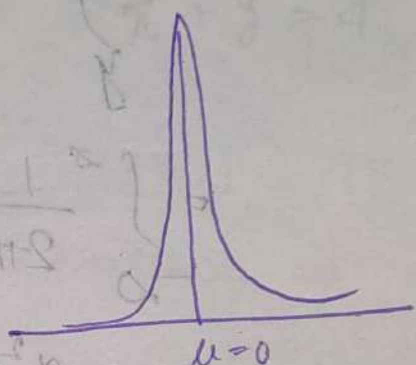
(i) $\mu = 0, \sigma = 1$ (iii) $\mu = -1, \sigma = 1$

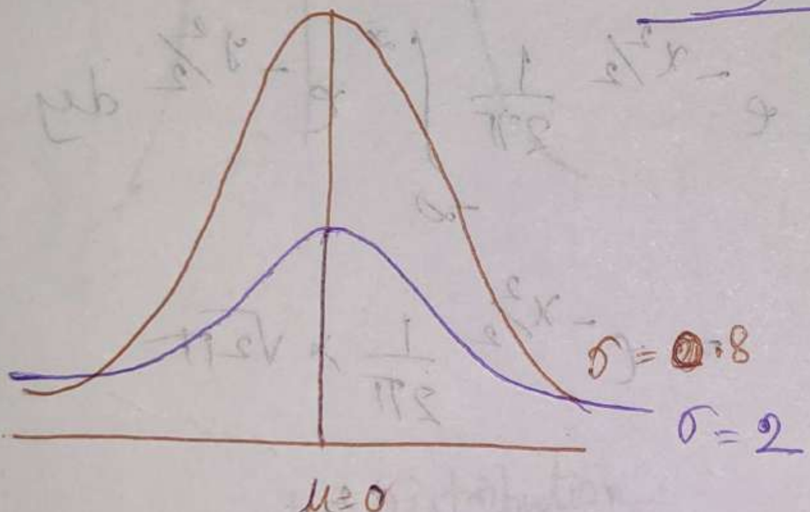
(ii) $\mu = 1, \sigma = 1$



(iv) $\mu=0,\ \sigma=2$

(v) $\mu=0,\ \sigma=0.8$





Standard bivariate normal distribution

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x^2-2\rho xy+y^2\right]\right\}$$

where $\rho \in (-1, 1)$

$-\infty < x, y < \infty$

Now, if $\rho = 0$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \times \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$f_X(x) = \int_y f(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dy$$

$$= e^{-x^2/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

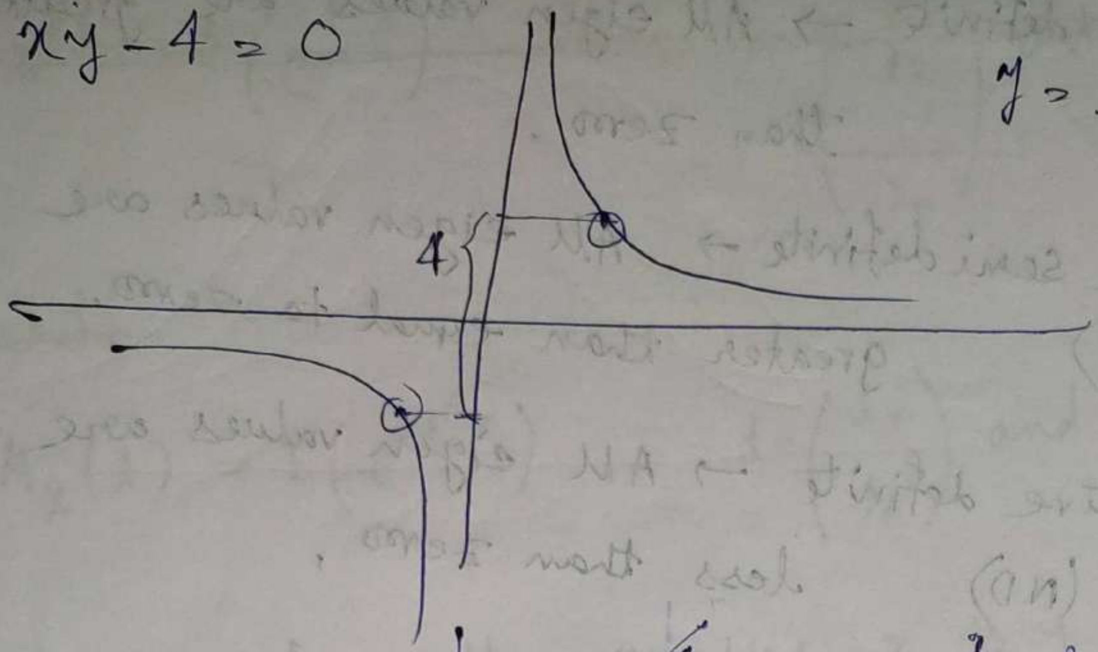
$$= e^{-x^2/2} \frac{1}{2\pi} \times \sqrt{2\pi}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$ = Marginal dist^n of x

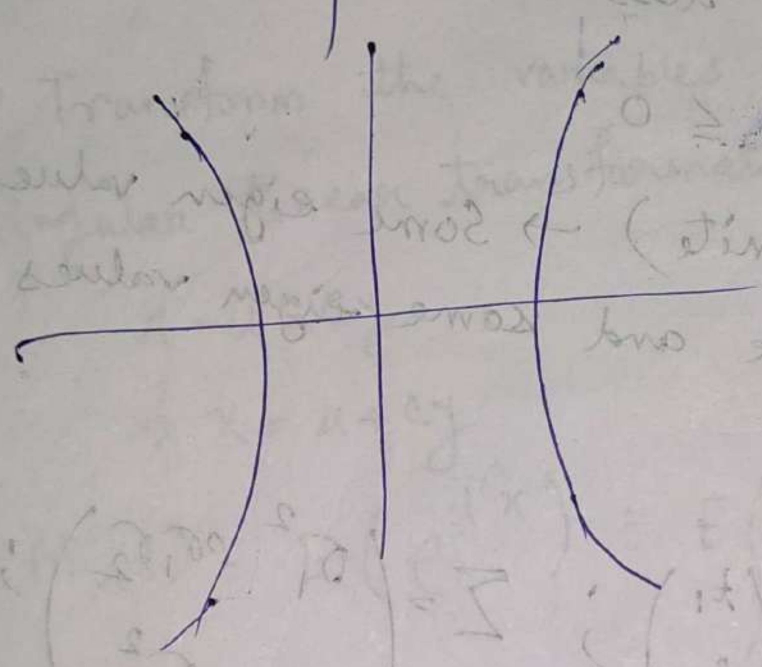
$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$ = Marginal dist^n of y

$xy - 4 = 0$

$y = \frac{4}{x}$



$x^2 + y^2 = 4$



Surface plot
———
volume plot

~~Bia~~ Bivariate Normal Distribution

$$f_{X,Y}(x, y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\left[\frac{1}{-2(1-\rho^2)} \left\{ \frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1 \sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \right\} \right]$$

Positive definite → All eigen values are greater
(PD) than zero.

Positive semi definite → All eigen values are
(PSD) greater than equal to zero.

Negative definite → All eigen values are
(ND) less than zero.

NSD → ≤ 0

ID (Indefinite) → Some eigen values are ~~positive~~ positive and some eigen values are negative.

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ; \; t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} ; \; \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} ;$$

$$|\Sigma| = \sigma_1^2 \sigma_2^2 (1-\rho^2)$$

$$t'\mu = (t_1 \;\; t_2) \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = t_1\mu_1 + t_2\mu_2$$

$$t'\Sigma t = (t_1 \;\; t_2) \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$= t_1^2 \sigma_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + t_2^2 \sigma_2^2 .$$

~~$M_x(t) = E\left(e^{t'x}\right)$ where $t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_p \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \end{pmatrix}$~~

~~Therefor~~

$$M_X(t) = E\left(e^{t'x}\right), \quad t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \text{ and } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

~~The~~ Transform the variables x to y by non singular linear transformation

$$x - \mu = Cy$$

$$\Rightarrow x = \mu + Cy$$

$$M_X(t) = E\left(e^{t'x}\right) = E\left(e^{t'(\mu + Cy)}\right)$$

$$= e^{t'\mu} E\left(e^{t'Cy}\right)$$

$$= e^{t'\mu} E\left(e^{(C't)'y}\right)$$

$$= e^{t'\mu} E\left(e^{u'y}\right), \quad u = C't$$

Mgf of y,

$$M_y(u) = E\left(e^{u'y}\right), \quad u' = (u_1, u_2)$$

$$= E\left(e^{u_1 y_1 + u_2 y_2}\right)$$

$$= E\left(\prod_{i=1}^{2} e^{u_i y_i}\right) = \prod_{i=1}^{2} E\left(e^{u_i y_i}\right)$$

$$= \prod_{i=1}^{2} \left(M_{y_i}(u_i)\right)$$

$$= \prod_{i=i}^{2} e^{\frac{1}{2} u_i^2}$$

Transform (x_1, x_2) to (y_1, y_2) by the non-singular linear transformation :

$$x - \mu = C_{2\times 2}\, y_{2\times 1} \quad \text{where } y' = (y_1 \;\; y_2)$$

$$\begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\therefore x_1 - \mu_1 = c_{11} y_1 + c_{12} y_2$$

$$x_2 - \mu_2 = c_{21} y_1 + c_{22} y_2$$

$$M_{X,Y}(t_1, t_2) = E\left(e^{t_1 x + t_2 y}\right)$$

$$= E\left(e^{t_1 x'} \cdot e^{t_2 y}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)}\, dx\, dy$$

$$= \int_y \int_x e^{t_1 x + t_2 y} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[(x-\rho y)^2 + (1-\rho^2) y^2\right]}\, dx\, dy$$

H.W

1. Bivariate normal density function proof.
2. Marginal distⁿ of $f_X(x)$, $f_Y(y)$ of bivariate normal
3. Bivariate mgf
4. Bivariate ~~pg~~f
5. Realisation of Multinomial distⁿ

$$\cancel{\begin{pmatrix} X \\ Y \end{pmatrix} \sim N \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}};$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right)$$

Moment generating function of bivariate normal dist^n :

$$M_{X,Y}(t_1, t_2) = E\left(e^{t_1 x + t_2 y}\right)$$

$$= \int_y \int_x e^{t_1 x + t_2 y} f(x,y)\, dx\, dy$$

$$= \int_y \int_x e^{t_1 x + t_2 y} f\left(\tfrac{y}{x}\right) \cdot f(x)\, dx\, dy$$

$P\left(\frac{Y}{X}\right) = \frac{P(Y \cap X)}{P(X)}$

$f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)}$

$$= \int_y e^{t_2 y} \cdot f\left(\tfrac{y}{x}\right) dy \cdot \int_x e^{t_1 x} f(x)\, dx$$

$X \sim N(\mu, \sigma^2)$

$M_X(t) = e^{t\mu + \frac{1}{2} t^2 \sigma^2}$

$f\left(\frac{Y}{X}\right) \sim N\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2 (1-\rho^2)\right)$

$$= e^{t_2\left[\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1)\right] + \frac{1}{2} t_2^2 \sigma_2^2 (1-\rho^2)} \cdot \int_x e^{t_1 x} f(x)\, dx$$

$$= e^{t_2 \mu_2 + t_2 \rho \frac{\sigma_2}{\sigma_1} x - t_2 \mu_1 \rho \frac{\sigma_2}{\sigma_1} + \frac{1}{2} t_2^2 \sigma_2^2 (1-\rho^2)} \cdot \int_x e^{t_1 x} f(x)\, dx$$

$$= e^{t_2 \mu_2 - t_2 \mu_1 \rho \frac{\sigma_2}{\sigma_1} + \frac{1}{2} t_2^2 \sigma_2^2 (1-\rho^2)} \cdot \int_x e^{t_1 x + t_2 \rho \frac{\sigma_2}{\sigma_1} x} f(x)\, dx$$

$$= e^{t_2 \mu_2 - t_2 \mu_1 \rho \frac{\sigma_2}{\sigma_1} + \frac{1}{2} t_2^2 \sigma_2^2 (1-\rho^2)} \cdot \int_x e^{\left(t_1 + t_2 \rho \frac{\sigma_2}{\sigma_1}\right) x} f(x)\, dx$$

$$= e^{t_2 \mu_2 - t_2 \mu_1 \rho \frac{\sigma_2}{\sigma_1} + \frac{1}{2} t_2^2 \sigma_2^2 (1-\rho^2)} \cdot e^{\left(t_1 + t_2 \rho \frac{\sigma_2}{\sigma_1}\right)\mu_1 + \frac{1}{2}\left(t_1 + t_2 \rho \frac{\sigma_2}{\sigma_1}\right)^2 \sigma_1^2}$$

$$= e^{t_1 \mu_1 + t_2 \mu_2 - \cancel{t_2 \mu_1 \rho \frac{\sigma_2}{\sigma_1}} + \frac{1}{2} t_2^2 \sigma_2^2 (1-\rho^2) + \cancel{t_2 \mu_1 \rho \frac{\sigma_2}{\sigma_1}} + \frac{1}{2} t_1^2 \sigma_1^2 + \frac{1}{2} t_2^2 \rho^2 \frac{\sigma_2^2}{\sigma_1^2} + t_1 t_2 \rho \sigma_2 \sigma_1}$$

$$= e^{t_1\mu_1 + t_2\mu_2 + \frac{1}{2}(t_1^2\sigma_1^2 + 2t_1t_2\rho\sigma_1\sigma_2 + t_2^2\sigma_2^2)} = \boxed{M_{XY}(t_1, t_2)} = L(t_1, t_2)$$

$$E(X^n Y^m) = \frac{L^{m+n}(t_1, t_2)}{\partial^m t_1 \, \partial^n t_2}$$

- Bivariate Normal dist$^{\underline{n}}$, density function proof!

We have to show, $\int_x \int_y f(x,y)\, dy\, dx = 1$

$$\int_y \int_x \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right]} dx\, dy$$

Let, $\frac{x-\mu_1}{\sigma_1} = t$, $\frac{dx}{\sigma_1} = dt$ | $\frac{y-\mu_2}{\sigma_2} = w$, $\frac{dy}{\sigma_2} = dw$

$$= \int_w \int_t \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[t^2 + w^2 - 2\rho t w]} dt\, dw$$

$$= \int_w \int_t \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[(t-\rho w)^2 + w^2 - \rho^2 w^2]} dt\, dw$$

$$= \int_w \int_t \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}w^2} \cdot e^{-\frac{1}{2}\frac{(t-\rho w)^2}{1-\rho^2}} dt\, dw$$

$$= \int_w \int_v \frac{e^{-w^2/2}}{2\pi} e^{-\frac{1}{2}v^2} dv\, dw$$

$\frac{t-\rho w}{\sqrt{1-\rho^2}} = v$

$\frac{dt}{\sqrt{1-\rho^2}} = dv$

$$\cancel{= \int_w \frac{e^{-w^2/2}}{2\pi}} = \int_w \frac{e^{-w^2/2}}{\cancel{2\pi}} \times 2 \int_0^\infty e^{-\frac{1}{2}v^2} dv\, dw$$

$$= \int_{\omega} \frac{e^{-\omega^2/2}}{\pi} \times \frac{\sqrt{\pi}}{\sqrt{2}} \, d\omega$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}} \times \frac{2}{\pi} \int_0^{\infty} e^{-\omega^2/2} \, d\omega$$

$$= \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}} = 1 \quad \text{(Proved)}$$

- ~~Formula~~ Formula

$$E\left(X^m Y^n\right) = \frac{\partial^{m+n}}{\partial t_1^m \partial t_2^n} M(t_1, t_2) \bigg|_{t_1 = t_2 = 0}$$

$$E\left(X^2 Y^0\right) = \frac{\partial^2}{\partial t_1^2 \partial t_2^0} M(t_1, t_2) \bigg|_{t_1 = t_2 = 0}$$

$$= \frac{\partial^2}{\partial t_1^2} \left[\mu_1 + t_2 \mu_2 + \frac{1}{2}\left(2t_1 \sigma_1^2 + 2t_2 \rho \sigma_1 \sigma_2 + t_2^2 \sigma_2^2\right)\right] \cdot e^{\left(\quad\right)}$$

$$= \mu_1 + \sigma_1^2$$

$$E(XY) = \frac{\partial^2}{\partial t_1 \partial t_2} M(t_1, t_2) \Big|_{t_1 = t_2 = 0}$$

$$= \frac{\partial}{\partial t_1} \left[\mu_2 + t_1 \rho \sigma_1 \sigma_2 + 2 t_2 \sigma_2^2 \right] \cdot e^{(\quad)}.$$

$$= \mu_2 + \rho \sigma_1 \sigma_2$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$

$$(a+b+c)^3 = a^3+b^3+c^3+3a^2b+3ab^2+3ac^2+3cb^2+6abc+3ca^2+3bc^2$$

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ad+2ac+2ad+2bc+2bd$$

Pascal triangle

1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$(a_1 + a_2 + \cdots + a_n)^n = \sum_{x_1, x_2 \cdots x_n} \frac{n!}{x_1!\, x_2! \cdots x_n!} a_1^{x_1} \cdot a_2^{x_2} \cdots a_n^{x_n}$$

where, $x_1 + x_2 + \cdots + x_n = n$

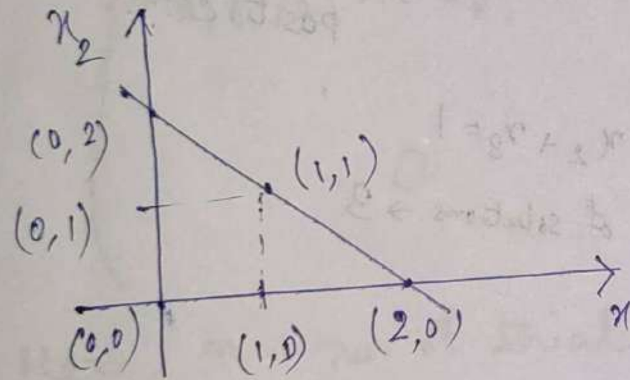
$$(a_1 + a_2)^n = \sum_{x_1=0}^{n} \frac{n!}{x_1! \cdot (n-x_1)!} \cdot a_1^{x_1} \cdot a_2^{n-x_1}$$

$x_1 + x_2 = n$

$x_2 = n - x_1$

$$(a_1 + a_2 + a_3)^n = \sum_{x_1, x_2 = 0}^{n} \frac{n!}{x_1! x_2! (n - x_1 - x_2)!} a_1^{x_1} \cdot a_2^{x_2} \cdot a_3^{n - x_1 - x_2}$$

$$(a_1 + a_2 + a_3)^2 = \sum_{x_1, x_2 = 0}^{2} \frac{2!}{x_1!\, x_2!\, (2 - x_1 - x_2)!} a_1^{x_1} \cdot a_2^{x_2} \cdot a_3^{2 - x_1 - x_2}$$



$x_1 + x_2 \leq 2$

$x_1 = 0,\ x_2 = 2$

$x_1 = 1,\ x_2 = 1$

$x_1 = 2,\ x_2 = 0$

$x_1 = 0,\ x_2 = 0$

$x_1 = 1,\ x_2 = 0$

$x_1 = 0,\ x_2 = 1$

$^{n+2}C_{r-1}$

$= a_2^2 + 2a_2a_3 + a_3^2 + 2a_1a_3 + a_1^2 + 2a_1a_2$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0;\ x_1 + x_2 + x_3 \leq 3$

~~$x_1 = 0$~~, ① $(0,0,0)$; ② $(0,1,0)$; ③ $(1,0,0)$

④ $(0,0,1)$; ⑤ $(0,1,1)$; ⑥ $(1,1,0)$; ⑦ $(1,0,1)$

⑧ $(2,0,0)$; ⑨ $(0,2,0)$; ⑩ $(0,0,2)$; ⑪ $(1,1,1)$

(12) $(1,2,0)$; (13) $(1,0,2)$; (14) $(0,1,2)$ (15) $(2,1,0)$;

(16) $(2,0,1)$; (17) $(0,2,1)$; (18) $(3,0,0)$

(19) $(0,3,0)$; (20) $(0,0,3)$

- $x_1 + x_2 + x_3 + \cdots + x_n = n$

  no. of solution, $\binom{n+n-1}{n-1} = \binom{2n-1}{n-1}$

- $x_1 + x_2 + x_3 + \cdots + x_n = r$

  $\binom{n+r-1}{n-1}$ ← no. of solutions including 0 & positive int.

- $x_1 + x_2 + x_3 = 0$

  no. of solutions → 1

  $x_1 + x_2 + x_3 = 1$

  no. of solutions → 3

  $x_1 + x_2 + x_3 = 2 \rightarrow 6$

  $x_1 + x_2 + x_3 = 3 \rightarrow 10$

- $x_1 + x_2 + x_3 + x_4 = \begin{cases} 0 \rightarrow 1 \\ 1 \rightarrow 4 \\ 2 \rightarrow 10 \\ 3 \rightarrow 20 \\ 4 \rightarrow 35 \end{cases}$

  total → 70 solutions

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!} \quad ; \quad \binom{n}{x_1 \; x_2} = \frac{n!}{x_1!\, x_2!\, (n-x_1-x_2)!}$$

$$(a_1 + \dots a_n)^n = \sum_{x_1, \dots x_{n-1}}^{n} \binom{n!}{x_1 \; x_2 \; \dots \; (n-x_1-x_2-\dots x_{n-1})} a_1^{x_1} \dots a_{n-1}^{x_{n-1}} \cdot a_n^{(n-x_1 \dots x_{n-1})}$$

Multinomial distribution pmf

The joint pmf of multinomial distribution is given by,

$P(X_1 = x_1, X_2 = x_2, \dots, X_{k-1} = x_{k-1}; X_k = x_k) =$

$$= \begin{cases} \dfrac{n!}{x_1!\, x_2! \dots x_k!} P_1^{x_1} P_2^{x_2} P_3^{x_3} \dots P_k^{x_k} & \text{if } \sum x_i = n \\ 0 & \text{o.w} \end{cases}$$

Here, n = no. of trial, $P_1, P_2, \dots P_k$ are probabilities s.t $\sum_{i=1}^{k} P_i = 1$

Now, ~~it~~ ~~ess~~.

$$\sum_x P(x) = \sum_x \frac{n!}{x_1!\, x_2! \dots x_n!} P_1^{x_1} \cdot P_2^{x_2} \dots P_k^{x_k}$$

$$= (P_1 + P_2 + \dots + P_k)^n = 1^n = 1$$

Mgf of multinomial distribution

$$M_X(t) = M_{(x_1, x_2, \dots, x_k)}(t_1, t_2, \dots t_k)$$

$$= E\left(e^{t_1 x_1 + t_2 x_2 + \dots + t_k x_k}\right)$$

$$= \sum_x e^{t_1 x_1 + t_2 x_2 + \dots + t_k x_k} \frac{n!}{x_1!\, x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \dots p_k^{x_k}$$

$$= \sum_x e^{t_1 x_1} \cdot e^{t_2 x_2} \dots e^{t_k x_k} \frac{n!}{x_1!\, x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$$

$$= \sum_x \frac{n!}{x_1!\, x_2! \dots x_k!} (p_1 e^{t_1})^{x_1} (p_2 e^{t_2})^{x_2} \dots (p_2 e^{t_k})^{x_k}$$

$$= (p_1 e^{t_1} + p_2 e^{t_2} + \dots + p_k e^{t_k})^n \quad \forall (t_1, t_2 \dots t_k) \in R$$

$$= \left(\sum_{i=1}^{k} p_i e^{t_i}\right)^n \quad \forall (t_1, t_2, \dots, t_k) \in R$$

Pgf of multinomial distribution

$$G_X(z) = E(z^x)$$

$$= \sum_x z^x \cdot \frac{n!}{x_1!\, x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$$

$$= \sum_x z_1^{x_1} \cdot z_2^{x_2} \dots z_k^{x_k} \cdot p_1^{x_1} p_2^{x_2} \dots p_k^{x_k} \cdot \frac{n!}{x_1!\, x_2 \dots x_k!}$$

$$= (P_1 Z_1 + P_2 Z_2 + \dots + P_K Z_K)^n \quad \text{where, } n = \sum_{i=1}^{K} x_i$$

$$= \left(\sum_{i=1}^{K} P_i Z_i\right)^n \quad \forall (Z_1, Z_2, Z_3, \dots Z_K) \in \mathbb{C}^K.$$

X_n = Outcome of n^{th} throw, $(n \geq 1)$

$\{X_n, n \geq 1\} = \{X_1, X_2, X_3, \dots, X_n\}$

$= \{H, T, H, H, T, T, H \dots T\}$

$\Omega = \{H, T\}$

X_n : sequence of random variable / stochastic process

X_n = Outcome of n th throw of a die

where $n \geq 1$

$\{X_n ; n \geq 1\} = \{X_1, X_2, \dots X_n\}$

Here, $\Omega = \{1, 2, 3, 4, 5, 6\}$

$\{X_n ; n \geq 1\} = \{6, 5, 4, 3, 2, 1, \dots\}$

Suppose X_n is the no. of heads upto n^{th} throw $(n \geq 1)$ of a ~~money~~ coin.

1 → head

0 → fail

$\cancel{X_n} = \{1, 1, 1, 0, 1, 0, 0, 1, 1, 0\}$

$\{X_{10}\} = \{1, 2, 3, 3, 4, 4, 4, 5, 5, 5\}$ → no. of heads upto 10 th throw

Suppose X_t is the no. of telephone calls received in an interval $(0,t)$

Application : stochastic processes has many applications such that phy, chem, bio, ecology, image processing, signal processing, control theory, information theory, finances etc.

Key Processes : Applications and the study of phenomena have in term inspired proposal of new stochastic process.

1. Wiener Process or Brownian motion process
2. Louis Bachelier – to study price changes on the Paris Bourse.
3. Poisson Process used by A K Erlang – to study the no. of phone calls occuring in a certain period of time.

This two stochastic processes are considered most important and central in the theory of stochastic process.

Types : Based on there mathematical properties stochastic processes can be grouped into various categories such as random work, martingles, markov processes, levy processes, Gaussian process, Renewal processes, branching processes.

A stochastic process (SP) $\underset{\sim}{X} = \{X(t), t \in T\}$ is a collection of random variables i.e. for each t in the index set T, $X(t)$ is a random variable. We often interpreate t as time and $X(t)$, the state of the process at time t.

Note: ① If the index set T is a countable set, we call X a discrete time stochastic process and if T is continuum we call it a continuous time SP.

② The set of all possible values of a single random variable of $X(t_1)$ of a stochastic process $\underset{\sim}{X}$ is known as its state space S_1. Also the set of all possible values of all the random variables $\{X(t), t \in T\}$ of a stochastic process X is known as the state space of the stochastic process X and denoted by S.

③ State space (SS) may be discrete or continuous.

A SP has two components ∴ state space S and time T

- **Discrete SS:** Let X_n be the total no. of heads appearing in the first n throws of a coin. The set of all possible values of X_n (SS) are $0, 1, 2, \ldots, n$. Here the state space is discrete.

We can write $X_n = Y_1 + Y_2 + \cdots + Y_n$, where Y_i is a discrete random variable takes value 1 or 0 according to ith throw shows head or not.

- <u>Continuous S S</u> : Let $X(n) = Y_1 + Y_2 + \cdots + Y_n$ where Y_i is a C.R.V takes ~~value~~ positive values in $[0, \infty)$ then the state space of X_n is $[0, \infty)$ in $[0, \infty)$ then the state space of X_n is $[0,\infty)$

Usually R.V's $X(t)$ are one dimensional but the process $\{X(t)\}$ may be multidimensional.

Consider $X(t) = (X_1(t), X_2(t), X_3(t))$

Where X_1 represents maximum, X_2 represents average and X_3 the minimum temperature at a place in an interval of time $(0, t)$. It is a three dimensional S.P in continuous time having continuous S.S.

(4) In general the R.V's within a SP, $\{X(t)\}$ are dependent.

A SP, $\{X(t), t \in T\}$ is with indepedent increments implies $\forall\ t_1, t_2, \ldots t_n$;

$t_1 < t_2 < \cdots < t_n$, the random variables

$X(t_2) - X(t_1),\ X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$

are independent.

Classification : Generally one dimensional SP $\{X_t, t \geq 1\}$ or $\{X(t), t \in T\}$ can be classified into four types – (1) discrete time discrete state space (coin toss) X_t = Outcome of t th throw of a coin. Here $S = \{0, 1\}$, $T = \{1, 2, \cdots\}$

(2) Discrete time continuous state space (inter arrival time)
Let $\{Y_1, Y_2, \cdots\}$ denotes the inter arrival time in queing system, defining a time until tth interval, $X_t = Y_1 + Y_2 + \cdots + Y_t$ where Y_i is a C.R.V takes positive values in $[0, \infty)$
Here $S = [0, \infty)$, $T = \{1, 2, 3, \cdots\}$

(3) Continuous time discrete state space (telephone calls), X_t = No. of phone calls within time interval $(0, t)$. Here $S = \{0, 1, 2 \cdots\}$, Theoratically $T = (0, \infty)$ but in practice suppose 1 to 4 PM then we consider the time between
$T = (1, 4)$

(4) Continuous time continuous SS (temperature)
X_t = Temperature at a place within time interval $(0, t)$. Here $S = (-10, 100)$, assume temperature range in cellocious, theoritically $S = (-\infty, \infty)$, $T = (0, \infty)$ but in

Practice suppose we want to consider the time 1 to 4 PM then $T = (1, 4)$

- **Sample Path** : Any realisation of $\underset{\sim}{X}$ is called a sample path. For example simple coin sample path ① $\{0, 1, 1, 0, \dots\}$
② $\{1, 1, 1, 1, 0, 0, 1, 1, 0, 0, \dots\}, \dots$ so on.

For example simple telephone calls

$[t : \{(0, 2), [2, 3), [3, 6), \dots\}]$

sample path 1 : $\{4, 1, \dots\}$

Markov Chain

- **Markov Process** : A SP $\{X(t) : t \in T\}$ is said to be (MP) if $\forall\ t_1, t_2, \dots t_n,\ t_1 < t_2 < \dots < t_n$

~~probability $\alpha \le X \le \beta$ given $| X(t_1) = x_1$~~

$$P\left(\alpha \le X \le \beta \mid X(t_1) = x_1, X(t_2) = x_2, \dots X(t_n) = x_n\right) = P\left(\alpha \le X \le \beta \mid X(t_n) =\right.$$

Here the first member has to be defined.

- **Markov chain** : A discrete parameter MP is known as markov chain (MC).

- **Chain** : If a SP has continuously infinite positions on which the process stands forms a chain.

∴ A SP $\{X(t) : t \in T\}$ is said to be a MC if

$$P(X_n = j \mid X_{n-1} = i, X_{n-2} = i_1, \ldots X_0 = i_{n-1})$$

$$\cancel{= P(X_{n-1} = i)} \quad = P(X_n = j \mid X_{n-1} = i)$$

$$= P_{ij}$$

Here $j, i, i_1, \ldots, i_{n-1} \in \mathbb{Z}$

- Transition Probability (One step) : To a pair of states (i,j) at the two successive trials (one step), the associated conditional probability (P_{ij}) is known as the probability of transition from the state i at $(n-1)$th trial to the state j at nth trial i.e. ~~probability~~

$$P(X_n = j \mid X_{n-1} = i) = P_{ij}$$

$$= P(X_{n+1} = j \mid X_n = i)$$

Suppose a markov chain has 3 ~~SP~~ SS. e.g. $\{1,2,3\}$ then $\mathbf{P} = (P_{ij}) = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}_{3\times 3}$

where $P_{ij} = P(X_n = j \mid X_{n-1} = i)$

- **Transition Probability (m step)** : To a pair of states (i, j) at the two non-successive trials (m-step) the ~~acc~~ associated conditional probability '$P_{ij}^{(m)}$' is known as probability of transition from the state i at ~~(n-1)th~~ n^{th} trial to the state j at $(n+m)^{th}$ trial

$$P_{ij}^{(m)} = P\left(X_{n+m} = j \mid X_n = i\right)$$

- **States** : The outcomes are called the states of the MC. Here $X_n = j$ means the process is at state j at n^{th} trial

- **Initial Probability** : Unconditional probability for the state of MC is called initial probability.

  e.g. $P(X_k = j) = P_j$ is the initial probability of the process of the state j

- **Homogeneous and ~~Hn~~ Non-homogeneous MC** :
  The transition probability may or may not be dependent of n. If the transition probability P_{ij} doesn't depend on n (i.e. step) implies is said to be homogeneous M.C (or to have stationary transition probability).

If P_{ij} depend on n the chain is said to be non-homogeneous MC.

Let $\{X_n\}$ be the R.V denotes the outcome of n^{th} toss $(n=1,2,\dots)$ of a coin where

$$X_n = \begin{cases} 1, & \text{if head appears} \\ 0, & \text{if tail appears} \end{cases}$$

with probability $P(X_n=1)=P$

$P(X_n=0) = 1-P$

Here $X_1, X_2, \dots, X_n$ are independent. Defined as

$S_n = X_1 + X_2 + X_3 + \dots + X_n$

i.e. Total no. of heads upto n^{th} trial.

S_n is a random variable takes values $0, 1, \dots, n$.

Here ~~probab~~ $P(S_{n+1} = j+1 \mid S_n = j) = P$

Again $P(S_{n+1} = j \mid S_n = j) = 1-P$

So the $(n+1)^{th}$ outcome ~~only~~ depends only on n^{th} outcome it implies a markov chain.

Also note that probabilities are not at all effected by the values of $S_1, S_2, \dots, S_{n-1}, S_n$

Since $P(S_{n+1} = j) = P(S_n + X_{n+1} = j)$

$= P(X_{n+1} = 1 \mid S_n = j-1)P(S_n = j-1) + P(X_{n+1} = 0 \mid S_n = j) P(S_n = j)$

$P(AB) = P(A|B) P(B)$

Transition Probability Matrix (TPM)

Let $\{X_n, n \geq 0\}$ be a MC with states $1, 2, \dots K$. Then, all the one step transition probabilities $P_{ij} = P(X_{n+1} = j \mid X_n = i)$, $i, j = 1, 2, \dots K$, can be written in the form of a square matrix is known as one step transition probability matrix (TPM) of the MC and it is denoted by

$$P = \begin{array}{c|c} & X_{n+1} = 1 \quad 2 \quad 3 \quad \cdots \quad K \\ \hline \begin{array}{c} X_n = 1 \\ 2 \\ 3 \\ \vdots \\ K \end{array} & \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1K} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2K} \\ \vdots & \vdots & \vdots & & \vdots \\ P_{K1} & P_{K2} & P_{K3} & \cdots & P_{KK} \end{pmatrix} \end{array}$$

Properties ① For TPM, row sum = 1 i.e.

$$\sum_{j=1}^{K} P_{ij} = 1 \quad , \; i = 1, 2, \dots K$$

② The states of TPM may be finite or infinite i.e. the square matrix may be finite or infinite.

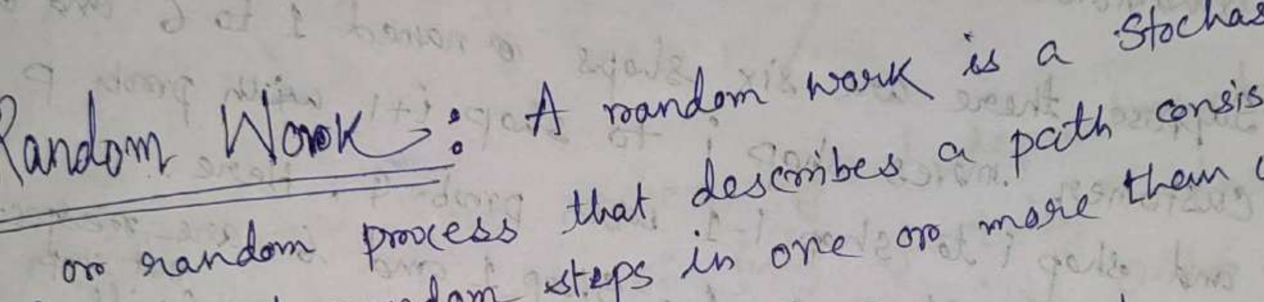
③ Any square matrix with non-negative elements and each row sum equals to unity (i.e. TPM) is called stochastic matrix.

Similarly m-step TPM of the MC denoted by,

$$
P^{(m)} = \begin{array}{c} \\ X_n=1 \\ 2 \\ 3 \\ \vdots \\ K \end{array}
\begin{array}{c}
\begin{array}{ccccc} X_{n+1}=1 & 2 & 3 & \cdots & K \end{array} \\
\begin{pmatrix}
P_{11}^{(m)} & P_{12}^{(m)} & P_{13}^{(m)} & \cdots & P_{1K}^{(m)} \\
P_{21}^{(m)} & P_{22}^{(m)} & P_{23}^{(m)} & \cdots & P_{2K}^{(m)} \\
\vdots & & \vdots & & \\
\vdots & & & & \\
P_{K1}^{(m)} & P_{K2}^{(m)} & P_{K3}^{(m)} & \cdots & P_{KK}^{(m)}
\end{pmatrix}
\end{array}
$$

□ <u>Random Work</u> : A random work is a stochastic or random process that describes a path consists of several random steps in one or more than one dimension.

<u>e.g.</u> <u>1D</u> : Random work starts at 0 and at each step moves +1 or −1 with equal probability from the ~~real~~ interger line



<u>2D</u> : A random work starts at the corner of a square and at each steps moves clockwise or anticlockwise with equal prob. with a corner of a square.

<u>3D</u> : The path traced by a molecule as it travels in a liquide or a gas i.e. Brownian motion.

e.g. The price ~~of a~~ fluctuation of a stock etc.

Random works are a fundamental topic in discussion of markov processes. We'll study several properties like dispersion dist^{n}, first passenge, eating times, encounter rates, recurrence or transience to quantify their behaviour.

- Suppose there are six shops named 1 to 6 and a customer moves shop i to shop $i+1$ with prob. p and shop i to shop $i-1$ with prob. q. Here $1 \le i \le 6$ and $p+q=1$. Shop 1 and 6 are respectively pizza and wine shop, one customer goes that shop will remain there i.e. absorbing state. Find one step TPM.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$p_{11} = 1$

$p_{12} = 0$

Let, X_n be the position of the people after n moves. So the states of X_n are $1, 2, \ldots, 6$. It is given that $P(X_{n+1} = 1 \mid X_n = 1) = 1$ and $P(X_{n+1} = 6 \mid X_n = 6) = 1$

Also, $P(X_{n+1} = i+1 \mid X_n = i) = p$ and $P(X_{n+1} = i-1 \mid X_n = i) = q$

for $1 \leq i \leq 6$.

Therefore the one-step TPM is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- It is known from meterological department that, for the month of July if today rains, the probability of rains tomorrow is 0.76. Also if today doesn't rain, the prob. of rains tomorrow is 0.43. Write down one step TPM for the SP.

$$\rightarrow \quad P = \begin{pmatrix} 0.76 & 0.24 \\ 0.57 & 0.43 \end{pmatrix}$$

A particle performs a random work with absorbing barriers say as 1 and 4. Whenever it is at any position $r (1 \le r \le 4)$ it moves r to $r+1$ with probability 0.4 or to $r-1$ with prob. 0.6. But if it reaches to 1 or 4 it remains there itself.

Let X_n be the position of particle after n moves. The different states of X_n are the different positions of particle. X_n is a MC, find its one step TPM.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Suppose there are two online shopping sites (Amazon & Filpkart), ~~Among them~~ have lots of customer. Among them 30% change their preferences from filpkart to Amazon in every month and 25% change their preferences from amazon to filpkart in every month. Let X_n be the preference of the customer in n month. Then for the MC X_n, find its one-step TPM.

$\rightarrow P(X_{n+1} = f \mid X_n = a) = 0.25$

$P(X_{n+1} = a \mid X_n = f) = 0.30$

$$P = \begin{array}{c|c} & \begin{array}{cc} a & f \end{array} \\ \begin{array}{c} a \\ f \end{array} & \begin{pmatrix} 0.70 & 0.30 \\ 0.25 & 0.75 \end{pmatrix} \end{array}$$

Prob-1

Suppose there are 3 popular online shopping sites (Amazon, Filpkart, Myntra) have lots of customers. Among them 20%, 15% change their preferences from filpkart to amazon, myntra respectively in every month; 24%, 18% change their preferences from Myntra to amazon, filpkart respectively in every month and 12%, 9% change their preferences from amazon to filpkart, myntra respectively in every month. Let X_n be the preference of the customer in n month. Then for the MC $\{X_n\}$, find its one step TPM.

$$\rightarrow \begin{array}{c|c} & \begin{array}{ccc} a & f & m \end{array} \\ \begin{array}{c} a \\ f \\ m \end{array} & \begin{pmatrix} 0.79 & 0.12 & 0.09 \\ 0.2 & 0.65 & 0.15 \\ 0.24 & 0.18 & 0.58 \end{pmatrix} \end{array}$$

Show that a ~~complete~~ MC is completely defined by initial and transitional probability.

→ Let $\{X_n\}$ is a MC having states $i_0, i_1, i_2, i_3, i_4, \ldots, i_n$. Now ~~probo~~

$$P\{X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= P(X_n = i_n \mid X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0) \cdot P(X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= P(X_n = i_n \mid X_{n-1} = i_{n-1}) \cdot P(X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= P_{i_{n-1} i_n} \cdot P(X_{n-1} = i_{n-1} \mid X_{n-2} = i_{n-2}) \cdot P(X_{n-2} = i_{n-2}, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= P_{i_{(n-1)} i_n} \cdot P_{i_{(n-2)} i_{(n-1)}} \cdot P(X_{n-2} = i_{n-2}, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= P_{i(n-1) i_n} \cdot P_{i(n-2) i(n-1)} \cdots\cdots P_{i_1 i_2} \cdot P(X_1 = i_1, X_0 = i_0)$$

$$= P_{i(n-1) i_n} \cdot P_{i(n-2) i(n-1)} \cdots\cdots P_{i_1 i_2} \, P_{i_0 i_1} \, P(X_0 = i_0)$$

$$= P_{i(n-1) i_n} \cdot P_{i(n-2) i(n-1)} \cdots\cdots P_{i_1 i_2} \, P_{i_0 i_1} \cdot P_{i_0}$$

$$= (\text{transitional probabilities}) \cdot (\text{Initial probability})$$

The TPM of a MC $\{X_n ; n = 1, 2, \dots\}$ having three states 1, 2, 3 is :

$$P = \begin{pmatrix} 0.2 & 0.5 & 0.3 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{pmatrix}$$ and the initial distribution is $\pi_0 = (0.5 \quad 0.4 \quad 0.1)$

Find ① $P(X_1 = 1)$

② $P(X_2 = 1, X_1 = 1)$

~~③ $P(X_2 = 3)$~~ ~~④ $P(X_3 = 2)$~~ ; ~~⑤ $P(X_2 = 3)$~~

~~⑥ $P(X_0 = 1)$~~ ~~⑦ $P(X_1 = 2)$~~

③ $P(X_2 = 3)$ ④ $P(X_3 = 2, X_2 = 3, X_1 = 2, X_0 = 1)$

→ ~~① $P(X_1 = 1) = 0.2$~~

~~② $P(X_2 = 1, X_1 = 1) =$~~

$P(X_1 = 1)$

$= P(X_1 = 1, X_0 = 1) + P(X_1 = 1, X_0 = 2) + P(X_1 = 1, X_0 = 3)$

$= P(X_1 = 1 | X_0 = 1)\, P(X_0 = 1) + P(X_1 = 1 | X_0 = 2) \cdot P(X_0 = 2) + P(X_1 = 1 | X_0 = 3) \cdot P(X_0 = 3)$

$= P_{11} \times 0.5 + P_{21} \times 0.4 + P_{31} \times 0.1$

$= (0.2 \times 0.5) + (0.3 \times 0.4) + (0.1 \times 0.1)$

$= 0.23$

③ $P(X_2 = 3)$

$= P(X_2 = 3, X_1 = 1) + P(X_2 = 3, X_1 = 2) + P(X_2 = 3, X_1 = 3)$

$= P(X_2 = 3 \mid X_1 = 1) \cdot P(X_1 = 1) + P(X_2 = 3 \mid X_1 = 2) \cdot P(X_1 = 2)$

$+ P(X_2 = 3 \mid X_1 = 3) \cdot P(X_1 = 3)$

$= 0.3 \times 0.15 + 0.4 \times 0.4 + 0.4 \times 0.1$

$= 0.35$

② $P(X_2 = 1, X_1 = 1)$

$= P(X_2 = 1 \mid X_1 = 1) \cdot P(X_1 = 1) = \theta_{11} \cdot P(X_1 = 1)$

$= 0.2 \times 0.23 = 0.046$

④ $P(X_3 = 2, X_2 = 3, X_1 = 2, X_0 = 1)$

$= P(X_3 = 2 \mid X_2 = 3) \cdot P(X_2 = 3, X_1 = 2, X_0 = 1)$

$= 0.5 \times 0.4 \times P(X_1 = 2 \mid X_0 = 1) \times P(X_0 = 1)$

$= 0.05$

Find the probability of

① rain on 3rd day ② No rain on 4th day.

$P(X_3 = 1)$ $P(X_4 = 0)$

~~$P(X_3 = 1)$~~

~~$= P(X_3 = 1, X_2 = 1) + P(X_3 = 1$~~

$P(X_1 = 2)$

$= P(X_1 = 2, X_0 = 1) + P(X_1 = 2, X_0 = 2) + P(X_1 = 2, X_0 = 3)$

$= 0.5 \times 0.5 + 0.3 \times 0.4 + 0.5 \times 0.11$

$= 0.42$

$P(X_1 = 3) =$

* The TPM of a MC $\{X_n ; n = 1, 2, \dots\}$ having three states 1, 2, 3 is:

$$P = \begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.6 & 0.3 & 0.1 \end{pmatrix}$$

and the initial distribution is

$$\Pi_0 = (0.3 \quad 0.4 \quad 0.3)$$

Find ① $P(X_2 = 1)$

② $P(X_2 = 1 \mid X_1 = 1)$ ③ $P(X_3 = 2, X_2 = 3 \mid X_1 = 2, X_0 = 1)$

④ $P(X_3 = 2, X_2 = 3, X_1 = 2 \mid X_0 = 1)$

$\rightarrow$ ① $P(X_2 = 1)$

$= P(X_2 = 1 \mid X_1 = 1) \cdot P(X_1 = 1) + P(X_2 = 1 \mid X_1 = 2) \cdot P(X_1 = 2)$

$+ P(X_2 = 1 \mid X_1 = 3) \cdot P(X_1 = 3)$ —— ①

$= \cancel{0.3 \times 0.3 + 0.2 \times 0.4 + 0.6 \times 0.3}$

$\cancel{= 0.35}$ Now

$P(X_1 = 1)$

$= P(X_1 = 1 \mid X_0 = 1) \cdot P(X_0 = 1) + P(X_1 = 1 \mid X_0 = 2) \cdot P(X_0 = 2) + P(X_1 = 1 \mid X_0 = 3) \cdot P(X_0 = 3)$

$= 0.35$

$$P(X_1 = 2) = P(X_1 = 2 \mid X_0 = 1) \cdot P(X_0 = 1) + P(X_1 = 2 \mid X_0 = 2) \cdot$$
$$P(X_0 = 2) + P(X_1 = 2 \mid X_0 = 3) \cdot P(X_0 = 3)$$

$$= 0.4$$

$$P(X_1 = 3) = 0.25$$

from ① $\Rightarrow P(X_2 = 1)$
$$= 0.335$$

② $P(X_2 = 1 \mid X_1 = 1) = 0.3$

③ $P(X_3 = 2, X_2 = 3 \mid X_1 = 2, X_0 = 1)$

$$= \frac{P(X_3 = 2, X_2 = 3, X_1 = 2, X_0 = 1)}{P(X_1 = 2, X_0 = 1)}$$

$$= \frac{P(X_3 = 2 \mid X_2 = 3) \cdot P(X_2 = 3 \mid X_1 = 2) \cdot P(X_1 = 2 \mid X_0 = 1) \cdot P(X_0 = 1)}{P(X_1 = 2 \mid X_0 = 1) \cdot P(X_0 = 1)}$$

$$= 0.12$$

④ $P(X_3 = 2, X_2 = 3, X_1 = 2 \mid X_0 = 1)$

$$= \frac{P(X_3 = 2 \mid X_2 = 3) \cdot P(X_2 = 3 \mid X_1 = 2) \cdot P(X_1 = 2 \mid X_0 = 1)\, P(X_0 = 1)}{P(X_0 = 1)}$$

$$= 0.06$$

Prob-1 Find the probability that a customer visits (i) Amazon on first month, (ii) Filpkart on second month and

(iii) Myntra on ~~second~~ third month.

(iv) Filpkart on first month, amazon on second month, myntra on third month.

→ Let, Initatial probability dist = $(\alpha', \beta', \gamma')$

Let, $P(X_3 = m) = \gamma$

(1) $P(X_1 = a, X_2 = f, X_3 = m)$

$= \cancel{0.12 \times} = 0.2 \times 0.18 \times \gamma = 0.036\gamma$

(2) $P(X_1 = f, X_2 = a, X_3 = m)$

$= 0.12 \times 0.24 \times \gamma = 0.0288\gamma$

* Markov chain as graphs: We can represent

(i) markov chain as graphs –

(1) The nodes/vertices of the graph is represents of the states of the MC

(2) Using one step TPM the transition between states represented by directed arcs with a direction arrow.

- If the states $S = \{1, 2, \ldots m\}$ is the set of vertices, ~~then~~ and α is the set of directed arcs between these vertices, then the graph $G = \{S, \alpha\}$ is the directed graph or transition graph of the chain.

- A directed graph/di-graph with positive arc weights and unit sum of the arc weights from each node (arrow away from each node) is called a stochastic graph.

  ▶ The directed graph of a MC is a stochastic graph.

- The TPM of a MC $\{X_n, n=1,2\ldots\}$ having three states 0, 1, 2 is

$$
P = \begin{array}{c|ccc}
 & X_{n+1}=0 & 1 & 2 \\
\hline
X_n=0 & 0 & 1 & 0 \\
1 & 0.3 & 0.7 & 0 \\
2 & 0.6 & 0.3 & 0.1
\end{array}
$$

Represent a TPM as a stochastic graph.

→ Here, the MC has states 0, 1, 2, so the graph has three nodes 0, 1 and 2.

State 0 goes to state 1 with prob. 1.

State 1 goes to state 0 with prob 0.3 and

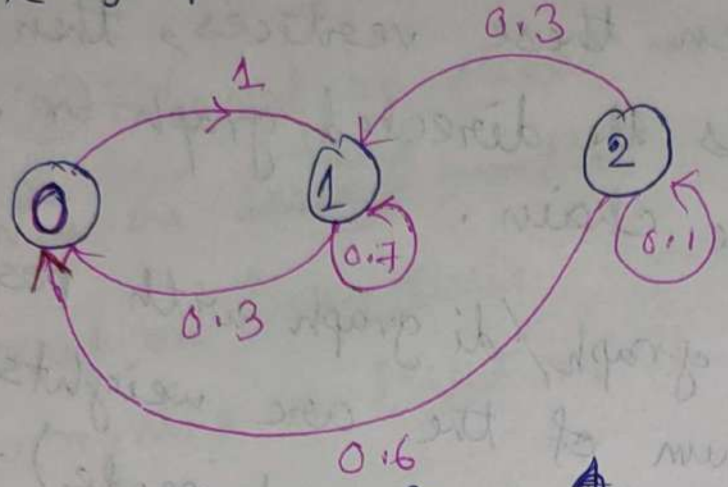
state 1 " " 1 " " 0.7

State 2 goes to state 0 with prob. 0.6,

State 2 " " 1 " " 0.3,

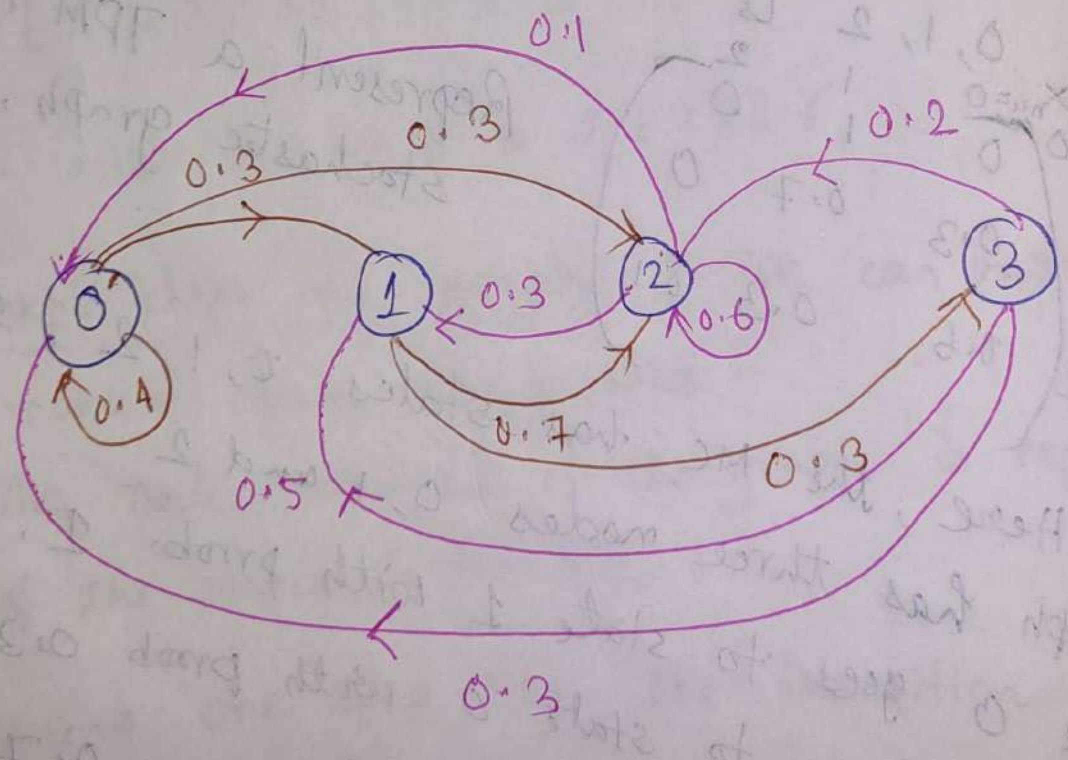
State 2 goes to state 2 " " 0.1

Then the given TPM can be drawn as a Stochastic graph,

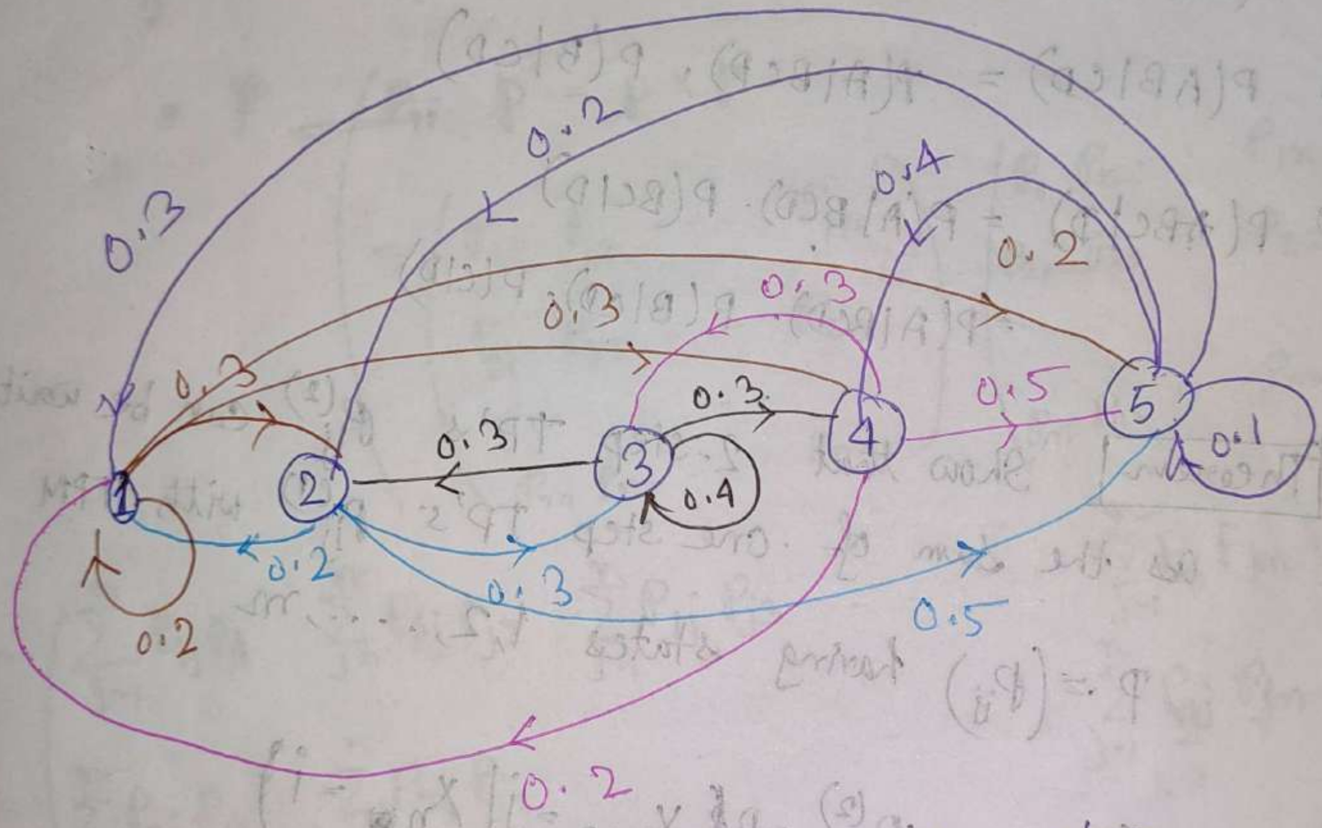


$$P = \begin{array}{c} \\ X_n=0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccc} X_{n+1}=0 & 1 & 2 & 3 \\ \end{array}$$

$$P = \begin{array}{c|cccc} & X_{n+1}=0 & 1 & 2 & 3 \\ \hline X_n=0 & 0.4 & 0.3 & 0.3 & 0 \\ 1 & 0 & 0 & 0.7 & 0.3 \\ 2 & 0.1 & 0.3 & 0.6 & 0 \\ 3 & 0.3 & 0.5 & 0.2 & 0 \end{array}$$



$$
\begin{array}{c c}
 & \begin{array}{ccccc} X_{n+1} = & 1 & 2 & 3 & 4 & 5 \end{array} \\
P = \begin{array}{c} X_n = 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0.2 & 0.3 & 0 & 0.3 & 0.2 \\ 0.2 & 0 & 0.3 & 0 & 0.5 \\ 0 & 0.3 & 0.4 & 0.3 & 0 \\ 0.2 & 0 & 0.3 & 0 & 0.5 \\ 0.3 & 0.2 & 0 & 0.4 & 0.1 \end{pmatrix}
\end{array}
$$



Higher Transition Probabilities: ① One step transition probabilities $\{X_{n+1}=j \mid X_n=i\}$ are denoted by P_{ij} or $P_{ij}^{(1)}$.

② Two step transition probabilities $\{X_{n+2}=j \mid X_n=i\}$ are denoted by P_{ij} or $P_{ij}^{(2)}$

(3) m-step transition probabilities $\{X_{n+m} = j \mid X_n = i\}$ are denoted by P_{ij} or $P_{ij}^{(m)}$

- $P(A.B) = P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A)$
- $P(ABC) = P(AB \mid C) \cdot P(C) = P(AC \mid B) \cdot P(B) = P(BC \mid A) \cdot P(A)$
- $P(ABC) = P(A \mid BC) \cdot P(BC) = P(B \mid AC) \cdot P(AC) = P(C \mid AB) \cdot P(AB)$
- $P(AB \mid C) = P(A \mid BC) \cdot P(B \mid C)$
- $P(AB \mid CD) = P(A \mid BCD) \cdot P(B \mid CD)$
- $P(ABC \mid D) = P(A \mid BCD) \cdot P(BC \mid D)$
  $= P(A \mid BCD) \cdot P(B \mid CD) \cdot P(C \mid D)$

$\boxed{\text{Theorem}}$ Show that 2-step TP's $P_{ij}^{(2)}$ can be written as the sum of one step TP's $P_{ij}^{(1)}$ with TPM $P = (P_{ij})$ having states $1, 2, \ldots, m$

$\rightarrow$ $P_{ij}^{(2)} = P\{X_{n+2} = j \mid X_n = i\}$

$$= \sum_{K=1}^{m} P(X_{n+2} = j, X_{n+1} = K \mid X_n = i)$$

$$= \sum_{K=1}^{m} P(X_{n+2} = j \mid X_{n+1} = K, X_n = i) \cdot P(X_{n+1} = K \mid X_n = i)$$

$$= \sum_{K=1}^{m} P(X_{n+2} = j \mid X_{n+1} = K) \cdot P_{iK}$$

$$= \sum_{k=1}^{m} p_{kj} \cdot p_{ik} = \sum_{k=1}^{m} p_{ik} \cdot p_{kj}$$

Q Using the above theorem, show that two step TPM $Q = \left(p_{ij}^{(2)}\right)$ can be written as $Q = P^2$ where Q is the square of one step TPM P

$$\rightarrow \quad p_{ij}^{(2)} = P\left(X_{n+2} = j \mid X_n = i\right)$$

$$P^2 = P \times P$$

$$= \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{22} \\ \vdots & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} p_{1j} p_{j1} & \sum_{j=1}^{n} p_{1j} p_{j2} & \sum_{j=1}^{n} p_{1j} p_{j3} & \cdots & \sum_{j=1}^{n} p_{1j} p_{jn} \\ \sum_{j=1}^{n} p_{2j} \cdot p_{j1} & \sum_{j=1}^{n} p_{2j} p_{j2} & & \cdots & \sum_{j=1}^{n} p_{2j} p_{jn} \\ \vdots & \vdots & & & \vdots \\ \sum_{j=1}^{n} p_{nj} p_{j1} & \sum p_{nj} p_{j2} & & \cdots & \sum_{j=1}^{n} p_{nj} p_{jn} \end{pmatrix}$$

Note : A chain is said to be regular if all entries of p^m are positive for some $m \geq 1$

* Q1 Show that the m-step TPM $Q = P_{ij}^{(m)}$ can be written as $Q = P^m = P_{ij}^{(m)}$. Here $Q = P^m$ is the m^{th} power of the one-step TPM P.

$\rightarrow$ $P^m /= P(X_{n+r} = j \mid X_n = i) = P_{ij}^{(r)}$

$r = 1 \rightarrow$ True

$r = 2 \rightarrow$ True

$r = \alpha \longrightarrow P(X_{n+\alpha} = j \mid X_n = i) = P_{ij}^{(\alpha)}$

for : $r = \alpha + 1$

$$P(X_{n+\alpha+1} = j \mid X_n = i)$$

$$= \sum_{K=1}^{m} P(X_{n+\alpha+1} = j, X_{n+\alpha} = K \mid X_n = i)$$

$$= \sum_{K=1}^{m} P(X_{n+\alpha+1} = j \mid X_{n+\alpha} = K, X_n = i) \cdot P(X_{n+\alpha} = K \mid X_n = i)$$

$$= \sum_{K=1}^{m} P_{Kj} \cdot \cancel{P_{Kj}^{(\alpha)}} \cdot P_{iK}^{(\alpha)}$$

$$p_{ij}^{(2)} = \sum_{k=1}^{n} p_{ik} \cdot p_{kj}$$

$$= p_{i1} p_{1j} + p_{i2} p_{2j} + \cdots + \cancel{p_{n2} p_{2n}} \, p_{in} p_{nj}$$

$$= p_{i1} = \begin{pmatrix} p_{11} \\ p_{21} \\ p_{31} \\ \vdots \\ p_{n1} \end{pmatrix}_{n\times 1} \quad p_{1j} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \end{pmatrix}_{1\times n}$$

$$p_{i2} = \begin{pmatrix} p_{12} \\ p_{22} \\ \vdots \\ p_{n2} \end{pmatrix}, \quad p_{2j} = \begin{pmatrix} p_{21} & p_{22} & \cdots & p_{2n} \end{pmatrix}$$

$$p_{i1} \times p_{1j} = \begin{pmatrix} p_{11}^2 & p_{11}p_{12} & p_{11}p_{13} & \cdots & p_{11}p_{1n} \\ p_{21}p_{11} & p_{21}p_{12} & p_{21}p_{13} & \cdots & p_{21}p_{1n} \\ \vdots & & & & \\ p_{n1}p_{11} & p_{n1}p_{12} & & \cdots & p_{n1}p_{1n} \end{pmatrix}_{n\times n}$$

$$p_{i2} \times p_{2j} = \begin{pmatrix} p_{12}p_{21} & p_{12}p_{22} & \cdots & \cdots & p_{12}p_{2n} \\ p_{22}p_{21} & p_{22}^2 & p_{22}p_{23} & & p_{22}p_{2n} \\ \vdots & & & & \vdots \\ p_{n2}p_{21} & p_{n2}p_{22} & \cdots & \cdots & p_{n2}p_{2n} \end{pmatrix}_{n\times n}$$

$\cancel{p_{ij}^{\alpha} = p_{11}}$ In usual notations

$$p_{ij}^{(\alpha)} = \sum_{i=1}^{n} p_{ir}^{(\alpha-1)} \cdot p_{ij} = \sum_{i=1}^{n} p_{ir}\, p_{ij}^{(\alpha-1)}$$

where the SP having states $1, 2, \ldots, m$ and α is a positive integer.

$$\cancel{p_{ij}^{(\alpha)} = P(X_{n+\alpha} = j \mid X_n = i)}$$

$$\cancel{= \sum_{i=1}^{n} P(X_{n+\alpha} = j, X_n = i}$$

→ Let us assume the result is true for $\alpha = l$

Then, $p_{ij}^{(l)} = \cancel{\sum_{i=1}^{n} P(X_{l}=i} = \sum_{i=1}^{n} p_{ij}^{l-1} \cdot p_{ij}$

Now, $p_{ij}^{(l+1)} = P(X_{n+l+1} = j \mid X_n = i)$

$$= \sum_{r=1}^{m} P(X_{n+l+1} = j, X_{n+l} = r \mid X_n = i)$$

$$= \sum_{r=1}^{m} P(X_{n+l+1} = j \mid X_{n+l} = r, X_n = i) \cdot P(X_{n+l} = r \mid X_n = i)$$

$$\cancel{= \sum p_{ij}^{(\alpha-1)} \cdot p_{ir}} = \sum_{r=1}^{m} p_{rj} \cdot p_{ir}^{(l)} \quad \cancel{= \sum_{r=1}^{m} p_{ir}^{(l)} \cdot p_{rj}}$$

$$\cancel{= \sum_{i=1}^{m} p_{ir}^{(l)} \cdot p_{ij}^{(l)}}$$

~~show~~ so the result is true for $\alpha = l$

Therefore from mathematical induction we can say that the result is true.

Now, for $\alpha = 1$.

$$p_{ij}^{(\alpha)} = P(X_{n+\alpha} = j \mid X_n = i)$$

$$= \sum_{i=1}^{n} P(X_{n+\alpha} = j, X_{n+}$$

$$p_{ij}^{(\alpha)} = \sum_{i=1}^{n} p_{ij}^{\alpha-1} \cdot p_{ij}$$

now, $p_{ij}^{(\alpha+1)} = P(X_{n+\alpha+1} = j \mid X_n = i)$

$$= \sum_{i=1}^{n} P(X_{n+\alpha+1} = j, X_{n+\alpha} = r \mid X_n = i)$$

$$= \sum_{i=1}^{n} P(X_{n+\alpha+1} = j \mid X_{n+\alpha} = r, X_n = i) \cdot P(X_{n+\alpha} = r \mid X_n = i)$$

$$= \sum_{r=1}^{n} p_{rj} \cdot p_{ir}^{(\alpha)}$$

$=$:

$\boxed{\text{P.T.O}}$

• In Usual notations show that $p_{ij}^{(\alpha+\beta)} = \sum_{r=1}^{m} p_{ir}^{(\alpha)} p_{rj}^{(\beta)}$

H.W

$$= \sum_{r=1}^{m} p_{ir}^{(\beta)} p_{rj}^{(\alpha)}$$

where the SP having states $1, 2, \ldots m$. α and β positive integers.

• NOTE The equation $\boxed{p_{ij}^{(\alpha+\beta)} = \sum_{r=1}^{m} p_{ir}^{\alpha} p_{rj}^{\beta}}$

is known as Chapman-Kolmogorov equation.

H.W In usual notations show that

$$p_{ij}^{(\alpha)} = \sum_{r=1}^{m} p_{ir}^{(\alpha-\beta)} \cdot p_{rj}^{\beta} = \sum_{r=1}^{m} p_{ir}^{(\beta)} p_{rj}^{(\alpha-\beta)}$$

• Let $\{X_n, n \geq 0\}$ be a MC has three states 1, 2, 3 with TPM is $P = \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.7 & 0.1 \end{pmatrix}$

and ~~an~~ initial distrubion

$$P(X_0 = i) = \frac{1}{3}, \quad i = 1, 2, 3$$

Find the two and three step TPMs. Hence find $P(X_4 = 2 \mid X_2 = 1)$, $P(X_8 = 3 \mid X_6 = 1)$,

$P(X_7 = 2 \mid X_4 = 1)$

$P(X_3 = 3 \mid X_0 = 1)$

$\cancel{P(X_4 = 2 \mid X_2 = 1)}$

$= \cancel{P(X_4 = 2} := \dfrac{P(X_4 = 2, X_2 = 1) \cdot \cancel{P(X_2 = 1)}}{P(X_2 = 1)}$

$\cancel{= P(X_4 = 2 \mid X_2}$

$$P^2 = \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.7 & 0.1 \end{pmatrix} \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.7 & 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.23 & 0.53 & 0.24 \\ 0.22 & 0.54 & 0.24 \\ 0.22 & 0.5 & 0.28 \end{pmatrix}$$

$P(X_4 = 2 \mid X_2 = 1)$

$= P_{12}^{(2)} = 0.53$

$P(X_8 = 3 \mid X_6 = 1)$

$= P_{13}^{(2)} = 0.24$

$P(X_7 = 2 \mid X_4 = 1)$

$= P_{12}^{(3)} = 0.525$

$$P^3 = \begin{pmatrix} 0.223 & 0.525 & 0.252 \\ & & \\ & & \end{pmatrix}$$

$P(X_3 = 3 \mid X_0 = 1)$ $\cancel{= \dfrac{P(X_3 = 3, X_0 = 1)}{P(X_0 = 1)}} = \cancel{\dfrac{P_{13}^{(3)}}{1/3}}$

$= 0.252$

Find the probability that there will be

1. rain on 3rd day given no rain on 1st day.
2. rain on 8th day given no rain on 6th day.
3. No rain on 4th day given no rain on 1st day
4. " " 12th day given no rain on 9th day

$$\rightarrow P = \begin{pmatrix} 0.76 & 0.24 \\ 0.43 & 0.57 \end{pmatrix} \quad P^2 = \begin{pmatrix} 0.76 & 0.24 \\ 0.43 & 0.57 \end{pmatrix} \begin{pmatrix} 0.76 & 0.24 \\ 0.43 & 0.57 \end{pmatrix}$$

$$x_n = \begin{array}{c|cc} x_{n+1} & 0 & 1 \\ 0 & 0.6808 & 0.3192 \\ 1 & 0.5719 & 0.4281 \end{array}$$

$P(X_3 = 1 \mid X_1 = 0)$

$= P(X_8 = 1 \mid X_6 = 0) = 0.571$

$$P^3 = \begin{pmatrix} 0.6808 & 0.3192 \\ 0.5712 & 0.4281 \end{pmatrix} \begin{pmatrix} 0.76 & 0.24 \\ 0.43 & 0.57 \end{pmatrix}$$

$$\begin{array}{c|cc} & 1 & 0 \\ \rightarrow & 0.654664 & 0.345336 \\ 0 & 0.618727 & 0.381273 \end{array}$$

$P(X_4 = 0 \mid X_1 = 0)$

$= 0.381273$

$= P(X_{12} = 0 \mid X_9 = 0)$

Q. In usual notations show that,

$$p_{ij}^{(\alpha)} = \sum_{i=1}^{m} p_{ir}^{(\alpha-1)} \cdot p_{rj} = \sum_{r=1}^{m} p_{ir} \cdot p_{rj}^{(\alpha-1)}$$

→ It's true for $\alpha = 1, 2$

Let us assume the result is true for $\alpha = l$

$$p_{ij}^{(l)} = \sum_{i=1}^{m} p_{ir}^{(l-1)} p_{rj}$$

Now, $p_{ij}^{(l+1)} = P(X_{n+l+1} = j \mid X_n = i)$

$$= \sum_{r=1}^{m} P(X_{n+l+1} = j, X_{n+l} = r \mid X_n = i)$$

$$= \sum_{r=1}^{m} P(X_{n+l+1} = j \mid X_{n+l} = r) \cdot p_{ir}^{(l)}$$

$$= \sum_{r=1}^{m} p_{rj} \, p_{ir}^{(l)}$$

So the result is true for $\alpha = l+1$

From the theory of mathematical induction we can say the result is true for $\forall \alpha$

Let us assume the result is true for $\alpha = l$

$$p_{ij}^{(l)} = \sum_{r=1}^{m} p_{ir} \, p_{rj}^{(l-1)}$$

now, $p_{ij}^{l+1} = P(X_{n+l+1} = j \mid X_n = i)$

$$= \sum_{r=1}^{m} P(X_{n+l+1} = j, X_{n+1} = r \mid X_n = i)$$

$$= \sum_{r=1}^{m} P(X_{n+l+1} = j \mid X_{n+1} = r)\, p_{ir}$$

$$= \sum_{r=1}^{m} p_{rj}^{(l)} \cdot p_{ir}$$

$$p_{ij}^{(\alpha+\beta)} = P(X_{n+\alpha} = j \mid X_{n-\beta} = i)$$

$$= \sum_{r=1}^{m} P(X_{n+\alpha} = j, X_n = r \mid X_{n-\beta} = i)$$

$$= \sum_{r=1}^{m} P(X_{n+\alpha} = j \mid X_n = r, X_{n-\beta} = i) \cdot P(X_n = r \mid X_{n-\beta} = i)$$

$$= \sum_{r=1}^{m} P(X_{n+\alpha} = j \mid X_n = r) \cdot p_{ir}^{(\beta)}$$

$$= \sum_{r=1}^{m} p_{rj}^{(\alpha)}\, p_{ir}^{(\beta)}$$

$\alpha + \beta = \boxed{3+4}$ $\quad n = 1, 2, \quad X_0, X_1, X_2$

$1-4 \quad X_{-3}$ $\quad n \geq \beta$

$\beta = 1$

Suppose $\{X_n, n \geq 1\}$ is sp having states $1, 2, \dots m$ and initial distribution $P(X_0 = i) = p_i$. So, that the unconditional probability $P(X_n = i) = p_i^{(n)}$ can be expressed as $p_i^{(n)}$

$$p_i^{(n)} = \sum_{r=1}^{m} p_{ri}\, p_r^{(n-1)}, \quad i = 1, 2, \dots, m$$

Also show that, the vector $\underset{\sim}{p}^{(n)}$ can be expressed as $\underset{\sim}{p}^{(n)} = \underset{\sim}{p}\, P^n$, here P is the one-step TPM and $\underset{\sim}{p}$ is the vector of initial distribution $\underset{\sim}{p} = [p_1\ p_2 \dots p_m]$.

→ Hint: $P(X_n = i) = \cancel{P(X_n =}$

$$= \sum_{j=1}^{m} P(X_n = i \mid X_{n-1} = j)\, P(X_{n-1} = j)$$

$$= \sum_{j=1}^{m} p_{ji}\, p_j^{(n-1)}. \qquad \left[\because P(X_n = i) = p_i^{(n)} \Rightarrow P(X_{n-1} = i) = p_i^{(n-1)} \right]$$

Again, $P(X_n = i)$

$$= \sum_{r=1}^{m} P(X_n = i \mid X_0 = r) \cdot P(X_0 = r)$$

$$= \sum_{r=1}^{m} p_{ri}^{(n)} \cdot p_r, \quad i = 1, 2, \dots, n$$

Also, $\underset{\sim}{p} = [p_1\ p_2 \dots p_m] = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}^{c}$

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & & & \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix}$$

So, $Q = P^{n} = \begin{pmatrix} p_{11}^{(n)} & p_{12}^{(n)} & \cdots & p_{1m}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & \cdots & p_{2m}^{(n)} \\ \vdots & \vdots & & \vdots \\ p_{m1}^{(n)} & p_{m2}^{(n)} & \cdots & p_{mm}^{(n)} \end{pmatrix}$

$$\underset{\sim}{p}^{(n)} = \begin{pmatrix} p_1^{(n)} \\ p_2^{(n)} \\ \vdots \\ p_n^{(n)} \end{pmatrix}' = \begin{pmatrix} P(X_n = 1) \\ P(X_n = 2) \\ \vdots \\ P(X_n = m) \end{pmatrix} = \begin{pmatrix} \sum_{r=1}^{m} p_{r1}^{(n)} \, p_r \\ \sum_{r=1}^{m} p_{r2}^{(n)} \, p_r \\ \vdots \\ \sum_{r=1}^{m} p_{rm}^{(n)} \, p_r \end{pmatrix}' = \underset{\sim}{p} P^{n}$$

$$P = \begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.6 & 0.3 & 0.1 \end{pmatrix}_{3\times 3}, \quad \pi_0 = (0.3 \quad 0.4 \quad 0.3)$$

Find ① $P(X_3 = 2)$ ② $P(X_4 = 1)$, ③ $P(X_6 = 2 \mid X_2 = 1)$

④ $P(X_7 = 1 \mid X_0 = 1)$

$\rightarrow$ ① $P(X_3 = 2) = \sum_{r=1}^{3} P(X_3 = 2 \mid X_0 = r) \cdot P(X_0 = r)$

$$P^3 = \begin{pmatrix} 0.343 & 0.403 & 0.254 \\ 0.334 & 0.414 & 0.252 \\ 0.33 & 0.405 & 0.265 \end{pmatrix}$$

$\underset{\sim}{p} = (0.3 \quad 0.4 \quad 0.3)$

$\underset{\sim}{p}\, P_0^3 = [0.3355 \quad 0.408 \quad 0.2565]$

$$\boxed{P(X_3 = 2) = 0.408}$$

② $$P^4 = \begin{pmatrix} 0.336071 & 0.407941 & 0.255988 \\ 0.335998 & 0.408058 & 0.255944 \\ 0.33591 & 0.407985 & 0.256105 \end{pmatrix}$$

$\underset{\sim}{p} = (0.3 \quad 0.4 \quad 0.3)$

$\underset{\sim}{p} P^4 = [0.3359935 \quad 0.408001 \quad \cancel{2.60}\ 0.2560055]$

$$\boxed{P(X_4 = 1) = 0.336}$$

(3) $P(X_6 = 2 \mid X_2 = 1)$

$= P_{12}^{4} = 0.4089$

(4) $P(X_7 = 1 \mid X_0 = 1) = \cancel{P(X_4=1 \mid X_0=1) + P(X_3=1 \mid X_0=1)}$

$= p_{11}^{(7)} = \sum_{r=1}^{3} p_{1r}^{(3)} p_{r1}^{(4)} = p_{11}^{(3)} p_{11}^{(4)} + p_{12}^{(3)} p_{21}^{(4)} + p_{13}^{(3)} p_{31}^{(4)}$

$= 0.3360023$

Generalisation of independent bernoulli trials :

(1) Usually bernoulli trials have two outcomes e.g. success, failure and we denote it by 1, 0 respectively, for example of tossing a coin we may consider head as success and tail as failure. we also denote head by 1 and tail by 0, Usually bernoulli trials are independent, however a generalisation of bernoulli trials can be chain dependent trials i.e. dependence is ~~cable~~ connected by a simple M.C.

(2) Suppose the chain dependent bernoulli trials have the TPM, $P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ with initial distribution

~~$P_0 = X_0 = 1$~~ $\left(P(X_0 = 1) = P_0 \quad P(X_0 = 0) = 1 - P_0 \right)$

if $\beta = 1-\alpha$ then the chain dependent bernoulli trials again reduces to independent bernoulli trial.

The probabilities at n^{th} trial are $P(X_n = 1) = P_n$ and $P(X_n = 0) = 1 - P_n$

- $x(n+1) = a(n)x(n) + g(n)$, $x(0) = x_0$, $n \geq 0$

$\downarrow$

e.g $x(n+1) = n^2 x(n) + (n+1)$

The solution of the difference equation $x(n+1) = a(n)x(n) + g(n)$ is

$$x(n) = \left[\prod_{i=0}^{n-1} a(i)\right] x_0 + \sum_{k=0}^{n-1} \left[\prod_{i=k+1}^{n-1} a(i)\right] g(k)$$

- Finite geometric series, $\sum_{i=0}^{n-1} a^{-i} = 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \cdots$

$$= \frac{1 - \frac{1}{a^n}}{1 - \frac{1}{a}}$$

$$= \frac{1-a^n}{a^{n-1}(1-a)}$$

Suppose a MC have the TPM

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$ with states 0, 1 and initial distribution $\left(P(X_0=1)=P_0 , P(X_0=0)=1-P_0 \right)$

Show that $P(X_n=1)=P_n = (1-\alpha-\beta)^{n-1}\left[P_0 - \frac{\alpha}{\alpha+\beta}\right] + \frac{\alpha}{\alpha+\beta}$

→ Now, $P_n = P(X_n=1) = \cancel{P(X_n=1 \mid X_{n-1}=0) + P(X_n=1 \mid X_{n-1}=1)}$

$$= P(X_n=1 \mid X_{n-1}=0) \cdot P(X_{n-1}=0) + P(X_n=1 \mid X_{n-1}=1) \cdot P(X_{n-1}=1)$$

$$= \alpha(1-P_{n-1}) + (1-\beta)P_{n-1}$$

$$= (1-\alpha-\beta)P_{n-1} + \alpha$$

So, the solution is,

$$P_n = \prod_{i=0}^{n-2}(1-\alpha-\beta)\, P_0 + \sum_{k=0}^{n-2}\left[\prod_{j=k+1}^{n-2}(1-\alpha-\beta)\right]\alpha$$

$$= (1-\alpha-\beta)^{n-1} \cdot P_0 + \alpha \sum_{k=0}^{n-2} (1-\alpha-\beta)^{n-2-k}$$

$$= (1-\alpha-\beta)^{n-1} P_0 + \alpha(1-\alpha-\beta)^{n-2} \sum_{k=0}^{n-2} (1-\alpha-\beta)^{-k}$$

$$= (1-\alpha-\beta)^{n-1} P_0 + \alpha(1-\alpha+\beta)^{n-2} \cdot \frac{1-(1-\alpha-\beta)^{n-1}}{(1-\alpha-\beta)^{n-2}(1-1+\alpha+\beta)}$$

$$= (1-\alpha-\beta)^{n-1} P_0 + \alpha(1-\alpha-\beta)^{n-2}\left[\frac{1-(1-\alpha-\beta)^{n-1}}{(1-\alpha-\beta)^{n-2}(\alpha+\beta)}\right]$$

$= (1-\alpha-\beta)^{n-1} P_0 + \frac{\alpha}{\alpha+\beta}[1-(1-\alpha-\beta)^{n-1}]$

$= (1-\alpha-\beta)^{n-1}\left[P_0 - \frac{\alpha}{\alpha+\beta}\right] + \frac{\alpha}{\alpha+\beta}$ (proved)

If $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a+b=1$, then find A^n.

$\Rightarrow |A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = 0$

$\Rightarrow (a-\lambda)^2 - b^2 = 0$

$\Rightarrow (a-\lambda)^2 = b^2$

$\Rightarrow a-\lambda = \pm b$

$\Rightarrow \lambda = a-b,\ a+b$

eigen values are

$a+b,\ a-b$

eigen vector, for $\lambda = a+b$,

$Ax = \lambda x$

$\Rightarrow \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (a+b)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} ax_1+bx_2 \\ bx_1+ax_2 \end{pmatrix} = \begin{pmatrix} ax_1+bx_1 \\ ax_2+bx_2 \end{pmatrix}$

$\Rightarrow ax_1+bx_2 = ax_1+bx_1$

$\Rightarrow x_1 = x_2$

$bx_1+ax_2 = ax_2+bx_2$

$x_1 = x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

eigen vector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

eigen vector, for $\lambda = a+b$

$$Ax = \lambda x$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (a-b)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} ax_1 + bx_2 \\ bx_1 + ax_2 \end{pmatrix} = \begin{pmatrix} ax_1 - bx_1 \\ ax_2 - bx_2 \end{pmatrix}$$

$$\Rightarrow ax_1 + bx_2 = ax_1 - bx_1 \qquad \Big| \qquad bx_1 + ax_2 = ax_2 - bx_2$$

$$\Rightarrow x_2 = -x_1 \qquad \Big| \qquad \Rightarrow x_1 = -x_2$$

Therefore, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \therefore$ eigen vector $= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Define, $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ by spretral decompositon,

$$A = P^{-1} D P \Rightarrow A^n = P^{-1} D^n P \qquad \Big| \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

where, $D = \begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}$

$$\text{so, } A^n = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\Bigg[\underbrace{\begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}\begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}\cdots}_{n \text{ times}}\Bigg]\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{bmatrix} (a-b)^n & 0 \\ 0 & (a+b)^n \end{bmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{bmatrix} (a-b)^n & (a-b)^n \\ -(a+b)^n & (a+b)^n \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (a-b)^n + (a+b)^n & (a-b)^n - (a+b)^n \\ (a-b)^n - (a+b)^n & (a-b)^n + (a+b)^n \end{bmatrix}$$

given, $a+b=1$

$$A^n = \frac{1}{2}\begin{bmatrix} (a-b)^n + 1 & (a-b)^n - 1 \\ (a-b)^n - 1 & (a-b)^n + 1 \end{bmatrix}$$

Markov Bernoulli chain with TPM,

$$P = \begin{bmatrix} 1-(1-\alpha)p & (1-\alpha)p \\ (1-\alpha)(1-p) & (1-\alpha)p+\alpha \end{bmatrix}$$ with states $[0, 1]$

and initial distribution $P(X_0 = 1) = p$, $P(X_0 = 0) = q = 1-p$.

Show that the P(event occurs in all trials are same at 1) i.e. $P(X_n = 1) = p_n \ \forall n$.

Hence find $E(X_n)$, $V(X_n)$, $E[X_n(X_n-1)]$, $E[X_n(X_n-2)]$.

r_{X_{n-1}, X_n}, $E(X_n X_{n-1})$, $E(X_n X_{n-2})$

$$\rightarrow P(X_n = 1) = p^n = \left[1-(1-\alpha)p-(1-\alpha)(1-p)\right]^{n-1}\left[p - \frac{(1-\alpha)p}{(1-\alpha)p+(1-\alpha)(1-p)}\right]$$

$$+ \frac{(1-\alpha)p}{(1-\alpha)p+(1-\alpha)(1-p)}$$

$$= \left[1-(1-\alpha)p-(1-\alpha)(1-p)\right]^{n-1}\left[p-\frac{(1-\alpha)p}{(1-\alpha)[p+1-p]}\right]$$

$$+ \frac{(1-\alpha)p}{(1-\alpha)[p+1-p]}$$

$$= 0 + p = p$$

$$E(X_n) = 0 \cdot P(X_n = 0) + 1 \cdot P(X_n = 1) = p$$

$$E(X_n^2) = 0^2 \cdot P(X_n = 0) + 1^2 P(X_n = 1) = p$$

$$V(X_n) = p - p^2$$

$$E(X_n(X_n - 1)) = E(X_n^2 - X_n) = E(X_n^2) - E(X_n)$$

$$= p - p = 0$$

$$E(X_n(X_n - 2)) = E(X_n^2) - 2E(X_n) = p - 2p = -p$$

$$E(X_n \cdot X_{n-1}) = \sum\sum x_n x_{n-1} P(X_n = x_n, X_{n-1} = x_{n-1})$$

Now, X_n, X_{n-1} takes the values 0 & 1

$$E(X_n \cdot X_{n-1}) = 0\cdot 0\, P(X_n=0, X_{n-1}=0) + 0\cdot 1\cdot P(X_n=0, X_{n-1}=1)$$
$$+ 1\cdot 0\cdot P(X_n=1, X_{n-1}=0) + 1\cdot 1\cdot P(X_n=1, X_{n-1}=1)$$

$$= P(X_n=1, X_{n-1}=1)$$

$$= P_{11} = 1-(1-\alpha)P$$

$$E(X_n \cdot X_{n-2}) = P(X_n=1, X_{n-2}=1)$$
$$= P_{11}^{(2)}$$

$$\rho_{X_{n-1}, X_n} = \frac{\cancel{\text{cov}}(X_{n-1}, X_n)}{V(X_n)} = \frac{E(X_n \cdot X_{n-1}) - E(X_n)\cdot E(X_{n-1})}{V(X_n)}$$

$$= \frac{1-(1-\alpha)P - P\cdot P}{P-P^2} = \frac{1-P+\alpha P - P^2}{P-P^2}$$

H.W Consider a sequence of random variables X_n, $n \geq 1$ such that each of X_n assumes only two values -1 and 1 with (~~conditional probabilities and~~) TPM

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

If X_n denotes the direction of movement to the left or right corresponding to the value -1 or 1 respectively at

$$\begin{array}{c|cc} & X_{n+1}=-1 & 1 \\ \hline X_n=-1 & 1-\alpha & \alpha \\ 1 & \beta & 1-\beta \end{array}$$

the n^{th} step i.e. a correlated random walk, the initial distribution $P(X_0=1)=p_0$, $P(X_0=-1)=q_0$

Show that the probability of events occurs in all trials are same i.e. $P(X_n=1)=p_n$

$$=\frac{\alpha}{\alpha+\beta}+\left(p_0-\frac{\alpha}{\alpha+\beta}\right)(1-\alpha-\beta)^{n-1}$$

Hence find $E(X_n)$, $V(X_n)$, $E(X_n X_{n-1})$, $E(X_n \cdot X_{n-2})$

- **Classification of state and chains**: Suppose X_n, $n \geq 1$ is a MC having states $1, 2, \ldots, m$ and initial distribution $P(X_0=i)=p_i$, $i=1(1)m$

Accessible: The state j is accessible from state i if $p_{ij}^{(n)}>0$ for some $n \geq 1$. It is denoted by $i \to j$

Not-accessible: The state j is not accessible from state i if $p_{ij}^{(n)}=0$ for some $n \geq 1$. It is denoted by $i \not\to j$

Communicate - Two states i, j are said to be communicate each other if $i \to j$ and $j \to i$ and it is denoted by $i \leftrightarrow j$.

If two states communicates to each other then there exists integers a, b such that

$$p_{ij}^{(a)}>0, \quad p_{ji}^{(b)}>0$$

The relation ~~i, j, j~~ $\rightarrow$, $\leftrightarrow$ are transitive

i.e. $i \rightarrow j, j \rightarrow k \Rightarrow i \rightarrow k$ and $i \leftrightarrow j, j \leftrightarrow k \Rightarrow i \leftrightarrow k$

Class : A class of states is a subset of the state space and within that subset every state of the class communicates themselves. Also no other state outside the class can communicate any state within the class.

If one state in a class has a property then all the states of that class has the same property. It is known as class property.

Theorem : Show that communication is an equivalence relation. In other words (i) reflexive i.e. $i \leftrightarrow i$ (ii) symmetric i.e. $i \leftrightarrow j \Rightarrow j \leftrightarrow i$, (iii) transitive i.e. $i \leftrightarrow j$, $j \leftrightarrow k \Rightarrow i \leftrightarrow k$

Closed states : Suppose we have n states and $S = \{i_1, i_2, \ldots, i_p\}$ is a set of states such that no outside state can be reached from any state of S, then we can say that S is closed.

- For a closed set of states S if $i \in S$, $j \notin S$

  $\Rightarrow P_{ij}^{(n)} = 0 \quad \forall n$

- The above point implies $\sum_{i \in S} P_{ik} = 1 \quad \forall k$

- So the TPM P can be expressed as

$$P = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}, \text{ where } ~~A = (ij~~ A = (a_{ij}) \quad i, j \in S$$

Absorbing States: A closed state usually contain one or more state. If it contains only one state i then then i is said to be absorbing states. The state i is absorbing iff

$$P_{ii} = 1, \quad P_{ij} = 0 \quad \forall \, i \neq j$$

Irreducible Markov chain: Every finite MC Contains atleast one closed set if the chain does not contain any proper closed subset other than the state space, then the chain is called irreducible.

① The TPM of irreducible chain is an irreducible matrix.

② In an irreducible MC, every state can be reached from every other state.

③ The irreducible matrices may be subdivided into two classes — Periodic and Aperiodic.

Reducible MC: chain which are not irreducible, are said to be reducible and TPM is reducible matrix.

Some more definitions

① First time reach probability

For an arbitrary state i, we define for each interger $n \geq 1$

$$f_{ii}^{(n)} = P(X_n = i, X_\alpha \neq i, \alpha = 1, 2, \dots n-1 \mid X_0 = i)$$

i.e. $f_{ii}^{(n)}$ is the probability that starting from state i first return to state j occurs at the n^{th} transition.

<u>Difference between $P_{ij}^{(n)}$ and $f_{ij}^{(n)}$</u>

The basic difference is $f_{ij}^{(n)}$ is the probability that starting from state i, the first return to state j occurs at the n^{th} transition (not limited to the first return)

- <u>Ex</u>: Consider the one-step TPM of a SP with three states 1, 2, 3, then find $P_{12}^{(2)}$ and $f_{12}^{(2)}$

$$P = P_{ij}$$

$$\rightarrow P_{12}^{(2)} = P(X_{n+2} = 2 \mid X_n = 1) = \sum_{r=1}^{3} P(X_{n+2} = 2, X_{n+1} = r \mid X_n = 1)$$

$$= P(X_{n+2} = 2, X_{n+1} = 1 \mid X_n = 1) + P(X_{n+2} = 2, X_{n+1} = 2 \mid X_n = 1)$$

$$+ P(X_{n+2} = 2, X_{n+1} = 3 \mid X_n = 1)$$

$$= \cancel{P(X_{n+2} = 2 \mid X_{n+1} = 1, X_n = 1)}$$

$$\cancel{= P_{11} P_{12} + P_{12} P_{22} + P_{13} P_{32}}$$

$$= P(X_{n+2} = 2 \mid X_{n+1} = 1; X_n = 1)\, P(X_{n+1} = 1 \mid X_n = 1)$$

$$+ P(X_{n+2} = 2 \mid X_{n+1} = 2, X_n = 1)\, P(X_{n+1} = 2 \mid X_n = 1)$$

$$+ P(X_{n+2} = 2 \mid X_{n+1} = 3, X_n = 1)\, P(X_{n+1} = 3 \mid X_n = 1)$$

$$= P_{11} P_{12} + P_{22} P_{12} + P_{32} P_{13}$$

$$f_{12}^{(2)} = P(X_{n+2} = 2 \mid X_n = 1)$$

$$= \sum_{r=1(\neq 2)}^{3} P(X_{n+2} = 2, X_{n+1} = r \mid X_n = 1)$$

$$= P(X_{n+2} = 2, X_{n+1} = 1 \mid X_n = 1) + P(X_{n+2} = 2, X_{n+1} = 3 \mid X_n = 1)$$

$$= P(X_{n+2} = 2 \mid X_{n+1} = 1, X_n = 1) \cdot P(X_{n+1} = 1 \mid X_n = 1)$$
$$+ P(X_{n+2} = 2 \mid X_{n+1} = 3; X_n = 1) \cdot P(X_{n+1} = 3 \mid X_n = 1)$$

$$= p_{11} p_{12} + p_{13} p_{32}$$

Q. Let $\{X_n, n \geq 0\}$ be a MC having state space 1,2,3,4 with TPM

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Find out $f_{11}^{(n)}, f_{22}^{(n)}, f_{33}^{(n)}, f_{44}^{(n)}$ where

$n = 1, 2, 3, 4$

$\rightarrow f_{11}^{(1)} = P(X_{n+1} = 1 \mid X_n = 1) = \frac{1}{3}$

$$\cancel{\sum_{r=2(\neq 1)}^{4} P(X_{n+1} = 1, X_n =}$$

$$f_{11}^{(2)} = P(X_{n+2}=1 \mid X_n = 1)$$

$$= \sum_{r\neq 1}^{4} P(X_{n+2}=1, X_{n+1}=r \mid X_n=1)$$

$$= P(X_{n+2}=1, X_{n+1}=2 \mid X_n=1) + P(X_{n+2}=1, X_{n+1}=3 \mid X_n=1)$$

$$+ P(X_{n+2}=1, X_{n+1}=4 \mid X_n=1)$$

$$= P(X_{n+2}=1 \mid X_{n+1}=2, X_n=1) \cdot P(X_{n+1}=2 \mid X_n=1)$$

$$+ P(X_{n+2}=1 \mid X_{n+1}=3, X_n=1) \cdot P(X_{n+1}=3 \mid X_n=1)$$

$$+ P(X_{n+2}=1 \mid X_{n+1}=4, X_n=1) \cdot P(X_{n+1}=4 \mid X_n=1)$$

$$= P_{21} P_{12} + P_{13} P_{31} + P_{14} \cdot P_{41}$$

$$f_{11}^{(3)} = \sum_{k,r\neq 1}^{4} P_{1k} P_{kr} P_{r1}$$

$$f_{11}^{(4)} = \sum_{\substack{k,r,j=1 \\ (\neq 1)}}^{4} P_{i\kappa} P_{\kappa r} P_{rj} P_{ji}$$

<u>First Entrance Theorem</u> : For any two states i, j

$$P_{ij}^{(n)} = \sum_{r=0}^{n} f_{ij}^{(r)} P_{ij}^{(n-r)}, \text{ with } P_{jj}^{(0)} = 1, \ f_{ij}^{(0)} = 0,$$

$$f_{ij}^{(1)} = P_{ij}$$

The above theorem can also be written as,

$$P_{ij}^{(n)} = f_{ij}^{(n)} + \sum_{r=1}^{n-1} f_{ij}^{(r)} P_{jj}^{(n-r)}$$

for $n=2$

$$p_{ij}^{(2)} = f_{ij}^{(2)} + \sum f_{ij}^{(1)} p_{jj}^{(1)}$$

$$p_{ij}^{(2)} = f_{ij}^{(2)} + f_{ij}^{(1)} p_{jj}^{(1)}$$

so, for $n=3$

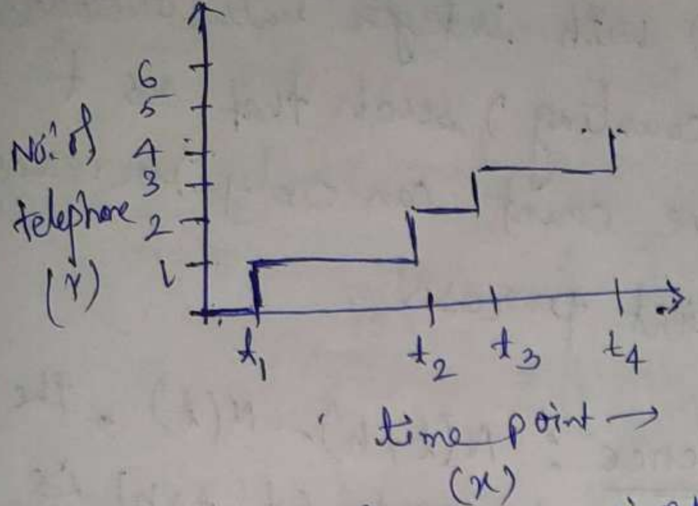
$$p_{ij}^{(3)} = f_{ij}^{(3)} + f_{ij}\, p_{jj}^{(2)} + f_{ij}^{(2)}\, p_{jj}$$

$$= f_{ij}^{(3)} + p_{ij}\left(f_{jj}^{(2)} + p_{jj}\, p_{jj}\right) + f_{ij}^{(2)}\, p_{jj}$$

and so on.

We can easily find $f_{ij}^{(n)}$

Poisson Process: Here we shall deal with stochastic processes with discrete state space and continuous time. One such process is poisson process. Consider a random event E such as, ① Incoming telephone calls, ② No. of accidents, ~~there~~. Let us consider the total no. $N(t)$ of the occurrences of the event E in an interval of duration t. For example, if an event actually occurs at instance of time $t_1, t_2, t_3, \cdots$; then $N(t)$ jumps from 0 to 1 at $t=t_1$, from 1 to 2 at $t=t_2$, so on. The situation can be represented graphically as follows —



The values of $N(t)$ given here are observed values of the random variable $N(t)$. Let $P_n(t)$ be the probability that the random variable $N(t)$ takes the value n i.e. $P_n(t) = P[N(t) = n]$, $n = 0, 1, 2, \ldots, \infty$

The probability is a function of the t. Since the only possible values of n are $0, 1, 2, \ldots$

therefore $\sum_{n=0}^{\infty} P_n(t) = 1$

Thus $\{P_n(t)\}$ represents the probability distribution of a random variable $N(t)$ for every value of t the family of random variables $\{N(t), t \geq 0\}$ is a SP.

$\boxed{\text{NOTE}}$ We will show that under certain conditions ~~N(t) follows~~, $N(t) \sim Poi(\lambda t)$

(~~In certain situations~~/conditions) In case of many empherical phenomena, these conditions are approximately true and the corresponding SP $\{N(t), t \geq 0\}$ follows the Poisson Law.

A Stochastic process $X(t)$ with integer value & state space (associated with counting) such that as t increases, the cumulative count can only increase is called a counting or point process.

$\boxed{\text{Postulates}}$ for poisson process

① Independence : $N(t+h) - N(t)$, the no. of occurence in the interval $(t, t+h)$ is independent of the no. of occurence prior to that ~~event~~. interval.

② Homogeneity ~~of that event~~ in time :

$P_n(t)$ depends only one the length t of the interval and is independent of where the interval is situated. i.e. $P_n(t)$ gives the probability of the no. of occurences (of E) in the interval t_1 to $t+t_1$ (i.e. of length t) for every t_1.

③ Regularity : Infinitesimal in an interval in a small interval of length h the probability of exactly one occurence is $\lambda h + o(h)$ and for more than one occurence is order of h, $o(h)$.

$\boxed{\text{Note}}$ As $h \to 0$, $\frac{o(h)}{h} \to 0$ (Mathematically if the interval $(t, t+h)$ is very small then $P_1(h) = \lambda h + o(h)$

and $P_K(h) = o(h) \; \forall \; K > 1$

Consequently $\sum_{K=2}^{\infty} P_K(h) = o(h)$

Again $\sum_{n=0}^{\infty} P_n(h) = 1$ it follows that

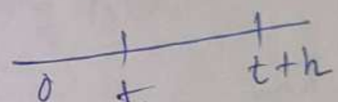
$$P_0(h) = 1 - \lambda h - o(h) - o(h)$$
$$= (1-\lambda h) + o(h)$$

<u>theorem</u>: Under the postualates ①, ②, ③, $N(t)$ follows poisson distribution with parameter λt

$N(t) \sim Poi(\lambda t)$ i.e. $P_n(t) = \dfrac{e^{-\lambda t} (\lambda t)^n}{n!}$,

$n = 0, 1, 2, \ldots$

→ Consider $P_n(t+h)$ $\forall n \geq 0$ can happen in n events by interval $t+h$ in the following mutually exclusive ways,



$A_1, A_2, \ldots A_{n+1}$.

For $n \geq 1$, A_1 : n evenents in the interval $(0, t)$ and no event in $(t, t+h)$

Therefore, $P(A_1) = P[N(t) = n]\, P[N(h) = 0 \mid N(t) = n]$

$$= P_n(t) \cdot P_0(h)$$
$$= P_n(t) - P_n(t)\lambda h + P_n(t)\, oh$$
$$= P_n(t)\left[(1-\lambda h) + o(h)\right]$$

A_2 : $(n-1)$ events in the interval $(0, t)$ and 1 event in $(t, t+h)$

$$P(A_2) = P[N(t) = n-1]\ P[N(h) = 1 \mid N(t) = n-1]$$

$$= P_{n-1}(t) \cdot P_1(h)$$

$$= P_{n-1}(t)\left(\lambda\{h\} + o(h)\right)$$

A_3 : $(n-2)$ events in the interval $(0,t)$ and 2 events in $(t, t+h)$

$$P(A_3) = P[N(t) = n-2]\ P[N(h) = 2 \mid N(t) = n-2]$$

$$= P_{n-2}(t) \cdot P_2(h)$$

$$= P_{n-2}(t) \cdot o(h) = o(h)$$

Therefore we have $P_n(t+h)$

$$= P_n(t)(1-\lambda h) + P_{n-1}(t)\lambda h + o(h) \quad , n \geq 1$$

$$\frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}$$

as $h \to 0$,

$$P_n'(t) = -\lambda\left[P_n(t) - P_{n-1}(t)\right] \quad \text{—(1)}$$

for $n = 0$, we get

$$P_0(t+h) = P_0(t) \cdot P_0(h)$$

$$= P_0(t)\left(1 - \cancel{\lambda(t)}^{\lambda h} + o(h)\right)$$

$$P_0(t+h) - P_0(t) = -\lambda h P_0(t) + o(h)$$

as $h \to 0$

$$P_0'(t) = -\lambda P_0(t) \quad \text{--- (2)}$$

at $t = 0$, $\left.\begin{array}{l} P_0(0) = 1 \\ \text{and } P_n(0) = 0 \end{array}\right\}$ initial conditions. — (3)

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

$$\frac{dP_0(t)}{P_0(t)} = -\lambda dt$$

$$\ln P_0(t) = -\lambda t + C$$

$$\Rightarrow \ln 1 = -0 + C$$

so, $C = 0$

at $t = 0$

$P_0(t) = 1$

$$\log P_0(t) = -\lambda t$$

$$P_0(t) = e^{-\lambda t}$$