Test for the mean of a normal population when σ^2 not known.

$H_0: \mu = \mu_0, \quad 0 < \sigma^2 < \infty$

$H_1: \mu \neq \mu_0, \quad 0 < \sigma^2 < \infty$

$\Theta = \{(\mu, \sigma^2) : -\infty < \mu < \infty, 0 < \sigma^2 < \infty\}$

$\Theta_0 = \{(\mu, \sigma^2) : \mu = \mu_0, 0 < \sigma^2 < \infty\}$

The likelihood function of the sample observations $x_1, x_2, \ldots$ is given by

$$L = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2}$$

The maximum likelihood estimates of μ and σ^2 are given by

$$\hat{\mu} = \frac{1}{n}\sum x_i, \quad \hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \bar{x})^2 = s^2$$

$$\max_{\substack{\mu,\sigma^2 \\ \in\Theta}} L(\theta) = L(\hat{\Theta}) = \left(\frac{1}{2\pi s^2}\right)^{n/2} e^{-\frac{1}{2s^2}\sum(x_i - \bar{x})^2} = \frac{1}{(2\pi s^2)^{n/2}} e^{-\frac{ns^2}{2s^2}} = \frac{1}{(2\pi s^2)^{n/2}} e^{-n/2}$$

In Θ_0, the estimate of σ^2 for given $\mu = \mu_0$ is given by

$$\hat{\sigma}^2_{H_0} = \frac{1}{n}\sum(x_i - \mu_0)^2 = s_0^2$$
$$= \frac{1}{n}\sum(x_i - \bar{x})^2 + (\bar{x} - \mu_0)^2 + \frac{2}{n}\sum(x_i - \bar{x})\cdot(\bar{x} - \mu_0)$$
$$= \frac{1}{n}\sum(x_i - \bar{x})^2 + (\bar{x} - \mu_0)^2 + 0$$

$$\hat{\sigma}^2_{H_0} = \hat{\sigma}^2 + (\bar{x} - \mu_0)^2 = s_0^2$$

$$\max_{\substack{\mu,\sigma^2 \\ \in\Theta_0}} L(\theta) = L(\hat{\Theta}_0) = \frac{1}{(2\pi s_0^2)^{n/2}} e^{-\frac{1}{2s_0^2}\cdot n s_0^2} = \frac{1}{(2\pi s_0^2)^{n/2}} e^{-n/2}$$

$$\lambda = \frac{\max_{\mu,\sigma^2 \in \Theta_0} \hat{L}(\Theta_0)}{\max_{\mu,\sigma^2 \in \Theta} \hat{L}(\Theta)} = \left(\frac{s^2}{s_0^2}\right)^{n/2}$$
$$= \left\{\frac{s^2}{s^2 + (\bar{x} - \mu_0)^2}\right\}^{n/2}$$
$$= \left\{\frac{1}{1 + (\bar{x} - \mu_0)^2/s^2}\right\}^{n/2} \quad \text{—— (A)}$$

Now what is the distribution of $(\bar{x} - \mu_0)^2/s^2$?

$\cancel{\bar{x} - \mu_0}$ Under H_0, $\bar{x} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$, $\frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} \sim N(0, 1)$

$s'^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2 = \frac{ns^2}{n-1}$

Also, $\frac{(n-1)s'^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\Rightarrow \frac{(n-1)\times\frac{ns^2}{n-1}}{\sigma^2} \sim \chi^2_{n-1} \quad \therefore \frac{ns^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\left.\begin{array}{l}\dfrac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \sim N(0,1) \\[2ex] \dfrac{ns^2}{\sigma^2} \sim \chi^2_{n-1}\end{array}\right\rangle \text{ind}$$

Now
$$\frac{\dfrac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}}{\sqrt{\dfrac{ns^2}{\sigma^2}/n-1}} = \frac{N(0,1)}{\sqrt{\chi^2_{n-1}/n-1}} \sim t_{n-1}$$

upon simplification,
$$\frac{\bar{x}-\mu_0}{s/\sqrt{n-1}} \sim t_{n-1}, \quad \text{so } \frac{(\bar{x}-\mu_0)^2}{s^2} = \frac{t^2}{n-1}$$

Next, (A),
$$\lambda = \frac{1}{\left(1+\frac{t^2}{n-1}\right)^{n/2}}$$

$$W: \{\underline{x} : \lambda(\underline{x}) < \lambda_0\}$$
$$\equiv W: \left\{\underline{x} : \left(1+\frac{t^2}{n-1}\right)^{-n/2} < \lambda_0\right\}$$
$$\equiv W: \left\{\underline{x} : \left(1+\frac{t^2}{n-1}\right) > \lambda_0^{-2/n}\right\}$$
$$\equiv W: \left\{\underline{x} : t^2 \geq (\lambda^{-2/n}-1)(n-1)\right\} \equiv W: \{\underline{x} : t^2 \geq A^2\}$$

Thus the critical region may well be defined by
$$W: |t| = \left|\frac{\sqrt{n}(\bar{x}-\mu_0)}{s}\right| > A.$$

where $P\{|t| \geq A \mid H_0\} = \alpha$

From t-table, A is determined ~~by~~ as $t_{\alpha/2, n-1}$.

So, for testing $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$ (σ^2 unknown) we have the critical region as follows.

if $|t| > t_{\alpha/2, n-1}$, we reject H_0.

Case II $H_0: \mu = \mu_0,\ 0 < \sigma^2 < \infty$
against $H_1: \mu > \mu_0,\ 0 < \sigma^2 < \infty$.

$$\Theta = \{(\mu, \sigma^2): -\infty < \mu < \infty,\ 0 < \sigma^2 < \infty\}$$

$$\Theta_0 = \{(\mu, \sigma^2): \mu = \mu_0,\ 0 < \sigma^2 < \infty\}$$

The maximum likelihood estimates of μ and σ^2 belonging to Θ are given by

$$\hat{\mu} = \begin{cases} \bar{x} & \text{if } \bar{x} \geq \mu_0 \\ \mu_0 & \text{if } \bar{x} < \mu_0 \end{cases}$$

$$\hat{\sigma}^2 = \begin{cases} s^2 & \text{if } \bar{x} \geq \mu_0 \\ s_0^2 & \text{if } \bar{x} < \mu_0 \end{cases}$$

$$s_0^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2$$

$$L(\hat{\Theta}) = \left(\frac{1}{2\pi s^2}\right)^{n/2} e^{-n/2} \quad \text{if } \bar{x} \geq \mu_0$$

$$L(\hat{\Theta}_0) = \left(\frac{1}{2\pi s_0^2}\right)^{n/2} e^{-n/2} \quad \text{if } \bar{x} < \mu_0$$

$$\lambda = \begin{cases} \left(\dfrac{s^2}{s_0^2}\right)^{n/2} & \text{if } \bar{x} \geq \mu_0 \\ 1 & \text{if } \bar{x} < \mu_0 \end{cases}$$

Thus the sample observations $(x_1, x_2, \cdots, x_n)$ for which $\bar{x} < \mu_0$ are to be included in the acceptance region.
Hence for the sample observations for which $\bar{x} \geq \mu_0$, the likelihood ratio criterion becomes.

$$\lambda = \left(\frac{s^2}{s_0^2}\right)^{n/2}, \quad \bar{x} \geq \mu_0$$

Proceeding similarly as in $H_1: \mu \neq \mu_0$ we get the rejection region

$$t^2 = \frac{n(\bar{x} - \mu_0)^2}{s^2} \geq A^2 \text{ or by } t = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s} \geq A.$$

where $A = t_{n-1}^{(\alpha)}$.

Reject H_0 if. $t > t_{n-1, \alpha}$.