## 1.1 Probability Distributions

### 1.1.1 Generating functions

### 1.1.2 Bivariate probability generating function

## 1.2 Stochastic Process

### 1.2.1 Introduction

In probability theory and related fields, a stochastic or, random process is a mathematical object usually defined as a family of random variables. In other words, a stochastic process (SP) deals with the description of a random sequence originates from a common random experiments associated with a probability structure. Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner.

► Simple Example: Coin: Suppose that $X_n$ is the outcome (assume $H = 1, T = 0$) of the $n^{th}$ throw $(n \geq 1)$ of a coin. Here all possible outcomes are $\Omega = \{H, T\}$, then $\{X_n, n \geq 1\}$ is a family or, sequence of random variables. Depends on $n = 1, 2, \cdots$, we have a distinct random variable $X_1, X_2, \cdots$. Here $\{X_n, n \geq 1\}$ is a stochastic process.
► Simple Example: Dice: Suppose that $X_n$ is the outcome of the $n^{th}$ throw $(n \geq 1)$ of a die. Here all possible outcomes are $\Omega = \{1, 2, 3, 4, 5, 6\}$, then $\{X_n, n \geq 1\}$ is a family or, sequence of random variables. Depends on $n = 1, 2, \cdots$, we have a distinct random variable $X_1, X_2, \cdots$. Here $\{X_n, n \geq 1\}$ is a stochastic process.
► Composite Example: Coin: Suppose that $X_n$ is the number of heads upto $n^{th}$ throw $(n \geq 1)$ of a coin. Depends on $n = 1, 2, \cdots$, we have a distinct random variable $X_1, X_2, \cdots$. Here $\{X_n, n \geq 1\}$ is a stochastic process.
► Simple Example: Telephone: Suppose that $X(t)$ is the number of telephone calls received in an interval $(0, t)$ of duration $t$ units. Here $\{X(t), t \in T\}$ is a stochastic process for $T = [0, \infty)$.
► Example: The growth of a bacterial population, an electrical current fluctuating due to thermal noise, the movement of a gas molecule, etc.
► Application: Stochastic processes have applications in many disciplines such as biology, chemistry, ecology, physics, image processing, signal processing, control theory, information theory, finance, etc.
► Key Processes: Applications and the study of phenomena have in turn inspired the proposal of new stochastic processes. *i)* Wiener process or, Brownian motion process used by Louis Bachelier to study price changes on the Paris Bourse. *ii)* Poisson process, used by A. K. Erlang to study the number of phone calls occurring in a certain period of time. These two stochastic processes are considered the most important and central in the theory of stochastic processes.
► The term random function is also used to refer to a stochastic or random process, because a stochastic process can also be interpreted as a random element in a function space.
► Types: Based on their mathematical properties, stochastic processes can be grouped into various categories, such as: random walks, martingales, Markov processes, Levy processes, Gaussian processes, renewal processes, branching processes, etc.

### 1.2.2 Definition

A stochastic process (SP) $\underset{\sim}{X} = \{X(t), t \in T\}$ is a collection of random variables. That is, for each $t$ in the index set $T$, $X(t)$ is a random variable. We often interpret $t$ as time and call $X(t)$ the state of the process at time $t$.

► If the index set $T$ is a countable set, we call $X$ a discrete-time stochastic process, and if $T$ is a continuum, we call it a continuous-time process.

► The set of possible values of a single random variable $X(t_1)$ of a stochastic process $\underset{\sim}{X}$ is known as its state space $S_1$. Also the set of all possible values of all the random variable $\{X(t), t \in T\}$ of a stochastic process $\underset{\sim}{X}$ is known as the state space of the stochastic process $\underset{\sim}{X}$ and denoted by $S$.

► State space (SS) may be discrete or, continuous.

► A SP has two components: State space $S$ and Time $T$.

**Example 1.1.** *Discrete SS: Let $X_n$ be the total number of heads appearing in the first $n$ throws of a coin, the set of possible values of $X_n$ (i.e. state space) are $0, 1, \cdots, n$. Here, the state space of $X_n$ is discrete.*

► *We can write $X_n = Y_1 + \cdots + Y_n$, where $Y_i$ is a discrete RV takes value 1 or, 0 according as the $i^{th}$ throw shows head or not.*

**Example 1.2.** *Continuous SS: Let $X_n = Y_1 + \cdots + Y_n$, where $Y_i$ is a continuous RV takes value takes positive values in $[0, \infty)$. Then the state space of $X_n$ is $[0, \infty)$.*

► Usually RVs $X(t)$ are one-dimensional, but the process $\{X(t)\}$ may be multi-dimensional. Consider $X(t) = (X_1(t), X_2(t), X_3(t))$, where $X_1$ represents the maximum, $X_2$ represents average and $X_3$ the minimum temperature at a place in an interval of time $(0, t)$. It is a three-dimensional stochastic process in continuous time having continuous state space.

► In general the RVs within a SP $\{X(t)\}$ are dependent.

► A SP $\{X(t), t \in T\}$ is with independent increments implies $\forall t_1, \cdots, t_n,\ t_1 < t_2 < \cdots < t_n$, the random variables $X(t_2) - X(t_1),\ X(t_3) - X(t_2), \cdots, X(t_n) - X(t_{n-1})$ are independent.

### 1.2.3 Classification

Generally, one-dimensional SP $\{X_t, t \geq 1\}$ or, $\{X(t), t \in T\}$ can be classified into four types:

1. Discrete time, discrete state space (Coin Toss): $X_t$ = Outcome of $t^{th}$ throw of a coin. Here $S = \{0, 1\}$, $T = \{1, 2, 3, \cdots\}$.

2. Discrete time, continuous state space (Interarrival Time): Let $\{Y_1, Y_2, \cdots\}$ denotes the inter-arrival times in a queing system. Define the time until $t^{th}$ arrival, $X_t = Y_1 + \cdots + Y_t$, where $Y_i$ is a continuous RV takes positive values in $[0, \infty)$. Here $S = [0, \infty)$, $T = \{1, 2, 3, \cdots\}$.

3. Continuous time, discrete state space (Telephone Call): $X_t$ = No. of phone calls within time interval $(0,t)$. Here $S = \{0,1,2,\cdots\}$, Theoretically $T = (0,\infty)$ but in practice, suppose we want to consider the time between $1-4$ pm then $T = (1,4)$.

4. Continuous time, continuous state space (Temperature): $X_t$ = Temperature at a place within time interval $(0,t)$. Here $S = (-10,100)$, assume temperature range in celsius, theoretically $S = (-\infty,\infty)$, $T = (0,\infty)$ but in practice, suppose we want to consider the time between $1-4$ pm then $T = (1,4)$.

■ Sample Path: Any realization of $\underset{\sim}{X}$ is called a sample path.

► For example Simple Coin:
Sample Path 1: $\{1,1,0,0,0,0,1,1,0,1,\cdots\}$, Sample Path 2: $\{1,1,0,1,0,1,0,0,0,1,\cdots\}$ and so on.

► For example Simple Telephone $[Say\ t : [0,2),[2,3),[3,6),\cdots]$:
Sample Path 1: $\{6,3,8,\cdots\}$, Sample Path 2: $\{4,5,5,\cdots\}$ and so on.