

Stationarity Conditions for an AR(2) Process

We can define the characteristic equation as $C(z) = 1 - \phi_1 z - \phi_2 z^2 = 0$, and require the roots to lie *outside* the unit circle, or we can write it as $C(z) = z^2 - \gamma_1 z - \gamma_2 = 0$, and require the roots to lie *inside* the unit circle. The latter approach is slightly simpler in this case.

$C(z) = z^2 - \gamma_1 z - \gamma_2 = 0$ implies that $z = [\gamma_1 \pm \sqrt{\gamma_1^2 + 4\gamma_2}] / 2$. Now, the roots will be *real* if $\gamma_1^2 + 4\gamma_2 \geq 0$. If $\gamma_1^2 + 4\gamma_2 < 0$, the roots will be *complex*, and this specifies the parabolic boundary for the oscillatory case.

When the roots are real, the larger root must be less than unity, and the smaller root must exceed negative unity, if they are both to lie inside the unit circle. So, we require:

- (i) $[\gamma_1 + \sqrt{\gamma_1^2 + 4\gamma_2}] / 2 < 1$, which implies that $\sqrt{\gamma_1^2 + 4\gamma_2} < 2 - \gamma_1$. This in turn implies that $\gamma_1^2 + 4\gamma_2 < 4 + \gamma_1^2 - 4\gamma_1$, or $\gamma_1 + \gamma_2 < 1$.
- (ii) $[\gamma_1 - \sqrt{\gamma_1^2 + 4\gamma_2}] / 2 > -1$, which implies that $\gamma_1 - \sqrt{\gamma_1^2 + 4\gamma_2} > -2$. This in turn implies that $(\gamma_1 + 2)^2 > \gamma_1^2 + 4\gamma_2$, or $\gamma_1^2 + 4 + 4\gamma_1 > \gamma_1^2 + 4\gamma_2$, or $\gamma_2 - \gamma_1 < 1$.

Finally, note that if $C(z)$ has two real roots, say λ_1 and λ_2 , then

$$C(z) = (z - \lambda_1)(z - \lambda_2) = 0, \text{ or } C(z) = [z^2 - (\lambda_1 + \lambda_2)z + \lambda_1\lambda_2] = 0.$$

Stationarity requires that $|\lambda_i| < 1$; $i = 1, 2$. So, equating coefficients, we have:

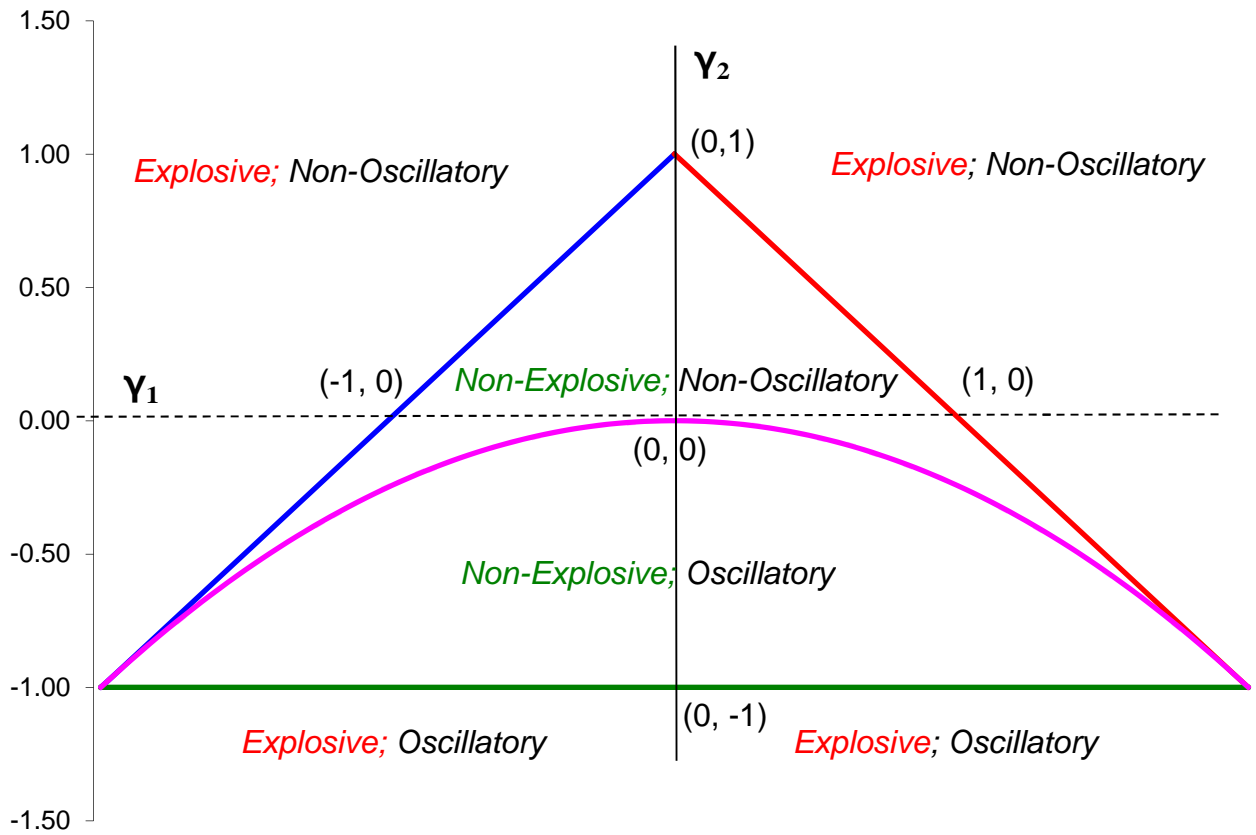
$$\gamma_1 = (\lambda_1 + \lambda_2), \text{ and } \gamma_2 = -\lambda_1\lambda_2.$$

Now, if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then $|\lambda_1\lambda_2| < 1$, which implies that $|\gamma_2| < 1$.

To summarize, in the case of real roots, the stationarity conditions for an AR(2) process are:

$$(\gamma_1 + \gamma_2) < 1, (\gamma_2 - \gamma_1) < 1, \text{ and } |\gamma_2| < 1.$$

All of this is illustrated in the diagram on the next page, where stationarity requires that the roots have to be inside the red, blue and green triangle.



Source: This diagram is based on Figure 7.1, on p.196 of A. Zellner, *An Introduction to Bayesian Inference in Econometrics*, Wiley, New York, 1971.

Exponential Decay

