Remark 1

Neyman - Pearson MP test procedure may be looked upon as a particular case of SPRT.

Remark 2

$$z_i = \log \frac{f(\theta_1, \underline{x})}{f(\theta_0, \underline{x})}$$

The trivial case $P(z_i = 0) = 1$ will be ignored. In thi[s] case $f(x;\theta_1) = f(x;\theta_0)$ and H_0 will be accepted ev[en] without making any observation.

Determination of A and B

The stopping bounds A and B (<A) and strength (α, ...) are used to determine an SPRT.

$$A = \frac{1-\beta}{\alpha} \; ; \; B = \frac{\beta}{1-\alpha}, \quad 0 < \alpha < 1, \; 0 < \beta < 1.$$

$\boxed{Pf}$ According to the rule, the event is stopped exactly after m observations if

$$\lambda_m \geq A., \text{ say the } \cancel{\text{space}} \text{ set is}$$

$$R_m^1 = \{(x_1, x_2, \cdots, x_m) : \lambda_m(\underline{x}) \geq A\}.$$

$$R_m^0 = \{(x_1, x_2, \cdots, x_m) : \lambda_m(\underline{x}) \leq B\}.$$

If an observed $\underline{x} \in R_m^1$, it means after taking the m th observation H_0 is rejected.

Hence

$$\alpha = P[\underline{x} \in \text{rejection region} / H_0 \text{ is true}]$$

$$= P[\underline{x} \in \bigcup_{n=1}^{\infty} R_m^1 \mid H_0]$$

$$= \sum_{n=1}^{\infty} P(R_m^1 / H_0) \quad (\text{since the sets } R_m^1, m=1,2,\cdots \text{ are mutually disjoint})$$

$$= \sum \int_{R_m^1} f_0(\underline{x}) d\underline{x} \leq \frac{1}{A} \sum_{m=1}^{\infty} \int_{R_m^1} f_1(\underline{x}) d\underline{x}$$

$$\left[\because \frac{f_1}{f_0} \geq A\right]$$

$$= \frac{1}{A} P\{\underline{x} \in \bigcup_{m=1}^{\infty} R_m^1 / H_1\}$$

$$= \frac{1}{A} P\{H_0 \text{ is rejected} / H_1\}$$

$$= \frac{1-\beta}{A}.$$

Similarly,

$$\beta = P\left\{ \bigcup_{m=1}^{\infty} R_m^0 / H_1 \right\}$$

$$= \sum_{m=1}^{\infty} \int_{R_m^0} f_{1m}(\underline{x})\, d\underline{x} \le B \sum_{m=1}^{\infty} \int_{R_m^0} f_0(\underline{x})\, d\underline{x} \qquad \left[\because \frac{f_1}{f_0} \le B\right]$$

$$= B \cdot P\{H_0 \text{ is accepted} / H_0\}$$

$$= B(1-\alpha)$$

$$\therefore \quad \alpha \le \frac{1-\beta}{A} \Rightarrow A \le \frac{1-\beta}{\alpha}.$$

$$\text{and} \quad \beta \le B(1-\alpha) \Rightarrow B \ge \frac{\beta}{1-\alpha}.$$

An appoximation is made by equating the stopping boundaries

$$A = \frac{1-\beta}{\alpha}$$

$$B = \frac{\beta}{1-\alpha}.$$

Remark 3

SPRT terminates with probability 1 under both H_0 and H_1.

[proof omitted].

Example

Give the SPRT for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1 (> \theta_0)$ in sampling from a normal density.

$$f(x,\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

where σ known.

$\boxed{\text{Ans}}$

$$\frac{f(x_i,\theta_1)}{f(x_i,\theta_0)} = e^{-\frac{1}{2\sigma^2}\left[(x_i-\theta_1)^2 - (x_i-\theta_0)^2\right]}$$

$$= e^{-\frac{1}{2\sigma^2}\left[(\theta_0-\theta_1)(2x_i-\theta_0-\theta_1)\right]}$$

$$z_i = \log \frac{f(x_i,\theta_1)}{f(x_i,\theta_0)} = \frac{\theta_1-\theta_0}{\sigma^2}\left(x_i - \frac{\theta_0+\theta_1}{2}\right)$$

$$\log \lambda_m = \sum_{i=1}^{m} z_i = \frac{\theta_1-\theta_0}{\sigma^2}\left[\Sigma x_i - \frac{m(\theta_0+\theta_1)}{2}\right]$$

Reject H_0 if

$$\frac{\theta_1-\theta_0}{\sigma^2}\left[\Sigma x_i - \frac{m(\theta_0+\theta_1)}{2}\right] \ge \log \frac{1-\beta}{\alpha}$$

$$\Rightarrow \sum_{i=1}^{m} x_i \ge \frac{\sigma^2}{\theta_1-\theta_0} \log\left(\frac{\beta}{1-\alpha}\right) + \frac{m(\theta_0+\theta_1)}{2}; \quad \theta_1 > \theta_0$$

Accept H_0 if.

$$\frac{\theta_1-\theta_0}{\sigma^2}\left[\sum x_i - \frac{m(\theta_0+\theta_1)}{2}\right] \le \log\frac{\beta}{1-\alpha}.$$

$$\Rightarrow \sum_{i=1}^{m} x_i \le \frac{\sigma^2}{\theta_1-\theta_0}\log\frac{\beta}{1-\alpha} + \frac{m(\theta_0+\theta_1)}{2} \quad ; (\theta_1>\theta_0)$$

continue taking additional observation as long as.

$$\log\frac{\beta}{1-\alpha} < \frac{\theta_1-\theta_0}{\sigma^2}\left[\sum x_i - \frac{m(\theta_0+\theta_1)}{2}\right] < \log\frac{1-\beta}{\alpha}.$$

$$\Rightarrow \frac{\sigma^2}{\theta_1-\theta_0}\log\frac{\beta}{1-\alpha} + \frac{m(\theta_0+\theta_1)}{2} < \sum x_i < \frac{\sigma^2}{\theta_1-\theta_0}\log\frac{1-\beta}{\alpha} + \frac{m(\theta_0+\theta_1)}{2}$$

$\theta_1>\theta_0$.

Operating characteristic (O.C.) function of SPRT

~~...~~ It may be that the true value of θ is neither θ_0 nor θ_1 although we are testing $H_0:\theta=\theta_0$ against $H_0:\theta=\theta_1$. Performance $(L(\theta))$ of SPRT for testing $H_0:(\theta=\theta_0)$ against $H_1:(\theta=\theta_1)$.

The OC function $L(\theta)$ is defined as.

$L(\theta)=$ Probability of accepting $H_0: \theta=\theta_0$ when θ is the true value of the parameter and since the power function.

$P(\theta)=$ Probability of rejecting H_0 where θ is (false null) the true value, we get

$$L(\theta)=1-P(\theta)$$

An approximate formula of OC function can be derived in terms of A and B as follows.

$$L(\theta)=\frac{A^{h(\theta)}-1}{A^{h(\theta)}-B^{h(\theta)}}$$

where for each value of θ, the value of $h(\theta)$ is to be determined so that

$$E\left[\frac{f(x,\theta_1)}{f(x,\theta_0)}\right]^{h(\theta)}=1. \qquad (*)$$

It is proved that there exists unique value of $h(\theta)\neq 0$ such that $(*)$ is satisfied.

Example 6 Refer back to the 1st example.

$X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli(θ).

$H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1\ (>\theta_0)$

$$E_\theta\left[\frac{f(x,\theta_1)}{f(x,\theta_0)}\right]^{h(\theta)} = E_\theta\left[\frac{\theta_1^{x}(1-\theta_1)^{1-x}}{\theta_0^{x}(1-\theta_0)^{1-x}}\right]^{h(\theta)}$$

$$= \theta\left(\frac{\theta_1}{\theta_0}\right)^{h(\theta)} + (1-\theta)\left(\frac{1-\theta_1}{1-\theta_0}\right)^{h(\theta)}$$

Now using (*), the equation is

$$\theta\left(\frac{\theta_1}{\theta_0}\right)^{h(\theta)} + (1-\theta)\left(\frac{1-\theta_1}{1-\theta_0}\right)^{h(\theta)} = 1.$$

$$\Rightarrow \quad \theta = \frac{1-\left(\frac{1-\theta_1}{1-\theta_0}\right)^{h}}{\left(\frac{\theta_1}{\theta_0}\right)^{h} - \left(\frac{1-\theta_1}{1-\theta_0}\right)^{h}}$$

$$L(\theta) \simeq \frac{\left(\frac{1-\beta}{\alpha}\right)^{h} - 1}{\left(\frac{1-\beta}{\alpha}\right)^{h} - \left(\frac{\beta}{1-\alpha}\right)^{h}}$$

various points on the o.c. curve are obtained by assigning arbitrary values of h and computing the corresponding value of θ and $L(\theta)$.

Example Refer back to normal problem.

First determine h.

$$E\left[\frac{f(x,\theta_1)}{f(x,\theta_0)}\right]^{h} = \int_{-\infty}^{\infty}\left[\frac{f(x;\theta_1)}{f(x,\theta_0)}\right]^{h} f(x;\theta)\,dx = 1.$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2} \cdot e^{-\left[\frac{1}{2\sigma^2}(\theta_1-\theta_0)(-2x+\theta_0+\theta_1)\right]^{h}}\,dx = 1$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2x\left((\theta_1-\theta_0)h+\theta\right) + \theta^2 + (\theta_1^2-\theta_0^2)h\right]}\,dx = 1.$$

$(\theta_1-\theta_0)^2h^2 + \theta^2$
$+ 2\theta h(\theta_1-\theta_0)$
$-\theta^2 - (\theta_1^2-\theta_0^2)h$

$$\cancel{\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x-(\theta_1-\theta_0)h+\theta\right]^2} \cdot \text{so}}$$

Try to make it in a perfect square, the adjustment factor will be

(solve it) $$\{(\theta_1-\theta_0)h+\theta\}^2 = (\theta_1^2-\theta_0^2)h+\theta.$$

$$\Rightarrow \quad h(\theta) = \frac{\theta_1+\theta_0-2\theta}{\theta_1-\theta_0}$$

Average Sample Number (A.S.N.)

The number of observation N required for a sequential test is a random variable. The expected value of N, $E_\theta(N)$ depends on the test procedure and distribution of X.

$E_\theta(N)$ is called average sample number. Smaller ASN better is the ple

For testing $H_0: \theta=\theta_0$ against $H_1: \theta=\theta_1$ is given by

$$E(N) = \frac{L(\theta)\log B + (1-L(\theta))\log A}{E(Z)}$$

$$Z = \log \frac{f(x;\theta_1)}{f(x,\theta_0)}.$$

Refer back to Bernoulli distribution $\quad f(x;\theta)=\theta^x(1-\theta)^{1-x}; \; x=0,1$, $0<\theta<1$

$$E(Z) = \sum_{x=0}^{1} \log\left[\frac{f(x,\theta_1)}{f(x,\theta_0)}\right]\cdot f(x;\theta)$$

$$= \sum_{x=0}^{1} \log\left[\left(\frac{\theta_1}{\theta_0}\right)^x \left(\frac{1-\theta_1}{1-\theta_0}\right)^{1-x}\right]\theta^x(1-\theta)^{1-x}.$$

$$= (1-\theta)\log\frac{1-\theta_1}{1-\theta_0} + \theta\cdot\log\frac{\theta_1}{\theta_0}.$$

$$= \theta\log\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)} + \log\frac{1-\theta_1}{1-\theta_0}.$$

$$E(N) = \frac{L(\theta)\log B + (1-L(\theta))\log A.}{E(Z)}$$

Refer back to Normal distribution

$$Z = \log\frac{f(x;\theta_1)}{f(x;\theta_0)} = \frac{\theta_1-\theta_0}{2}\left(x - \frac{\theta_0+\theta_1}{2}\right)$$

$$E(Z) = \frac{\theta_1-\theta_0}{2\sigma^2}\left(2E(x)-\theta_0-\theta_1\right) = \frac{\theta_1-\theta_0}{2\sigma^2}\left(2\theta-\theta_0-\theta_1\right)$$

Replace here $\quad E(N) = \dfrac{L(\theta)\log B + (1-L(\theta))\log A}{E(Z)}.$

$\boxed{\text{H.W.}}$ Let X have the p.d.f.

$$f(x,\theta) = \begin{cases} \theta e^{-\theta x}; & x\geq 0,\ \theta>0 \\ 0. \end{cases}$$

Construct the SPRT, obtain the O.C. function and ~~SPR~~ ASN.

Practical Problem

① Obtain the ASN and OC function for testing $H_0: \theta = \frac{1}{3}$ against $H_1: \theta = \frac{2}{3}$.

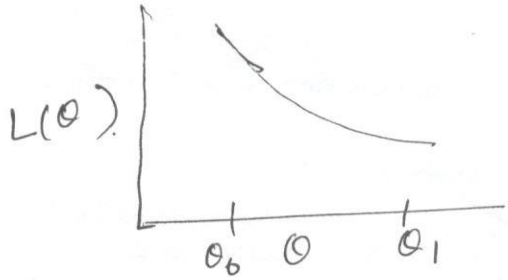
$$f(x;\theta) = \begin{cases} \theta^x (1-\theta)^{1-x}, & x = 0, 1 \\ 1 \end{cases}$$

$\alpha = .05$, $\beta = .05$, $h = -\infty, -1, 0, 1, 2, 3, 10, \infty$

Draw OC curve also.

Hint.
$$\theta = \frac{1 - \left(\frac{1-\theta_1}{1-\theta_0}\right)^h}{\left(\frac{\theta_1}{\theta_0}\right)^h - \left(\frac{1-\theta_1}{1-\theta_0}\right)^h}, \quad L(\theta) = \frac{\left(\frac{1-\beta}{\alpha}\right)^h - 1}{\left(\frac{1-\beta}{\alpha}\right)^h - \left(\frac{\beta}{1-\alpha}\right)^h}$$

Putting h, find θ, then find $L(\theta)$.



Remember $0 < \theta < 1$

For each θ, $L(\theta)$ find $E(N) = ASN$

② Let $X \sim N(\theta, 1)$, $H_0: \theta = 0$ $H_1: \theta = 2$, $\alpha = .05$, $\beta = .1$

Take $h = -\infty, -1, 0, 1, 2, 3, \infty$

Find $L(\theta)$ and $E(N)$.